A simple spectral condition implying separability for states of bipartite quantum systems

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Abstract. We give a simple spectral condition in terms of the ordered eigenvalues of the state of a bipartite quantum system which is sufficient for separability.

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We consider quantum systems where the underlying Hilbert space $\mathcal{H}$ is the tensor product of two finite dimensional Hilbert spaces. A state of such a system is identified with a density operator. A state is said to be separable if it can be written as a convex sum of pure product states of the system; that is to say vector states where the vectors are product vectors. The separable states form a convex subset of the states of the system.

For the simplest bipartite composite system we have the following result

Theorem 1 *If the eigenvalues* $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ *of the two qubit state* ρ *satisfy* $3\lambda_1 + \sqrt{2}\lambda_2 + (3-\sqrt{2})\lambda_3 \leq 2$*, then* ρ *is separable.*

The states satisfying the inequality have spectra in the simplex spanned by the spectra (always numbered taking into account multiplicities and nonincreasingly) $(1/2, 1/6, 1/6, 1/6)$, $((2+\sqrt{2})/8, (2+\sqrt{2})/8, (2-\sqrt{2})/8, (2-\sqrt{2})/8)$, $(1/3, 1/3, 1/3, 0)$ and $(1/4, 1/4, 1/4, 1/4)$.

The method used to prove this also gives a different proof of the following result given in [1] (Theorem 3)

Theorem 2 *If the eigenvalues* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$ *of the state* ρ *of a bipartite quantum system of dimension* d *satisfy* $3\lambda_d + (d-1)\lambda_{d-1} \geq 1$*, then* ρ *is separable.*

Both results provide simple spectral criteria ensuring separability. In the case of two qubits ($d = 4$) Theorem 2 is much weaker than Theorem 1 .

The proof to be given uses certain tools developped in [1] which we briefly present. Given a state ρ on a d-dimensional bipartite quantum system, we let $spec(\rho) = (\lambda_1, \lambda_2, \cdots, \lambda_d)$ denote the vector of repeated eigenvalues of ρ enumerated so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$. Consider the normalized trace τ over $\mathcal{H}$, then τ factorizes as the trace over the two factors of $\mathcal{H}$ so that τ is a separable state. Consider the segment with endpoints ρ and τ: $\rho_t = t \cdot \rho + (1-t) \cdot \tau$, $0 \leq t \leq 1$. The modulus of separability ℓ [1], measures how far can you go towards ρ beginning at τ until you lose separability: $\ell(\rho) = \sup\{t :\ \rho_t \text{ is separable}\}$. The quantity $(1/\ell) - 1$ was studied by Vidal and Tarrach [2] as the "random robustness of entanglement". It can be shown ([2, 1]) that the supremum is a maximum; that ρ_t is separable iff $t \leq \ell(\rho)$; that $\ell(\rho) > 0$; and that $1/\ell$ is a convex map on the states

$$\ell(s.\rho + (1-s)\cdot\phi) \geq \left(\frac{s}{\ell(\rho)} + \frac{1-s}{\ell(\phi)}\right)^{-1} \quad . \tag{1}$$

The other ingredient is the so called gap-representation of [1]. Let $spec(\rho) = (\lambda_1, \lambda_2, \cdots, \lambda_d)$ and $\rho = \sum_{j=1}^{d} \lambda_j \cdot \rho_j$ be the spectral decomposition of ρ where the ρ_j are pairwise orthogonal pure vector states. Define $\mu_j = j(\lambda_j - \lambda_{j+1})$, $j = 1, 2, \cdots, d-1$; $\widehat{\rho}_j = j^{-1}\sum_{m=1}^{j} \rho_j$, $j = 1, 2, \cdots, d$. Notice that $\sum_{j=1}^{d-1} \mu_j = 1 - d\lambda_d$; that $\widehat{\rho}_d = \tau$; and that

$$spec(\widehat{\rho}_j) = (\underbrace{1/j, 1/j, \cdots, 1/j}_{j}, 0, \cdots, 0) \ .$$

So $\widehat{\rho}_1$ is pure. Then a gap-representation of ρ is $\rho = \sum_{j=1}^{d-1} \mu_j \cdot \widehat{\rho}_j + d\ \lambda_d \cdot \tau$. Noticing that $\lambda_d = 1/d$ iff $\rho = \tau$, we assume that this is not the case and write

$$\rho = (1 - d\lambda_d)\cdot\omega + d\lambda_d \cdot \tau \ , \quad \omega = \sum_{j=1}^{d-1} \frac{\mu_j}{1 - d\lambda_d} \cdot \widehat{\rho}_j \ .$$

By the results mentioned, ρ is separable iff

$$(1 - d\lambda_d) \leq \ell(\omega) \ . \tag{2}$$

Applying (1) to the state ω in its gap representation, we have

$$\ell(\omega) \geq \left(\sum_{j=1}^{d-1} \frac{\mu_j}{(1 - d\lambda_d)\ell(\widehat{\rho}_j)}\right)^{-1} = (1 - d\lambda_d)\left(\sum_{j=1}^{d-1} \frac{\mu_j}{\ell(\widehat{\rho}_j)}\right)^{-1} \quad ;$$

thus (2) is satisfied (and thus ρ is separable) if $\sum_{j=1}^{d-1} \mu_j/\ell(\widehat{\rho}_j) \leq 1$. We can replace the $\ell(\widehat{\rho}_j)$ by lower bounds.

Proposition *If $\ell(\widehat{\rho}_j) \geq p_j \geq 0$ for $j = 1, 2, \cdots, d-1$ and $\sum_{j=1}^{d-1} \mu_j/p_j \leq 1$ then ρ is separable.*

The prime reason for introducing the gap-representation is that not only the last summand τ but also the second last $\widehat{\rho}_{d-1}$ are separable. This follows from a result of Gurvits and Barnum [3]: If $tr(\phi^2) \leq 1/(d-1)$ for a bipartite composite system of dimension d then ϕ is separable. Now indeed $tr(\widehat{\rho}_{d-1}^2) = 1/(d-1)$.

The least possible modulus of separability has been computed by Vidal and Tarrach [2]: $\inf\{\ell(\phi) : \ \phi \text{ a state }\} = 2/(2+d)$; the infimum is assumed at a pure state. To prove theorem 2, put $p_1 = p_2 = \cdots = p_{d-2} = 2/(2+d)$ and $p_{d-1} = 1$ in the proposition.

Turning to theorem 1, consider the numbers $\widehat{\ell}_j := \inf\{\ell(\phi) : \ spec(\phi) = e^{(j)}\}$, which give the minimal moduli of separability for the states spanning all possible gap-representations. Replacing p_j by $\widehat{l}_j$ in the Proposition gives us a general inequality providing a sufficient condition for separability. No general information is available for the $\widehat{l}_j$ except the calculation of [4] for two qubits where $\widehat{l}_1 = 1/3$, $\widehat{l}_2 = 1/\sqrt{2}$, and $\widehat{l}_3 = 1$. From this and the proposition one gets theorem 1. Since $\widehat{l}_1 = 1/3$, and $\widehat{l}_3 = 1$ follow from the results quoted above, we only give the calculation of $\widehat{l}_2$ in the appendix.

Appendix A. Calculation of ℓ for a two qubit state with spec=(1/2,1/2,0,0)

[4] gives a direct calculation of $\widehat{\ell}_1$, $\widehat{\ell}_2$ and $\widehat{\ell}_3$ using the Wootters Criterion [5]. Recall that if ρ is a state of a two qubit system, the Wootters operator W asssociated to it is

$$W = (\sqrt{\rho}(\sigma_y \otimes \sigma_y)\overline{\rho}(\sigma_y \otimes \sigma_y)\sqrt{\rho})^{1/2} .$$

Here all operators are taken as matrices with respect to a product orthonormal basis.

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and $\overline{\rho}$ is the complex conjugate of ρ taken with respect to the basis which is real. The Wootters Criterion is: ρ is separable if and only if the (repeated) eigenvalues $w_1 \geq w_2 \geq w_3 \geq w_4$ of W satisfy $w_1 \leq w_2 + w_3 + w_4$.
We will calculate the modulus of separability for any state ρ for which $spec(\rho) = (1/2, 1/2, 0, 0)$ by calculating the spectrum of the Wootters operator associated to $\rho_t = t\rho + (1-t)\tau$, $0 \leq t \leq 1$. The spectrum of ρ_t consists of two double eigenvalues $\alpha = (1+t)/4$ and $\beta = (1-t)/4$ (which coincide for $t = 0$ where $\rho_0 = \tau$). In order not to overload the notation we consider a density operator A with $spec(A) = (\alpha, \alpha, \beta, \beta)$ where $\alpha + \beta = 1/2$, and $\alpha \geq \beta \geq 0$; thus $1/4 \leq \alpha \leq 1/2$. The spectral decomposition of A reads $A = \alpha P + \beta P^{\perp}$, where P is an orthoprojection of rank 2 and $P^{\perp} = \mathbf{1} - P$ is its orthocomplement, another orthoprojection of rank 2. It follows that $(\sigma_y \otimes \sigma_y)\overline{A}(\sigma_y \otimes \sigma_y) = \alpha Q + \beta Q^{\perp}$ is the spectral decomposition where $Q = (\sigma_y \otimes \sigma_y)\overline{P}(\sigma_y \otimes \sigma_y)$ is an orthoprojection of rank 2 and $Q^{\perp} = \mathbf{1} - Q$. Using this one obtains for the square of the Wootters operator associated to A the formula

$$W^2 = \beta^2\mathbf{1} + \beta(\alpha - \beta)(P + Q) + (\alpha - \beta)(\sqrt{\alpha\beta} - \beta)(PQ + QP)$$

$$+(\alpha^2 - \beta^2 - 2\sqrt{\alpha\beta}(\alpha - \beta))PQP .$$

Now since P, $P^{\perp}$, Q and $Q^{\perp}$ are orthoprojections of rank 2 in a four dimensional Hilbert space, we have three mutually exclusive alternatives for the subspaces U and V spanned by P and Q respectively: (1) $U \cap V = \{0\}$ which happens when and only when $Q = \mathbf{1} - P$ which is equivalent to $tr(PQ) = 0$; (2) $dim(U \cap V) = 2$ which happens when and only when $Q = P$ which is equivalent to $tr(PQ) = 2$; and (3) $dim(U \cap V) = 1$ which happens when and only when there are unit vectors ψ, ϕ and χ in the four dimensional Hilbert space which satisfy $\langle\psi, \phi\rangle = \langle\psi, \chi\rangle = 0$ and $| \langle\chi, \phi\rangle | < 1$ such that $P = | \psi\rangle\langle\psi | + | \phi\rangle\langle\phi |$ and $Q = | \psi\rangle\langle\psi | + | \chi\rangle\langle\chi |$. One has $tr(PQ) = 1 + | \langle\chi, \phi\rangle |^2$. This alternative is equivalent to $tr(PQ) \in [1, 2)$.

The three alternatives are distingushed by the value of $tr(PQ)$. For convenience we introduce the following characteristic geometric parameter $\xi = tr(PQ) - 1$, which will determine the modulus of separability completely. We now distinguish the three possibilities.
(1) which occurs iff $\xi = -1$. Here $PQ = 0$ allows one to compute $W^2 = \alpha\beta\mathbf{1}$. The Wootters Criterion is satisfied and the associated state is separable.
(2) which occurs iff $\xi = 1$. Here $P = Q$ allows on to calculate directly $W = A$, and

the Wootters Criterion is just $\alpha \le 1/2$ so the associated state is separable.
(3) which occurs iff $0 \le \xi < 1$. We may assume that

$$\psi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \ , \ \phi = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \ , \ \chi = \begin{pmatrix} 0 \\ \sqrt{\xi} \\ \eta_1 \\ \eta_2 \end{pmatrix} \ , \ \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \ ,$$

where $\eta \neq 0$ because $\| \eta \|^2 = \| \chi \|^2 - \xi = 1 - \xi > 0$. We now partition $\mathbb{C}^4 = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^2$, so that

$$P = \begin{pmatrix} 1 & 0 & \langle 0 \mid \\ 0 & 1 & \langle 0 \mid \\ \mid 0 \rangle & \mid 0 \rangle & 0_2 \end{pmatrix} \ , \ Q = \begin{pmatrix} 1 & 0 & \langle 0 \mid \\ 0 & \xi & \sqrt{\xi} \langle \eta \mid \\ \mid 0 \rangle & \sqrt{\xi} \mid \eta \rangle & \mid \eta \rangle \langle \eta \mid \end{pmatrix} \ .$$

Doing the necessary matrix multiplications we get from our previous formula for W^2

$$W^2 = \begin{pmatrix} \alpha^2 & 0 & \langle 0 \mid \\ 0 & \alpha\beta + \alpha(\alpha - \beta)\xi & (\alpha - \beta)\sqrt{\xi\alpha\beta} \, \langle \eta \mid \\ \mid 0 \rangle & (\alpha - \beta)\sqrt{\xi\alpha\beta} \, \mid \eta \rangle & \beta^2 \mathbf{1}_2 + \beta(\alpha - \beta) \mid \eta \rangle \langle \eta \mid \end{pmatrix} \ .$$

It is now clear that α^2 is an eigenvalue of W^2. The eigenvalue condition for an eigenvalue ζ to the eigenvector $x \oplus \mu$ for the lower right 3×3 block on $\mathbb{C} \oplus \mathbb{C}^2$ is

$$(\zeta - \alpha\beta - \alpha(\alpha - \beta)\xi)x = (\alpha - \beta)\sqrt{\xi\alpha\beta}\langle \eta, \mu \rangle \tag{A.1}$$

$$(\zeta - \beta^2)\mu = (\alpha - \beta)(\sqrt{\xi\alpha\beta}\, x + \beta\langle \eta, \mu \rangle)\, \eta \ . \tag{A.2}$$

Putting $x = 0$ and taking as we may $\mu \neq 0$ orthogonal to η, (A.1) is satisfied and (A.2) reduces to $(\zeta - \beta^2)\mu = 0$, thus β^2 is an eigenvalue of W^2. We are now left with the problem of finding eigenvectors orthogonal to those already found. They are of the form $x \oplus c\eta$ with $x, c \in \mathbb{C}$. Inserting such eigenvectors into (A.1) and (A.2), the discussion of the solutions is tedious but straightforward. One obtains the two missing eigenvalues of W^2 to be

$$\zeta_{\pm}(\alpha, \xi) = \frac{\alpha}{2}(1 - 2\alpha) + \frac{\xi}{8}(4\alpha - 1)^2$$
$$\pm \frac{4\alpha - 1}{4}\sqrt{2\xi\alpha(1 - 2\alpha) + \xi^2(2\alpha - 1/2)^2} \ .$$

Having the four eigenvalues of A we must decide which is the largest. We have $\alpha \ge \beta$ by assumption, and clearly $\zeta_+(\alpha, \xi) \ge \zeta_-(\alpha, \xi)$. Moreover, $\xi \mapsto \zeta_+(\alpha, \xi)$ is increasing and $\zeta_+(\alpha, 1) = \alpha^2$. Thus, α is the largest eigenvalue of W and the Wootters Criterion reads: $\alpha \le \beta + \sqrt{\zeta_+(\alpha, \xi)} + \sqrt{\zeta_-(\alpha, \xi)}$. Manipulation of this inequality show that it is equivalent to $\alpha \le \left(1 + (1/\sqrt{2 - \xi})\right)/4$.

Recalling that $\alpha = (1 + t)/4$, we arrive at: If the two qubit state ρ has $spec(\rho) = (1/2, 1/2, 0, 0)$ then

$$\ell(\rho) = \begin{cases} 1 & , \quad \text{if } Q = \mathbf{1} - P \\ \frac{1}{\sqrt{3 - tr(PQ)}} & , \quad \text{otherwise} \end{cases}$$

where P is the spectral orthoprojection to the eigenavlue 1/2 and $Q = (\sigma_y \otimes \sigma_y)\overline{P}(\sigma_y \otimes \sigma_y)$. Since $tr(PQ) \in [1, 2]$ when $Q \neq \mathbf{1} - P$, we obtain $\widehat{\ell}_2 = \inf\{\ell(\rho) : \ spec(\rho) = (1/2, 1/2, 0, 0)\} = 1/\sqrt{2}$.

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