A simple spectral condition implying separability for states of bipartite quantum systems

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Abstract. We give a simple spectral condition in terms of the ordered eigenvalues of the state of a bipartite quantum system which is sufficient for separability.

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We consider quantum systems where the underlying Hilbert space \( H \) is the tensor product of two finite dimensional Hilbert spaces. A state of such a system is identified with a density operator. A state is said to be separable if it can be written as a convex sum of pure product states of the system; that is to say vector states where the vectors are product vectors. The separable states form a convex subset of the states of the system.

For the simplest bipartite composite system we have the following result

**Theorem 1.** If the eigenvalues \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \) of the two qubit state \( \rho \) satisfy

\[
3\lambda_1 + \sqrt{2}\lambda_2 + (3 - \sqrt{2})\lambda_3 \leq 2,
\]

then \( \rho \) is separable.

The states satisfying the inequality have spectra in the simplex spanned by the spectra (always numbered taking into account multiplicities and nonincreasingly)

\[
(1/2, 1/6, 1/6, 1/6), (2 + \sqrt{2})/8, (2 + \sqrt{2})/8, (2 - \sqrt{2})/8, (2 - \sqrt{2})/8, (1/3, 1/3, 1/3, 0) \quad \text{and} \quad (1/4, 1/4, 1/4, 1/4).
\]

The method used to prove this also gives a different proof of the following result given in [1] (Theorem 3)

**Theorem 2.** If the eigenvalues \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \) of the state \( \rho \) of a bipartite quantum system of dimension \( d \) satisfy

\[
3\lambda_d + (d-1)\lambda_{d-1} \geq 1,
\]

then \( \rho \) is separable.

Both results provide simple spectral criteria ensuring separability. In the case of two qubits \( (d = 4) \) Theorem 2 is much weaker than Theorem 1.

The proof to be given uses certain tools developed in [1] which we briefly present. Given a state \( \rho \) on a \( d \)-dimensional bipartite quantum system, we let \( \text{spec}(\rho) = (\lambda_1, \lambda_2, \ldots, \lambda_d) \) denote the vector of repeated eigenvalues of \( \rho \) enumerated so that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \). Consider the normalized trace \( \tau \) over \( H \), then \( \tau \) factorizes as the trace over the two factors of \( H \) so that \( \tau \) is a separable state. Consider the segment with endpoints \( \rho \) and \( \tau \): \( \rho_t = t \cdot \rho + (1 - t) \cdot \tau, \; 0 \leq t \leq 1 \). The modulus of separability \( \ell \) [1], measures how far can you go towards \( \rho \) beginning at \( \tau \) until you lose separability: \( \ell(\rho) = \sup\{t : \rho_t \text{ is separable}\} \). The quantity \( (1/\ell) - 1 \) was studied...
by Vidal and Tarrach [2] as the “random robustness of entanglement”. It can be shown \([2, 1]\) that the supremum is a maximum; that \(\rho_t\) is separable iff \(t \leq \ell(\rho)\); that \(\ell(\rho) > 0\); and that \(1/\ell\) is a convex map on the states

\[
\ell(s, \rho + (1 - s) \cdot \phi) \geq \left( \frac{s}{\ell(\rho)} + \frac{1 - s}{\ell(\phi)} \right)^{-1}.
\]  

The other ingredient is the so-called gap-representation of \([1]\). Let \(\text{spec}(\rho) = (\lambda_1, \lambda_2, \ldots, \lambda_d)\) and \(\rho = \sum_{j=1}^d \lambda_j \cdot \rho_j\) be the spectral decomposition of \(\rho\) where the \(\rho_j\) are pairwise orthogonal pure vector states. Define \(\mu_j = j(\lambda_j - \lambda_{j+1}), \ j = 1, 2, \ldots, d-1;\)

\[
\hat{\rho}_j = j^{-1} \sum_{m=1}^j \rho_j, \ j = 1, 2, \ldots, d.
\]

Notice that \(\sum_{j=1}^{d-1} \mu_j = 1 - d\lambda_d\); that \(\hat{\rho}_d = \tau\); and that

\[
\text{spec}(\hat{\rho}_j) = \left(\frac{1}{j}, \frac{1}{j}, \ldots, \frac{1}{j}, 0, \ldots, 0\right).
\]

So \(\hat{\rho}_1\) is pure. Then a gap-representation of \(\rho\) is \(\rho = \sum_{j=1}^{d-1} \mu_j \cdot \hat{\rho}_j + d \lambda_d \cdot \tau\). Noticing that \(\lambda_d = 1/\ell(\rho)\) iff \(\rho = \tau\), we assume that this is not the case and write

\[
\rho = (1 - d\lambda_d) \cdot \omega + d\lambda_d \cdot \tau, \quad \omega = \sum_{j=1}^{d-1} \frac{\mu_j}{1 - d\lambda_d} \cdot \hat{\rho}_j.
\]

By the results mentioned, \(\rho\) is separable iff

\[
(1 - d\lambda_d) \leq \ell(\omega).
\]

Applying (1) to the state \(\omega\) in its gap representation, we have

\[
\ell(\omega) \geq \left( \frac{d-1}{\sum_{j=1}^{d-1} (1 - d\lambda_d) \ell(\hat{\rho}_j)} \right)^{-1} = (1 - d\lambda_d) \left( \sum_{j=1}^{d-1} \frac{\mu_j}{\ell(\hat{\rho}_j)} \right)^{-1};
\]

thus (2) is satisfied (and thus \(\rho\) is separable) if \(\sum_{j=1}^{d-1} \mu_j / \ell(\hat{\rho}_j) \leq 1\). We can replace the \(\ell(\hat{\rho}_j)\) by lower bounds.

**Proposition** If \(\ell(\hat{\rho}_j) \geq p_j \geq 0\) for \(j = 1, 2, \ldots, d-1\) and \(\sum_{j=1}^{d-1} \mu_j / p_j \leq 1\) then \(\rho\) is separable.

The prime reason for introducing the gap-representation is that not only the last summand \(\tau\) but also the second last \(\hat{\rho}_{d-1}\) are separable. This follows from a result of Gurvits and Barnum [3]: If \(\text{tr}(\phi^2) \leq 1/(d - 1)\) for a bipartite composite system of dimension \(d\) then \(\phi\) is separable. Now indeed \(\text{tr}(\hat{\rho}_{d-1}^2) = 1/(d - 1)\).

The least possible modulus of separability has been computed by Vidal and Tarrach [2]:

\[
\inf \{ \ell(\phi) : \phi \text{ a state} \} = 2/(2 + d);
\]

the infimum is assumed at a pure state. To prove theorem 2, put \(p_1 = p_2 = \cdots = p_{d-2} = 2/(2 + d)\) and \(p_{d-1} = 1\) in the proposition.

Turning to theorem 1, consider the numbers \(\hat{\ell}_j := \inf \{ \ell(\phi) : \text{spec}(\phi) = e^{(j)}\}\), which give the minimal moduli of separability for the states spanning all possible gap-representations. Replacing \(p_j\) by \(\hat{\ell}_j\) in the Proposition gives us a general inequality providing a sufficient condition for separability. No general information is available for the \(\hat{\ell}_j\) except the calculation of \([4]\) for two qubits where \(\hat{\ell}_1 = 1/3, \hat{\ell}_2 = 1/\sqrt{2}\), and \(\hat{\ell}_3 = 1\). From this and the proposition one gets theorem 1. Since \(\hat{\ell}_1 = 1/3,\) and \(\hat{\ell}_3 = 1\) follow from the results quoted above, we only give the calculation of \(\hat{\ell}_2\) in the appendix.
Appendix A. Calculation of $\ell$ for a two qubit state with $\text{spec}=(1/2,1/2,0,0)$

[4] gives a direct calculation of $\hat{\ell}_1$, $\hat{\ell}_2$ and $\hat{\ell}_3$ using the Wootters Criterion [5]. Recall that if $\rho$ is a state of a two qubit system, the Wootters operator $W$ associated to it is

$$W = (\sqrt{\rho} (\sigma_y \otimes \sigma_y) \overline{\rho} (\sigma_y \otimes \sigma_y) \sqrt{\rho})^{1/2}.$$ 

Here all operators are taken as matrices with respect to a product orthonormal basis.

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and $\overline{\rho}$ is the complex conjugate of $\rho$ taken with respect to the basis which is real. The Wootters Criterion is: $\rho$ is separable if and only if the (repeated) eigenvalues $w_1 \geq w_2 \geq w_3 \geq w_4$ of $W$ satisfy $w_1 \leq w_2 + w_3 + w_4$.

We will calculate the modulus of separability for any state $\rho$ for which $\text{spec}(\rho) = (1/2,1/2,0,0)$ by calculating the spectrum of the Wootters operator associated to $\rho_t = t \rho + (1-t) \tau$, $0 \leq t \leq 1$. The spectrum of $\rho_t$ consists of two double eigenvalues $\alpha = (1 + t)/4$ and $\beta = (1-t)/4$ (which coincide for $t = 0$ where $\rho_0 = \tau$). In order not to overload the notation we consider a density operator $A$ with $\text{spec}(A) = (\alpha, \alpha, \beta, \beta)$ where $\alpha + \beta = 1/2$, and $\alpha \geq \beta \geq 0$; thus $1/4 \leq \alpha \leq 1/2$. The spectral decomposition of $A$ reads $A = \alpha P + \beta P^\perp$, where $P$ is an orthoprojection of rank 2 and $P^\perp = 1 - P$ is its orthocomplement, another orthoprojection of rank 2. It follows that $(\sigma_y \otimes \sigma_y) \overline{A} (\sigma_y \otimes \sigma_y) = \alpha Q + \beta Q^\perp$ is the spectral decomposition where $Q = (\sigma_y \otimes \sigma_y) \overline{P} (\sigma_y \otimes \sigma_y)$ is an orthoprojection of rank 2 and $Q^\perp = 1 - Q$. Using this one obtains for the square of the Wootters operator associated to $A$ the formula

$$W^2 = \beta^2 1 + \beta(\alpha - \beta)(P + Q) + (\alpha - \beta)(\sqrt{\alpha \beta} - \beta)(PQ + QP) + (\alpha^2 - \beta^2 - 2\sqrt{\alpha \beta}(\alpha - \beta))PQP.$$ 

Now since $P$, $P^\perp$, $Q$ and $Q^\perp$ are orthoprojections of rank 2 in a four dimensional Hilbert space, we have three mutually exclusive alternatives for the subspaces $U$ and $V$ spanned by $P$ and $Q$ respectively: (1) $U \cap V = \{0\}$ which happens when and only when $Q = 1 - P$ which is equivalent to $\text{tr}(PQ) = 0$; (2) $\text{dim}(U \cap V) = 2$ which happens when and only when $Q = P$ which is equivalent to $\text{tr}(PQ) = 2$; and (3) $\text{dim}(U \cap V) = 1$ which happens when and only when there are unit vectors $\psi$, $\phi$ and $\chi$ in the four dimensional Hilbert space which satisfy $\langle \psi, \phi \rangle = \langle \psi, \chi \rangle = 0$ and $| \langle \chi, \phi \rangle | < 1$ such that $P = | \psi \rangle \langle \psi | + | \phi \rangle \langle \phi |$ and $Q = | \psi \rangle \langle \psi | + | \chi \rangle \langle \chi |$. One has $\text{tr}(PQ) = 1 + | \langle \chi, \phi \rangle |^2$. This alternative is equivalent to $\text{tr}(PQ) \in [1, 2]$.

The three alternatives are distinguished by the value of $\text{tr}(PQ)$. For convenience we introduce the following characteristic geometric parameter $\xi = \text{tr}(PQ) - 1$, which will determine the modulus of separability completely. We now distinguish the three possibilities.

(1) which occurs iff $\xi = -1$. Here $PQ = 0$ allows one to compute $W^2 = \alpha \beta 1$. The Wootters Criterion is satisfied and the associated state is separable.

(2) which occurs iff $\xi = 1$. Here $P = Q$ allows on to calculate directly $W = A$, and
the Wootters Criterion is just $\alpha \leq 1/2$ so the associated state is separable.  

(3) which occurs iff $0 \leq \xi < 1$. We may assume that

$$
\psi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \chi = \begin{pmatrix} 0 \\ \sqrt{\xi} \\ \eta_1 \\ \eta_2 \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix},
$$

where $\eta \neq 0$ because $\|\eta\|^2 = \|\chi\|^2 - \xi = 1 - \xi > 0$. We now partition $\mathbb{C}^4 = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^2$, so that

$$
P = \begin{pmatrix} 1 & 0 & \langle 0 | \\ 0 & 1 & \langle 0 | \\ \langle 0 | & \langle 0 | & 0_2 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & \langle 0 | \\ 0 & \xi & \sqrt{\xi} \langle \eta | \\ \langle 0 | & \sqrt{\xi} \langle \eta | & \langle \eta | \langle \eta | \end{pmatrix}.
$$

Doing the necessary matrix multiplications we get from our previous formula for $W^2$

$$
W^2 = \begin{pmatrix} \alpha^2 & 0 & \langle 0 | \\ 0 & \alpha \beta + \alpha (\alpha - \beta) \xi & (\alpha - \beta) \sqrt{\xi} \langle \eta | \\ \langle 0 | & (\alpha - \beta) \sqrt{\xi} \langle \eta | & \beta^2 1_2 + \beta (\alpha - \beta) \langle \eta | \langle \eta | \end{pmatrix}.
$$

It is now clear that $\alpha^2$ is an eigenvalue of $W^2$. The eigenvalue condition for an eigenvalue $\zeta$ to the eigenvector $x \oplus \mu$ for the lower right $3 \times 3$ block on $\mathbb{C} \oplus \mathbb{C}^2$ is

$$
(\zeta - \alpha \beta - \alpha (\alpha - \beta) \xi) x = (\alpha - \beta) \sqrt{\xi} \alpha \beta \langle \eta, \mu \rangle 
$$

(A.1)

$$
(\zeta - \beta^2) \mu = (\alpha - \beta) (\sqrt{\xi} \alpha \beta x + \beta \langle \eta, \mu \rangle) \eta. 
$$

(A.2)

Putting $x = 0$ and taking as we may $\mu \neq 0$ orthogonal to $\eta$, (A.1) is satisfied and (A.2) reduces to $(\zeta - \beta^2) \mu = 0$, thus $\beta^2$ is an eigenvalue of $W^2$. We are now left with the problem of finding eigenvectors orthogonal to those already found. They are of the form $x \oplus c \eta$ with $x, c \in \mathbb{C}$. Inserting such eigenvectors into (A.1) and (A.2), the discussion of the solutions is tedious but straightforward. One obtains the two missing eigenvalues of $W^2$ to be

$$
\zeta_\pm(\alpha, \xi) = \frac{\alpha}{2} (1 - 2\alpha) + \frac{\xi}{8} (4\alpha - 1)^2 
$$

$$
\pm \frac{4\alpha - 1}{4} \sqrt{2\xi \alpha (1 - 2\alpha) + \xi^2 (2\alpha - 1)^2}.
$$

Having the four eigenvalues of $A$ we must decide which is the largest. We have $\alpha \geq \beta$ by assumption, and clearly $\zeta_+(\alpha, \xi) \geq \zeta_-(\alpha, \xi)$. Moreover, $\xi \mapsto \zeta_+(\alpha, \xi)$ is increasing and $\zeta_+(\alpha, 1) = \alpha^2$. Thus, $\alpha$ is the largest eigenvalue of $W$ and the Wootters Criterion reads: $\alpha \leq \beta + \sqrt{\zeta_+(\alpha, \xi) + \frac{1}{\sqrt{3}} \zeta_-(\alpha, \xi)}$. Manipulation of this inequality show that it is equivalent to $\alpha \leq (1 + (1/\sqrt{2} - \xi)) / 4$.

Recalling that $\alpha = (1 + t) / 4$, we arrive at: If the two qubit state $\rho$ has $\text{spec}(\rho) = (1/2, 1/2, 0, 0)$ then

$$
\ell(\rho) = \begin{cases} 1 & \text{if } Q = 1 - P \\ \frac{1}{\sqrt{3} - \text{tr}(PQ)} & \text{otherwise} \end{cases}
$$

where $P$ is the spectral orthoprojection to the eigenvalue $1/2$ and $Q = (\sigma_y \otimes \sigma_y)\overline{\text{tr}}(\sigma_y \otimes \sigma_y)$. Since $\text{tr}(PQ) \in [1, 2]$ when $Q \neq 1 - P$, we obtain $\ell_2 = \inf\{\ell(\rho) : \text{spec}(\rho) = (1/2, 1/2, 0, 0)\} = 1/\sqrt{2}$.
References