On the long-time asymptotics of quantum dynamical semigroups

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Abstract

We consider semigroups \( \{ \alpha_t : t \geq 0 \} \) of normal, unital, positive maps \( \alpha_t \) on a \( W^* \)-algebra \( M \). The (predual) semigroup \( \{ \rho_t(p) \} \) := \( \rho \circ \alpha_t \) on normal states \( \rho \) of \( M \) leaves invariant the face \( \mathcal{F}_p := \{ \rho : \rho(p) = 1 \} \) supported by the projection \( p \in M \), iff \( \alpha_t(p) \geq p \) (i.e., \( p \) is sub-harmonic). We complete the arguments showing that the sub-harmonic projections form a complete lattice. We then consider \( r_o \), the smallest projection which is larger than each support of a minimal invariant face; then \( r_o \) is subharmonic. In finite dimensional cases and when \( t \) is completely positive, \( \sup \alpha_t(r_o) = 1 \) and \( r_o \) is also the smallest projection \( p \) for which \( \alpha_t(p) \to 1 \). If \( \{ \nu_t : t \geq 0 \} \) admits a faithful family of normal stationary states then \( r_o = 1 \) is useless; if not, it helps to reduce the problem of the asymptotic behaviour of the semigroup for large times.

Keywords: Quantum dynamical semigroups; sub-harmonic projections; long-time asymptotics.

1 Introduction and preliminaries

We consider a von Neumann algebra \( M \) and denote its normal state space by \( S \). A quantum dynamical semigroup \( \{ \alpha_t : t \geq 0 \} \) is a family of normal, unital, positive, linear maps \( \alpha_t : M \to M \) with the property \( \alpha_t \circ \alpha_s = \alpha_{t+s} \) where \( \alpha_0 \) is the identity. Then, the map \( \nu_t : S \to S \) defined by \( \nu_t(p) = \rho \circ \alpha_t \) is affine, \( \nu_0 \) is the identity, and \( \nu_t \circ \nu_s = \nu_{t+s} \). Conversely, given a semigroup \( \{ \nu_t : t \geq 0 \} \) of affine maps on \( S \), the dual maps are a positive quantum dynamical semigroup.

One often demands on physical grounds, that \( \alpha_t \) be completely positive. When \( M \) is the algebra of all bounded linear operators on a Hilbert space, the generator of a one-parameter semigroup of completely positive, linear, normal, unital maps which is strongly continuous in \( t \) has the canonical GKS-Lindblad form.

The long-time asymptotics of such semigroups has been studied in the 1970’s and in the 1980’s, after pioneering papers of E.B. Davies[1], culminating with the work of Frigerio[2, 3, 4], and U. Groh [5]. More recent studies are due to Fagnola & Rebolledo[6, 7], Umanità[8], Mohari[10, 11] and Baumgartner & Narnhofer[12]. We refer to Ref. [7] for a recent overview. In pertinent cases, the asymptotics can be studied via the GKS-Lindblad generator.

In this note all states (positive linear functionals of unit norm) are normal. Limits of states are with respect to the distance induced by the norm. But recall that the norm-closure of a convex set of states coincides with its weak-closure. All projections are ortho-projections (self-adjoint idempotents). For a projection \( p \), \( p^2 = 1 - p \). Limits in \( M \) are invariably in the \( \sigma^* \)-topology. The support of a state \( \rho \) –written \( s_\rho \)– is the smallest projection \( p \in M \) such that \( \rho(p) = 1 \).

We will consider a quantum dynamical semigroup \( \{ \alpha_t : t \geq 0 \} \) and will always explicitly state additional positivity hypotheses. In particular, if each \( \alpha_t \) is completely positive, we say that the semigroup is CP. A state \( \omega \) is stationary if \( \nu_t(\omega) = \omega \circ \alpha_t = \omega \).
2 Invariant faces and sub-harmonic projections

A face is a convex subset \( F \) of \( S \) which is stable under convex decomposition: if \( tp + (1 - t)\mu \in F \) for \( 0 < t < 1 \) with \( \rho, \mu \in S \) then \( \rho, \mu \in F \). If \( p \in \mathcal{M} \) is a projection then \( \mathcal{F}_p := \{ \rho \in S : \rho(p) = 1 \} \) is a closed face; namely the face supported by \( p \). A fundamental result[13, 14] is that every closed face is of this form; i.e. it is the face supported by some projection. Clearly, \( \mathcal{F}_p \subset \mathcal{F}_q \) iff \( p \leq q \). The following result is implicit or partially explicit in the work of Fagnola & Rebolloed and Umantá.

**Proposition 1** Suppose \( \nu \) is an affine map of \( S \) into itself and let \( \alpha \) be the dual normal, linear, positive map of \( \mathcal{M} \) into itself. For a projection \( p \in \mathcal{M} \) the following conditions are equivalent: (1) the face \( \mathcal{F}_p \) supported by \( p \) is \( \nu \)-invariant; (2) \( \alpha(p) \geq p \); (3) \( \nu(\alpha(a)p) = \nu(\alpha(p)a)p \) for every \( a \in \mathcal{M} \); (4) \( \alpha(p^\perp a^\perp) = p^\perp \alpha(p^\perp a^\perp) p^\perp \) for every \( a \in \mathcal{M} \).

**Proof:** We first prove the chain \( (2) \Rightarrow (1) \Rightarrow (3) \Rightarrow (2) \). If \( \alpha(p) \geq p \) then, for any state \( \rho \) one has \( \nu(\rho)(p) = \rho(\alpha(p)) \geq \rho(p) \). Thus, \( \rho(p) = 1 \) implies \( \nu(\rho)(p) = 1 \), i.e. \( \nu(\mathcal{F}_p) \subset \mathcal{F}_p \). If \( \nu(\mathcal{F}_p) \subset \mathcal{F}_p \), we show that

\[
\omega(p\alpha(p)a)p = \omega(p\alpha(a)p) \quad \text{for every } \omega \in \mathcal{S}.
\]

Since every normal linear functional is the linear combination of at most four states, this then implies that \( p\alpha(p)p = \alpha(p)p \). To prove \((*)\) observe that, by the Cauchy-Schwarz inequality for states, the claim is trivially valid if \( \omega(p) = 0 \). Otherwise, consider the state \( \omega_p(a) := \omega(pap)/\omega(p) \). Clearly \( \omega_p \in \mathcal{F}_p \); thus,

\[
\omega(p)^{-1}\omega(p\alpha(p)a)p = \omega_p(\alpha(p)a) = \nu(\omega_p)(pap) = \nu(\omega_p)(a).
\]

which is \((*)\). Finally, if \( p\alpha(p)p = \alpha(p)p \), then \( p - \alpha(p)p = \alpha(p^\perp)p = 0 \) and Lemma 2 of the Appendix implies \( \alpha(p) \geq p \).

If \( 0 \leq x = p^\perp xp^\perp \leq 1 \) then, by Lemma 2 of the Appendix, \( x \leq p^\perp \) and \( \alpha(x) \leq \alpha(p^\perp) \); when \( (2) \) is the case \( \alpha(p^\perp) \leq p^\perp \) so that \( \alpha(x) \leq p^\perp \) which by the aforementioned Lemma, implies \( p^\perp \alpha(x)p^\perp = \alpha(x) \).

For general \( 0 \leq x = p^\perp xp^\perp \) we consider \( x/\|x\| \) and obtain \( p^\perp \alpha(x)p^\perp = \alpha(x) \). Since every \( a \in \mathcal{M} \) is a linear combination of at most four positive elements, we conclude that \( (2) \) implies \( (4) \). But \( (4) \) implies \( \alpha(p^\perp) = p^\perp \alpha(p^\perp)p^\perp \) which, by the same Lemma, gives \( \alpha(p^\perp) \leq p^\perp \) which is equivalent to \( \alpha(p) \geq p \).

In the context of positive one-parameter semigroups \( \{ \alpha_t : t \geq 0 \} \), a projection \( p \) satisfying \( \alpha_t(p) \geq p \) has been termed sub-harmonic[6]. We say the projection \( p \) is sub-harmonic for the linear, normal, unital and positive map \( \alpha \) on \( \mathcal{M} \) if \( \alpha(p) \geq p \).

**Proposition 2** If a family of projections is sub-harmonic for a linear, normal, unital and positive map \( \alpha \) on \( \mathcal{M} \), then the infimum of the family is sub-harmonic for \( \alpha \).

**Proof:** If \( \{ \mathcal{F}_\iota : \iota \in I \} \) is a family of closed faces \( \mathcal{F}_\iota \) of \( S \) then \( \bigcap \mathcal{F}_\iota \) is, clearly, a closed face and it is the largest closed face contained in each \( \mathcal{F}_\iota \). The support of \( \bigcap \mathcal{F}_\iota \) is exactly \( \inf \{ p_\iota : \iota \in I \} \), where \( p_\iota \) is the support of \( \mathcal{F}_\iota \). Moreover, if each \( \mathcal{F}_\iota \) is \( \nu \)-invariant then so is the intersection.

The corresponding statement for the supremum of such a family has been observed and proved (directly)[8]. For projections \( p \) that are super-harmonic, i.e. \( \alpha(p) \leq p \) (equivalently \( p^\perp \) is sub-harmonic), we have (in reply to a question posed in Ref. [9]):

**Corollary 1** If a family of projections is super-harmonic for a linear, normal, unital and positive map \( \alpha \) on \( \mathcal{M} \), then the supremum of the family is super-harmonic for \( \alpha \).

**Proof:** \( \sup \{ p : p \in \mathcal{F} \} = (\inf \{ p^\perp : p \in \mathcal{F} \})^\perp \) and \( \inf \{ p^\perp : p \in \mathcal{F} \} \) is sub-harmonic by the previous proposition.

The corresponding statement for the infimum of a super-harmonic family follows from the result for the supremum of a sub-harmonic family by orthocomplementation as above. Thus,
**Theorem 1** The set of sub-harmonic and the set of super-harmonic projections with respect to a linear, normal, unital and positive map on $\mathcal{M}$ are both complete lattices.

A minimal invariant face is a closed $\nu$-invariant face which does not properly contain another non-empty closed $\nu$-invariant face. Equivalently, it is a face whose support is a minimal sub-harmonic projection, i.e., a sub-harmonic projection that is not larger than a non-zero sub-harmonic projection other than itself. One can prove, and this goes back to at least Davies (see Ref. [1], Theorem 3.8 of Sect. 6.3), that if the minimal invariant face admits a stationary state then it is unique and its support is the support of the face. Moreover (Ref. [5], Proposition 3.4) the restriction of $\nu_t$ to the face is ergodic (the Cesàro means converge to the stationary state).

3 A “recurrent” projection

We define the minimal recurrent projection $r_o$ as the smallest projection which is larger than every minimal sub-harmonic projection. Equivalently, $r_o$ is the support of the smallest $\nu$-invariant face which contains every minimal $\nu$-invariant face. By virtue of its definition and the result mentioned above—to the effect that the supremum of a family of sub-harmonic projections is sub-harmonic—it follows that the minimal recurrent projection is sub-harmonic. Hence the directed family $\alpha_t(r_o)$ which is bounded above by 1 has a lowest upper bound in $\mathcal{M}$ denoted by $x$ which is positive and below 1. Since $x = \lim_{t \to \infty} \alpha_t(r_o)$ it follows that $\alpha_t(x) = x$ for every $t \geq 0$. Let $s(x)$ denote the support of $x$, that is the smallest projection $p \in \mathcal{M}$ with $xp = x$. The following treatment follows the lines of work by Mohari[11].

**Lemma 1** If $\{\alpha_t : t \geq 0\}$ is CP, then $s(x) = 1$.

**Proof:** $s(x)$ is the largest projection $q$ with $xq = 0$, and it is sub-harmonic by a result of Ref. [11] quoted in the appendix. Assume that $s(x) \neq 1$; then there is a minimal sub-harmonic non-zero projection $q$ with $q \leq s(x)$. One has $xq = 0$. By the definition of $r_o$, we have $q \leq r_o$ and thus $q = q \leq \alpha_t(r_o)q \leq xq = 0$, which contradicts the assumption.

Let $\mathcal{J} := \{a \in \mathcal{M} : \lim_{t \to \infty} \alpha_t(a^*a) = 0\}$. Since for each state $\rho$, one has the Cauchy-Schwarz inequality
\[|\rho(\alpha_t(a^*b^*))| = |\rho(\alpha_t(ba))| = |\nu_t(ba)| \leq \sqrt{\nu_t(b^*b)\nu_t(a^*a)} \leq \sqrt{||b||\sqrt{\rho(\alpha_t(a^*a))}}\],
we infer that $\mathcal{J}$ is a linear subspace of $\mathcal{M}$. If $c \in \mathcal{M}$ and $a \in \mathcal{J}$, the same inequality applied to $b = a^*c^*c$ shows that $ca \in \mathcal{J}$; thus $\mathcal{J}$ is a left-ideal.

If $\mathcal{M}$ is finite-dimensional (that is *-isomorphic to the direct sum of finitely many full matrix algebras) then, on the one hand, $s(x) = 1$ implies that $x$ is invertible, and Mohari has shown that if $x$ is invertible then $x = 1$; and on the other hand, $\mathcal{J}$ is closed and there is a projection such that $\mathcal{J} = \mathcal{M} \cdot z$.

**Theorem 2** If $\mathcal{M}$ is finite dimensional and $\{\alpha_t : t \geq 0\}$ is CP, then $sup\{\alpha_t(r_o) : t \geq 0\} = 1$. Moreover $\mathcal{J} = \mathcal{M} \cdot r_o^+ \subset \mathcal{M}$ and $r_o$ is the smallest projection $p \in \mathcal{M}$ with $\lim_{t \to \infty} \alpha_t(p) = 1$.

**Proof:** There is[13] a projection $z \in \mathcal{M}$ with $\mathcal{J} = \mathcal{M} \cdot z$. Lemma 1 implies that $x$ is invertible and Theorem 2.5 of Ref. [11] gives $x = 1$. Hence $r_o^+ \in \mathcal{J}$ and thus $r_o^+ \leq z$ or $r_o \geq z^+$. Suppose $p$ is a minimal sub-harmonic projection; there is a stationary state $\omega$ in the minimal invariant face supported by $p$ and it follows (see end of §2) that it is unique and $s_o = p$. Since $\omega(z) = \omega(\alpha_t(z)) \to 0$, we have $\omega(z^+) = 1$ and thus $p \leq z^+$. But then, by the definition of $r_o$, $r_o \leq z^+$. Thus $r_o = z^+$. □

It follows from the Cauchy-Schwarz inequality for states that $\lim_{t \to \infty} \alpha_t(a^*r_o^+) = \lim_{t \to \infty} \alpha_t(r_o^+a) = 0$ for every $a \in \mathcal{M}$ so that $\alpha_t(a) = \alpha_t(r_oar_o)$ for large $t$ and every $a \in \mathcal{M}$. If $\{\nu_t : t \geq 0\}$ admits a faithful family of stationary states, then the minimal recurrent projection is the identity. This happens because for a stationary state $\omega$, one has $\omega(r_o^+) = \omega(\alpha_t(r_o^+)) \Downarrow 0$ and thus $\omega(r_o^+) = 0$. However in this case there are results[4, 5, 10] on the asymptotic behaviour of the semigroup.

Other recurrent projections have been considered. For example (Ref. [5], p. 407; Ref. [8]), the supremum $r$ of the supports of the stationary states (if any are available), which is then sub-harmonic and above $r_o$.

There is no reason to expect that the above theorem 2 holds in infinite dimension.
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4 Appendix

We collect here a number of technical results used in the above proofs.

Lemma 2 For $x \in M$ satisfying $1 \geq x \geq 0$ and $p \in M$ a projection one has:

a) the following five conditions are equivalent:

1. $x \geq p$;  
2. $pxp = p$;  
3. $x = p + p^\perp xp^\perp$;  
4. $xp = p$;  
5. $px = p$.

b) the following four conditions are equivalent:

1. $p \geq x$;  
2. $pxp = x$;  
3. $xp = x$;  
4. $x = px$.

Proof: a): Given $1 \geq x \geq p$, multiplication from left and right by $p$ gives $p \geq pxp \geq p$ and thus $pxp = p$. If $pxp = p$ then $p(1 - x)p = 0$ which implies $(1 - x)^{1/2}p = 0$ and thus $(1 - x)p = 0$ or $xp = p$; taking adjoints $p = px$.

And $xp = p$ or $px = p$ implies $pxp = p$.

Finally either of the equivalent conditions (4) or (5) imply that $x - p = p^\perp xp^\perp \geq 0$.

b): $p \geq x$ iff $p^{\perp} \leq 1 - x$. Apply a). $\square$

The following crucial observation and the proof, repeated here for convenience, are due to Mohari[11].

Proposition 3 (Mohari) Suppose $\alpha : M \rightarrow M$ is linear, unital, normal and completely positive and $x \in M$ is positive with $\alpha(x) = x$. Then the support of $x$ is super-harmonic.

Proof: $\mathcal{M}$ acts on a Hilbert space $\mathcal{K}$. By the Stinespring Representation Theorem there is a normal $^*$-homomorphism $\pi$ of $\mathcal{M}$ into $\mathcal{B}(\mathcal{H})$ (the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$) and an isometry $V : \mathcal{K} \rightarrow \mathcal{H}$ such that $\alpha(a) = V^*\pi(a)V$ for all $a \in \mathcal{M}$. If $\mathcal{M} \subset \mathcal{B}(\mathcal{K})$ then the support $s[a]$ of the self-adjoint $a \in \mathcal{M}$ coincides with the smallest projection $q \in \mathcal{B}(\mathcal{K})$ such that $qa = a$ (Proposition 1.10.4 of [13]). Now if $x$ satisfies the hypothesis, $s \equiv s[x]$ and $z = s^\perp$, then $0 = xz = \alpha(x)z = zV^*\pi(x)Vz = (yVz)^*yVz$ where $y = \sqrt{\pi(y)}$. Thus $yVz = 0$ and hence $\pi(x)Vz = 0$.

The support of $\pi(x)$ is $\pi(s)$ and since $Vz$ maps $\mathcal{K}$ into the kernel of $\pi(x)$, we conclude that $\pi(s)Vz = 0$. But then, $\alpha(s)z = V^*\pi(s)Vz = 0$ or $\alpha(s) = \alpha(s)s$ which by the Lemma above implies $\alpha(s) \leq s$. $\square$

References


