UNIVERSIDAD NACIONAL DE CÓRDOBA

FACULTAD DE MATEMÁTICA, ASTRONOMÍA Y FÍSICA

SERIE “A”

TRABAJOS DE FÍSICA

Nº 10/10

On the long-time asymptotics of quantum dynamical semigroups

G.A. Raggio - P.R. Zangara

Editores: Miguel A. Chesta–Ricardo C. Zamar

CIUDAD UNIVERSITARIA – 5000 CÓRDOBA

REPÚBLICA ARGENTINA

On the long-time asymptotics of quantum dynamical semigroups*

G.A. Raggio† and P.R. Zangara
FaMAF, Universidad Nacional de Córdoba,
Córdoba, Córdoba X5000, Argentina

June 1, 2010

Abstract

We consider semigroups $\{\alpha_t : \ t \geq 0\}$ of normal, unital, positive maps α_t on a W^*-algebra $\mathcal{M}$. The (predual) semigroup $\nu_t(\rho) := \rho \circ \alpha_t$ on normal states ρ of $\mathcal{M}$ leaves invariant the face $\mathcal{F}_p := \{\rho : \ \rho(p) = 1\}$ supported by the projection $p \in \mathcal{M}$, iff $\alpha_t(p) \geq p$ (i.e., p is sub-harmonic). We complete the arguments showing that the sub-harmonic projections form a complete lattice. We then consider r_o, the smallest projection which is larger than each support of a minimal invariant face; then r_o is subharmonic. In finite dimensional cases and when α_t is completely positive, $\sup \alpha_t(r_o) = \mathbf{1}$ and r_o is also the smallest projection p for which $\alpha_t(p) \to \mathbf{1}$. If $\{\nu_t : \ t \geq 0\}$ admits a faithful family of normal stationary states then $r_o = \mathbf{1}$ is useless; if not, it helps to reduce the problem of the asymptotic behaviour of the semigroup for large times.

Keywords: Quantum dynamical semigroups; sub-harmonic projections; long-time asymptotics.

1 Introduction and preliminaries

We consider a von Neumann algebra $\mathcal{M}$ and denote its normal state space by $\mathcal{S}$. A *quantum dynamical semigroup* $\{\alpha_t : \ t \geq 0\}$ is a family of normal, unital, positive, linear maps $\alpha_t : \mathcal{M} \to \mathcal{M}$ with the property $\alpha_t \circ \alpha_s = \alpha_{t+s}$ where α_0 is the identity. Then, the map $\nu_t : \mathcal{S} \to \mathcal{S}$ defined by $\nu_t(\rho) = \rho \circ \alpha_t$ is affine, ν_0 is the identity, and $\nu_t \circ \nu_s = \nu_{t+s}$. Conversely, given a semigroup $\{\nu_t : \ t \geq 0\}$ of affine maps on $\mathcal{S}$, the dual maps are a positive quantum dynamical semigroup.

One often demands on physical grounds, that α_t be completely positive. When $\mathcal{M}$ is the algebra of all bounded linear operartors on a Hilbert space, the generator of a one-parameter semigroup of completely positive, linear, normal, unital maps which is strongly continuous in t has the canonical GKS-Lindblad form.

The long-time asymptotics of such semigroups has been studied in the 1970's and in the 1980's, after pioneering papers of E.B. Davies[1] , culminating with the work of Frigerio[2, 3, 4] , and U. Groh [5] . More recent studies are due to Fagnola & Rebolledo[6, 7] , Umanitá[8] , Mohari[10, 11] and Baumgartner & Narnhofer[12] . We refer to Ref. [7] for a recent overview. In pertinent cases, the asympotics can be studied via the GKS-Lindblad generator.

In this note all states (positive linear functionals of unit norm) are normal. Limits of states are with respect to the distance induced by the norm. But recall that the norm-closure of a convex set of states coincides with its weak-closure. All projections are ortho-projections (self-adjoint idempotents). For a projection p, $p^{\perp} = \mathbf{1} - p$. Limits in $\mathcal{M}$ are invariably in the w^*-topology. The *support* of a state ρ –written s_ρ– is the smallest projection $p \in \mathcal{M}$ such that $\rho(p) = 1$.
We will consider a quantum dynamical semigroup $\{\alpha_t : \ t \geq 0\}$ and will always explicitly state additional positivity hypotheses. In particular, if each α_t is completely positive, we say that the semigroup is CP. A state ω is *stationary* if $\nu_t(\omega) = \omega \circ \alpha_t = \omega$.

*Presented in the 30th Conference on Quantum Probability and Related Topics, held in Santiago, Chile, November 23-28, 2009.
†E-mail: raggio@famaf.unc.edu.ar

2 Invariant faces and sub-harmonic projections

A *face* is a convex subset $\mathcal{F}$ of $\mathcal{S}$ which is stable under convex decomposition: if $t\rho + (1-t)\mu \in \mathcal{F}$ for $0 < t < 1$ with $\rho, \mu \in \mathcal{S}$ then $\rho, \mu \in \mathcal{F}$. If $p \in \mathcal{M}$ is a projection then $\mathcal{F}_p := \{\rho \in \mathcal{S} : \rho(p) = 1\}$ is a closed face; namely the face *supported* by p. A fundamental result[13, 14] is that every closed face is of this form; i.e. it is the face supported by some projection. Clearly, $\mathcal{F}_p \subset \mathcal{F}_q$ iff $p \leq q$. The following result is implicit or partially explicit in the work of Fagnola & Rebolledo and Umanitá.

Proposition 1 *Suppose ν is an affine map of $\mathcal{S}$ into itself and let α be the dual normal, linear, positive map of $\mathcal{M}$ into itself. For a projection $p \in \mathcal{M}$ the following conditions are equivalent: (1) the face $\mathcal{F}_p$ supported by p is ν-invariant; (2) $\alpha(p) \geq p$; (3) $p\alpha(a)p = p\alpha(pap)p$ for every $a \in \mathcal{M}$; (4) $\alpha(p^\perp a p^\perp) = p^\perp \alpha(p^\perp a p^\perp) p^\perp$ for every $a \in \mathcal{M}$.*

Proof: We first prove the chain (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (2). If $\alpha(p) \geq p$ then, for any state ρ one has $\nu(\rho)(p) = \rho(\alpha(p)) \geq \rho(p)$. Thus, $\rho(p) = 1$ implies $\nu(\rho)(p) = 1$, i.e. $\nu(\mathcal{F}_p) \subset \mathcal{F}_p$. If $\nu(\mathcal{F}_p) \subset \mathcal{F}_p$, we show that

$$(^*) \quad \omega(p\alpha(pap)p) = \omega(p\alpha(a)p) \text{ for every } \omega \in \mathcal{S} .$$

Since every normal linear functional is the linear combination of at most four states, this then implies that $p\alpha(pap)p = p\alpha(a)p$. To prove (*) observe that, by the Cauchy-Schwarz inequality for states, the claim is trivially valid if $\omega(p) = 0$. Otherwise, consider the state $\omega_p(a) := \omega(pap)/\omega(p)$. Clearly $\omega_p \in \mathcal{F}_p$; thus,

$$\omega(p)^{-1}\omega(p\alpha(pap)p) = \omega_p(\alpha(pap)) = \nu(\omega_p)(pap) = \nu(\omega_p)(a)$$

$$= \omega_p(\alpha(a)) = \omega(p)^{-1}\omega(p\alpha(a)p) ;$$

which is (*). Finally, if $p\alpha(pap)p = p\alpha(a)p$, then $p - p\alpha(p)p = p\alpha(p^\perp)p = 0$ and Lemma 2 of the Appendix implies $\alpha(p) \geq p$.
If $0 \leq x = p^\perp x p^\perp \leq \mathbf{1}$ then, by Lemma 2 of the Appendix, $x \leq p^\perp$ and $\alpha(x) \leq \alpha(p^\perp)$; when (2) is the case $\alpha(p^\perp) \leq p^\perp$ so that $\alpha(x) \leq p^\perp$ which by the aforementioned Lemma, implies $p^\perp\alpha(x)p^\perp = \alpha(x)$. For general $0 \leq x = p^\perp x p^\perp$ we consider $x/\|x\|$ and obtain $p^\perp\alpha(x)p^\perp = \alpha(x)$. Since every $a \in \mathcal{M}$ is a linear combination of at most four positive elements, we conclude that (2) implies (4). But (4) implies $\alpha(p^\perp) = p^\perp\alpha(p^\perp)p^\perp$ which, by the same Lemma, gives $\alpha(p^\perp) \leq p^\perp$ which is equivalent to $\alpha(p) \geq p$. □

In the context of positive one-parameter semigroups $\{\alpha_t : t \geq 0\}$, a projection p satisfying $\alpha_t(p) \geq p$ has been termed *sub-harmonic*[6] . We say the projection p is *sub-harmonic* for the linear, normal, unital and positive map α on $\mathcal{M}$ if $\alpha(p) \geq p$.
Proposition 1 relates the sub-harmonic property of a projection to the more geometric notion of invariance of the supported face. This relationship can be immediately put to use:

Proposition 2 *If a family of projections is sub-harmonic for a linear, normal, unital and positive map α on $\mathcal{M}$, then the infimum of the family is sub-harmonic for α.*

Proof: If $\{\mathcal{F}_\iota : \iota \in I\}$ is a family of closed faces $\mathcal{F}_\iota$ of $\mathcal{S}$ then $\bigcap_\iota \mathcal{F}_\iota$ is, clearly, a closed face and it is the largest closed face contained in each $\mathcal{F}_\iota$. The support of $\bigcap_\iota \mathcal{F}_\iota$ is exactly $\inf\{p_\iota : \iota \in I\}$, where p_ι is the support of $\mathcal{F}_\iota$. Moreover, if each $\mathcal{F}_\iota$ is ν-invariant then so is the intersection. □

The corresponding statement for the supremum of such a family has been observed and proved (directly)[8] . For projections p that are *super-harmonic*, i.e. $\alpha(p) \leq p$ (equivalently $p^\perp$ is sub-harmonic), we have (in reply to a question posed in Ref. [9]):

Corollary 1 *If a family of projections is super-harmonic for a linear, normal, unital and positive map α on $\mathcal{M}$, then the supremum of the family is super-harmonic for α.*

Proof: $\sup\{p : p \in \mathcal{F}\} = (\inf\{p^\perp : p \in \mathcal{F}\})^\perp$ and $\inf\{p^\perp : p \in \mathcal{F}\}$ is sub-harmonic by the previous proposition. □

The corresponding statement for the infimum of a super-harmonic family follows from the result for the supremum of a sub-harmonic family by orthocomplementation as above. Thus,

Theorem 1 *The set of sub-harmonic and the set of super-harmonic projections with respect to a linear, normal, unital and positive map on $\mathcal{M}$ are both complete lattices.*

A *minimal invariant face* is a closed ν_t-invariant face which does not properly contain another non-empty closed ν_t-invariant face. Equivalently, it is a face whose support is a minimal sub-harmonic projection, i.e. a sub-harmonic projection that is not larger than a non-zero sub-harmonic projection other than itself. One can prove, and this goes back to –at least– Davies (see Ref. [1], Theorem 3.8 of Sect. 6.3), that if the minimal invariant face admits a stationary state then it is unique and its support is the support of the face. Moreover (Ref. [5], Proposition 3.4) the restriction of ν_t to the face is ergodic (the Cesàro means converge to the stationary state).

3 A "recurrent" projection

We define the *minimal recurrent* projection r_o as the smallest projection which is larger than every minimal sub-harmonic projection. Equivalently, r_o is the support of the smallest ν_t-invariant face which contains every minimal ν_t-invariant face. By virtue of its definition and the result mentioned above –to the effect that the supremum of a family of sub-harmonic projections is sub-harmonic– it follows that the minimal recurrent projection is sub-harmonic. Hence the directed family $\alpha_t(r_o)$ which is bounded above by $\mathbf{1}$ has a lowest upper bound in $\mathcal{M}$ denoted by x which is positive and below $\mathbf{1}$. Since $x = \lim_{t\to\infty} \alpha_t(r_o)$ it follows that $\alpha_t(x) = x$ for every $t \geq 0$. Let $s[x]$ denote the support of x, that is the smallest projection $p \in \mathcal{M}$ with $xp = x$. The following treatment follows the lines of work by Mohari[11] .

Lemma 1 *If $\{\alpha_t : t \geq 0\}$ is CP, then $s[x] = \mathbf{1}$.*

<u>Proof</u>: $s[x]^\perp$ is the largest projection q with $xq = 0$, and it is sub-harmonic by a result of Ref. [11] quoted in the appendix. Assume that $s[x] \neq \mathbf{1}$; then there is a minimal sub-harmonic non-zero projection q with $q \leq s[x]^\perp$. One has $xq = 0$. By the definition of r_o, we have $q \leq r_o$ and thus $q = qr_oq \leq q\alpha_t(r_o)q \leq qxq = 0$, which contradicts the assumption. □

Let $\mathcal{J} := \{a \in \mathcal{M} : \lim_{t\to\infty} \alpha_t(a^*a) = 0\}$. Since for each state ρ, one has the Cauchy-Schwarz inequality

$$|\rho(\alpha_t(a^*b^*))| = |\rho(\alpha_t(ba))| = |\nu_t(\rho)(ba)|$$
$$\leq \sqrt{\nu_t(\rho)(bb^*)\nu_t(\rho)(a^*a)} \leq \|b\| \sqrt{\rho(\alpha_t(a^*a))} \ ;$$

we infer that $\mathcal{J}$ is a linear subspace of $\mathcal{M}$. If $c \in \mathcal{M}$ and $a \in \mathcal{J}$, the same inequality applied to $b = a^*c^*c$ shows that $ca \in \mathcal{J}$; thus $\mathcal{J}$ is a left-ideal.

If $\mathcal{M}$ is finite-dimensional (that is *-isomorphic to the direct sum of finitely many full matrix algebras) then, on the one hand, $s[x] = \mathbf{1}$ implies that x is invertible, and Mohari has shown that if x is invertible then $x = \mathbf{1}$; and –on the other hand– $\mathcal{J}$ is closed and there is a projection such that $\mathcal{J} = \mathcal{M} \cdot z$. Then

Theorem 2 *If $\mathcal{M}$ is finite dimensional and $\{\alpha_t : t \geq 0\}$ is CP, then* $\sup\{\alpha_t(r_o) : t \geq 0\} = \mathbf{1}$. *Moreover $\mathcal{J} = \mathcal{M} \cdot r_o^\perp$ and r_o is the smallest projection $p \in \mathcal{M}$ with* $\lim_{t\to\infty} \alpha_t(p) = \mathbf{1}$.

<u>Proof</u>: There is[13] a projection $z \in \mathcal{M}$ with $\mathcal{J} = \mathcal{M} \cdot z$. Lema 1 implies that x is invertible and Theorem 2.5 of Ref. [11] gives $x = \mathbf{1}$. Hence $r_o^\perp \in \mathcal{J}$ and thus $r_o^\perp \leq z$ or $r_o \geq z^\perp$. Suppose p is a minimal sub-harmonic projection; there is a stationary state ω in the minimal invariant face supported by p and it follows (see end of §2) that it is unique and $s_\omega = p$. Since $\omega(z) = \omega(\alpha_t(z)) \to 0$, we have $\omega(z^\perp) = 1$ and thus $p \leq z^\perp$. But then, by the definition of r_o, $r_o \leq z^\perp$. Thus $r_o = z^\perp$. □

It follows from the Cauchy-Schwarz inequality for states that $\lim_{t\to\infty} \alpha_t(ar_o^\perp) = \lim_{t\to\infty} \alpha_t(r_o^\perp a) = 0$ for every $a \in \mathcal{M}$ so that $\alpha_t(a) \asymp \alpha_t(r_oar_o)$ for large t and every $a \in \mathcal{M}$. If $\{\nu_t : t \geq 0\}$ admits a faithful family of stationary states, then the minimal recurrent projection is the identity. This happens because for a stationary state ω, one has $\omega(r_o^\perp) = \omega(\alpha_t(r_o^\perp)) \downarrow 0$ and thus $\omega(r_o^\perp) = 0$. However in this case there are results[4, 5, 10] on the asymptotic behaviour of the semigroup.
Other recurrent projections have been considered. For example (Ref. [5], p. 407; Ref. [8]), the supremum r of the supports of the stationary states (if any are available), which is then sub-harmonic and above r_o.

There is no reason to expect that the above theorem 2 holds in infinite dimension.

Acknowledgements

We thank the organizers of the 30th Conference on Quantum Probability and Related Topics, held in Santiago, Chile. The support of CONICET (PIP 11220080101741) is acknowledged. The first author is grateful to M.E. Martín Fernández for generous support.

4 Appendix

We collect here a number of technical results used in the above proofs.

Lemma 2 *For $x \in \mathcal{M}$ satisfying $\mathbf{1} \geq x \geq 0$ and $p \in \mathcal{M}$ a projection one has:*

a) the following five conditions are equivalent:

$$(1)\ x \geq p\,;\ \ (2)\ pxp = p\,;\ \ (3)\ x = p + p^{\perp}xp^{\perp}\,;\ \ (4)\ xp = p\,;\ \ (5)\ px = p\,.$$

b) the following four conditions are equivalent:

$$(1)\ p \geq x\,;\ \ (2)\ pxp = x\,;\ \ (3)\ x = xp\,;\ \ (4)\ x = px\,.$$

Proof: a): Given $\mathbf{1} \geq x \geq p$, multiplication from left and right by p gives $p \geq pxp \geq p$ and thus $pxp = p$. If $pxp = p$ then $p(\mathbf{1} - x)p = 0$ which implies $(\mathbf{1} - x)^{1/2}p = 0$ and thus $(\mathbf{1} - x)p = 0$ or $xp = p$; taking adjoints $p = px$.
And $xp = p$ or $px = p$ implies $pxp = p$.
Finally either of the equivalent conditions (4) or (5) imply that $x - p = p^{\perp}xp^{\perp} \geq 0$.

b): $p \geq x$ iff $p^{\perp} \leq \mathbf{1} - x$. Apply a). $\square$

The following crucial observation and the proof, repeated here for convenience, are due to Mohari[11] .

Proposition 3 (Mohari) *Suppose $\alpha : \mathcal{M} \to \mathcal{M}$ is linear, unital, normal and completely positive and $x \in \mathcal{M}$ is positive with $\alpha(x) = x$. Then the support of x is super-harmonic.*

Proof: $\mathcal{M}$ acts on a Hilbert space $\mathcal{K}$. By the Stinespring Representation Theorem there is a normal $*$-homomorphism π of $\mathcal{M}$ into $\mathcal{B}(\mathcal{H})$ (the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$) and an isometry $V : \mathcal{K} \to \mathcal{H}$ such that $\alpha(a) = V^*\pi(a)V$ for all $a \in \mathcal{M}$. If $\mathcal{M} \subset \mathcal{B}(\mathcal{K})$ then the support $s[a]$ of the self-adjoint $a \in \mathcal{M}$ coincides with the smallest projection $q \in \mathcal{B}(\mathcal{K})$ such that $qa = a$ (Proposition 1.10.4 of [13]). Now if x satisfies the hypothesis, $s \equiv s[x]$ and $z = s^{\perp}$, then $0 = zxz = z\alpha(x)z = zV^*\pi(x)Vz = (yVz)^*(yVz)$ where $y = \sqrt{\pi(y)}$. Thus $yVz = 0$ and hence $\pi(x)Vz = 0$. The support of $\pi(x)$ is $\pi(s)$ and since Vz maps $\mathcal{K}$ into the kernel of $\pi(x)$, we conclude that $\pi(s)Vz = 0$. But then, $\alpha(s)z = V^*\pi(s)Vz = 0$ or $\alpha(s) = \alpha(s)s$ which by the Lemma above implies $\alpha(s) \leq s$. $\square$

References

[1] E. B. Davies: *Q*uantum Theory of Open Systems. (Academic Press, London 1976).

[2] A. Frigerio: *Quantum dynamical semigroups and approach to equilibrium. Lett. Math. Phys.* **2**, 79–87 (1977/78).

[3] A. Frigerio: *Stationary states of quantum dynamical semigroups. Comm. Math. Phys.* **63**, 269–276 (1978).

[4] A. Frigerio, and M. Verri: *Long-time asymptotic properties of dynamical semigroups on W^*–algebras. Math. Z.* **180**, 275–286 (1982).

[5] U. Groh: *Positive semigroups on C^*- and W^*-Algebras.* In *One-parameter Semigroups of Positive Operators*, edited by R. Nagel. Lecture Notes in Mathematics 1184, (Springer-Verlag, Berlin, 1986); pp. 369–425.

[6] F. Fagnola, and R. Rebolledo: *Subharmonic projections for quantum Markov semigroup. J. Math. Phys.* **43**, 1074–1082 (2002).

[7] F. Fagnola, and R. Rebolledo: *Notes on the Qulitative Behaviour of Quantum Markov semigroups.* In *Open Quantum Systems III. Recent Developments*, edited by S. Attal, A. Joye, and C.–A. Pillet. Lecture Notes in Mathematics 1882. (Springer-Verlag, Berlin, 2006); pp. 161-206.

[8] V. Umanitá: *Classification and decomposition of Quantum Markov Semigroups. Probab. Theory Relat. Fields* **134**, 603–623 (2006).

[9] F. Fagnola: *Quantum Markov semigroups: structure and asymptotics. Rend. Circ. Mat. Palermo serie II Suppl. No.* **73**, 35–51 (2004).

[10] A. Mohari: *Markov shift in non-commutative probability. J. Funct. Anal.* **199**, 189–209 (2003).

[11] A. Mohari: *A resolution of quantum dynamical semigroups.* Preprint arXiv:math/0505384v1, May 2005.

[12] B. Baumgartner, and H. Narnhofer: *Analysis of quantum semigroups with GKS–Lindblad generators: II. General. J. Phys. A: Math. Theor.* **41**, 395303 (2008).

[13] S. Sakai: *C^*-algebras and W^*-algebras.* (Springer-Verlag, Berlin, 1971).

[14] L. Asimow, and A.J. Ellis: *Convexity Theory and its Applications in Functional Analysis.* (Academic Press, London, 1980).