

UNIVERSIDAD NACIONAL DE CÓRDOBA
FACULTAD DE MATEMÁTICA, ASTRONOMÍA Y FÍSICA

SERIE “A”

TRABAJOS DE MATEMÁTICA

Nº 106/2011

**A presentation by generators and relations of nichols algebras
of diagonal type and convex orders on root systems**

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REPÚBLICA ARGENTINA

A presentation by generators and relations of Nichols algebras of diagonal type and convex orders on root systems

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September 29, 2011

Abstract

We obtain a presentation by generators and relations of any Nichols algebra of diagonal type with finite root system. We prove that the defining ideal is finitely generated. The proof is based in Kharchenko's theory of PBW basis of Lyndon words. We prove that the lexicographic order on Lyndon words is convex for such PBW generators and so the PBW basis is orthogonal with respect to the canonical non-degenerate form associated to the Nichols algebra.

Keywords. quantized enveloping algebras, Nichols algebras, pointed Hopf algebras.

1 Introduction

The consideration of pointed Hopf algebras has grown since the appearance of quantized enveloping algebras [Dr, Ji]. The finite-dimensional analogues, the so-called small quantum groups, were introduced and described by Lusztig [L1, L2].

The Lifting Method of Andruskiewitsch and Schneider is the leading method for the classification of finite-dimensional pointed Hopf algebras. Such method depends on the answers to some questions, including the following one:

Question 1.1. [And, Question 5.9]: Given a braided vector space of diagonal type, determine if the associated Nichols algebra is finite-dimensional, and in such case compute its dimension. Give a nice presentation by generators and relations.

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Mathematics Subject Classification (2010): Primary 16T05; Secondary 17B37, 17B22

The first part of this question has been answered by Heckenberger in [H2], where the author gives a list of all diagonal braidings whose associated Nichols algebra has a finite root system, but neither an explicit formula for the dimension nor a finite set of defining relations are given. Some of them are Lusztig's examples, which are associated with the so-called Cartan braidings and for which the dimension and a presentation by generators and relations are known. Standard braidings were introduced in [AA] and they constitute a family which includes properly the family of Cartan braidings. Nichols algebras with standard braidings have been presented by generators and relations in [Ang], where also an explicit formula for the dimension has been given. Another result about presentation of Nichols algebras is given in [Y] for quantized enveloping algebras associated with semisimple Lie superalgebras, and for quantized enveloping algebras of Lie algebras [K1]. Some other preliminaries considerations on the relations of a Nichols algebra of diagonal type appear in [He], and in [H3] for the rank-two case.

Andruskiewitsch and Schneider [AS3] have classified finite-dimensional pointed Hopf algebras whose group of group-like elements is abelian of order not divisible by some small primes using the Lifting method; all the possible such braidings are of finite Cartan type. They answered positively the following conjecture for $H_0 = k\Gamma$, Γ an abelian group as above:

Conjecture 1.2. [AS1, Conj. 1.4] Let H be a finite-dimensional pointed Hopf algebra over k . Then H is generated by group-like and skew-primitive elements.

This result was proved as a previous step of the main Theorem in [AS3] using the presentation by generators and relations. The conjecture was recently proved in a more general context [AnGa], when the braiding is of standard type. The proof follows also using the presentation by generators and relations.

Because of the braidings of Cartan type we see that there exists a close relation between pointed Hopf algebras and the classical Lie theory. In such direction the definition of the Weyl groupoid and the root system [H1, HS, HY] associated to a Nichols algebra $\mathfrak{B}(V)$ of diagonal type has shown to be a good extension of the idea of root systems and Weyl groups associated to semisimple Lie algebras. Such root system is obtained as the set of degrees of the generators of any PBW basis, and controls coideal subalgebras between other structures associated to $\mathfrak{B}(V)$ [HS].

In the classical case, convex orders over the root system were described in order to characterize quantized enveloping algebras $U_q(\mathfrak{g})$ for \mathfrak{g} semisimple [KhT, Le, R2], and to obtain Lusztig isomorphisms in the affine case [Be]. This kind of orders was first introduced in [Z]. The characterization of convex orders is in consequence necessary, and it has been done for finite [P] and affine [I] root systems.

Our main result is Theorem 4.9: we obtain a presentation by generators and relations for any Nichols algebra of diagonal type whose root system is finite. We obtain two kind of

relations that are enough to present $\mathfrak{B}(V)$: powers of root vectors (generators of a PBW basis), and some generalizations of quantum Serre relations which express the braided bracket of two root vectors as a linear combination of other root vectors in an explicit way, see Lemmata 4.7, 4.5.

Theorem 4.9 follows by consideration of PBW bases as in [K1]. Such PBW bases consist of homogeneous polynomials associated to Lyndon letters (which are called hyperletters) and inherit the lexicographical order. Another important element is the characterization of convex orders for generalized root systems. Such convex orders are related with reduced expressions of elements of the Weyl groupoid. These reduced expressions characterize also right coideal subalgebras of Nichols algebras, so we can relate convex orders and coideal subalgebras. In particular, the following result holds by Theorem 4.9:

Theorem 1.3. *Let V be a braided vector space of diagonal type whose associated root system is finite, and let $I(V)$ be the ideal of $T(V)$ such that $\mathfrak{B}(V) = T(V)/I(V)$. Then $I(V)$ is finitely generated.*

Theorem 4.9 extends the presentation by generators and relations of Nichols algebras of standard type, see Remark 5.4, and then gives a new proof for braidings of Cartan type. In particular we obtain the classical presentation of quantized enveloping algebras $U_q(\mathfrak{g})$ and Lusztig's small quantum groups $u_q(\mathfrak{g})$, with a different proof. We hope that this presentation helps to prove Conjecture 1.2 when the group of group-like elements is abelian, see Remark 5.3.

The plan of this article is the following. In Section 2 we recall the definition of Nichols algebra. We also consider results from [K1, R2] concerning a PBW basis for Nichols algebras of diagonal type.

In Section 3 we deal with root systems and coideal subalgebras of Nichols algebras of diagonal type. In Subsection 3.1 we recall the notion of Weyl groupoid and root system, and give some properties of these objects. In Subsection 3.2 we characterize convex orders on finite root systems generalizing the results in [P]. In Subsection 3.3 we recall some results from [HS] involving coideal subalgebras of Nichols algebras of diagonal type with finite root systems and use these results to characterize PBW bases of hyperletters. In particular we obtain that the lexicographical order on the hyperletters is convex.

In Section 4 we obtain the desired presentation by generators and relations. First we prove that the Kharchenko's PBW basis is orthogonal for the canonical non-degenerate bilinear form as in Proposition 2.1 when the braiding matrix is symmetric. Power root vector relations hold in $\mathfrak{B}(V)$ by Lemma 4.7 and generalized quantum Serre relations hold by Lemma 4.5. These two sets of relations are enough to give the presentation. We show in Section 5 how the main theorem allows to obtain explicitly the presentation of some Nichols algebras in some examples.

Notation. \mathbb{N} denotes the set of positive integers, and \mathbb{N}_0 the set of non-negative integers.

We fix an algebraically closed field k of characteristic 0; all vector spaces, Hopf algebras and tensor products are considered over k .

For each $N > 0$, \mathbb{G}_N denotes the group of N -th roots of 1 in k .

Given $n \in \mathbb{N}$, we set the following polynomials in q :

$$\binom{n}{j}_q = \frac{(n)_q!}{(k)_q!(n-k)_q!}, \quad \text{where } (n)_q! = \prod_{j=1}^n (k)_q, \quad \text{and } (k)_q = \sum_{j=0}^{k-1} q^j.$$

2 Preliminaries

We recall some definitions and results that we shall need in the subsequent sections. They are related with characterizations of Nichols algebras of diagonal type and PBW bases of such algebras.

Recall that a braided vector space is a pair (V, c) , where V is a vector space and $c \in \text{Aut}(V \otimes V)$ is a solution of the braid equation:

$$(c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c).$$

A braided vector space (V, c) is of *diagonal type* if there exists a basis x_1, \dots, x_θ and scalars $q_{ij} \in k^\times$ such that

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \quad 1 \leq i, j \leq \theta. \quad (2.1)$$

Following [K1] we describe an appropriate PBW-basis of a braided graded Hopf algebra $\mathfrak{B} = \bigoplus_{n \in \mathbb{N}} \mathfrak{B}^n$ such that $\mathfrak{B}^1 \cong V$, where (V, c) is of diagonal type. In particular we obtain PBW bases for Nichols algebras $\mathfrak{B}(V)$ of diagonal type. This construction is based in the notion of Lyndon words. Each Lyndon word has a canonical decomposition as a product of a pair of smaller Lyndon words, called the Shirshov decomposition. Using such decomposition and the braided bracket, we define inductively a set of hyperwords, which are the elements of a PBW basis for braided graded Hopf algebras of diagonal type. We recall also some properties of this PBW basis.

2.1 Braided vector spaces of diagonal type and Nichols algebras

Given a braided vector space (V, c) , this braiding can be extended to $c : T(V) \otimes T(V) \rightarrow T(V) \otimes T(V)$ canonically, see (2.3) for the diagonal case. We define for each pair $x, y \in T(V)$ the *braided commutator* as follows:

$$[x, y]_c := \text{multiplication} \circ (\text{id} - c)(x \otimes y). \quad (2.2)$$

Fix a braided vector space of diagonal type (V, c) and an ordered basis $X = \{x_1, \dots, x_\theta\}$ of V as in (2.1). Let \mathbb{X} be the corresponding vocabulary (the set of words with letters in X) and consider the lexicographical order on \mathbb{X} . We will identify the vector space $k\mathbb{X}$ with $T(V)$. We shall consider two different gradings of the algebra $T(V)$. First, the usual \mathbb{N}_0 -grading $T(V) = \bigoplus_{n \geq 0} T^n(V)$. If we denote by ℓ the length of a word in \mathbb{X} , then $T^n(V) = \bigoplus_{x \in \mathbb{X}, \ell(x)=n} kx$.

Second, let $\alpha_1, \dots, \alpha_\theta$ be the canonical basis of \mathbb{Z}^θ . Then $T(V)$ is \mathbb{Z}^θ -graded, where the degree is given by $\deg x_i = \alpha_i$, $1 \leq i \leq \theta$. Consider the bilinear form $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \rightarrow k^\times$ given by $\chi(\alpha_i, \alpha_j) = q_{ij}$, $1 \leq i, j \leq \theta$. Then

$$c(u \otimes v) = q_{u,v} v \otimes u, \quad u, v \in \mathbb{X}, \quad (2.3)$$

where $q_{u,v} = \chi(\deg u, \deg v) \in k^\times$. The braided commutator satisfies a ‘‘braided’’ derivation equation which gives place to a ‘‘braided’’ Jacobi identity, namely

$$[[u, v]_c, w]_c = [u, [v, w]_c]_c - \chi(\alpha, \beta) v [u, w]_c + \chi(\beta, \gamma) [u, w]_c v, \quad (2.4)$$

$$[u, v w]_c = [u, v]_c w + \chi(\alpha, \beta) v [u, w]_c, \quad (2.5)$$

$$[u v, w]_c = \chi(\beta, \gamma) [u, w]_c v + u [v, w]_c, \quad (2.6)$$

for any homogeneous $u, v, w \in T(V)$, of degrees $\alpha, \beta, \gamma \in \mathbb{N}^\theta$, respectively.

We denote by ${}^H_H\mathcal{YD}$ the category of Yetter-Drinfeld modules over H , where H is a Hopf algebra with bijective antipode. Any $V \in {}^H_H\mathcal{YD}$ becomes a braided vector space [Mo, Section 10.6]. If $H = k\Gamma$, where Γ is a finite abelian group, then any $V \in {}^H_H\mathcal{YD}$ is a braided vector space of diagonal type: if $V_g = \{v \in V \mid \delta(v) = g \otimes v\}$, $V^\chi = \{v \in V \mid g \cdot v = \chi(g)v \text{ for all } g \in \Gamma\}$ and $V_g^\chi = V^\chi \cap V_g$, then $V = \bigoplus_{g \in \Gamma, \chi \in \widehat{\Gamma}} V_g^\chi$. In this setting the braiding is given by

$$c(x \otimes y) = \chi(g)y \otimes x, \quad x \in V_g, g \in \Gamma, y \in V^\chi, \chi \in \widehat{\Gamma}.$$

Reciprocally, any braided vector space of diagonal type can be realized as a Yetter-Drinfeld module over the group algebra of many abelian groups. For example let (V, c) be a braided vector space of diagonal type. Call Γ the free abelian group of rank θ , with basis g_1, \dots, g_θ , and define the characters $\chi_1, \dots, \chi_\theta$ of Γ by

$$\chi_j(g_i) = q_{ij}, \quad 1 \leq i, j \leq \theta.$$

We can consider V as a Yetter-Drinfeld module over $k\Gamma$ for which $x_i \in V_{g_i}^{\chi_i}$.

Given $V \in {}^H_H\mathcal{YD}$, the tensor algebra $T(V)$ admits a unique structure of graded braided Hopf algebra in ${}^H_H\mathcal{YD}$ such that the elements of V are primitive. As in [AS2], we define the *Nichols algebra* $\mathfrak{B}(V)$ associated to V as the quotient of $T(V)$ by the maximal element

$I(V)$ of the family \mathfrak{S} of all the homogeneous two-sided ideals $I \subseteq \bigoplus_{n \geq 2} T(V)$ such that I is a Yetter-Drinfeld submodule of $T(V)$ and a Hopf ideal: $\Delta(I) \subset I \otimes T(V) + T(V) \otimes I$.

The following proposition characterizes the Nichols algebra associated to V in a very interesting way.

Proposition 2.1. [L2, Prop. 1.2.3], [AS2, Prop. 2.10]. *Given scalars $a_1, \dots, a_\theta \in \mathbb{k}^\times$, there exists a unique bilinear form $(\cdot | \cdot) : T(V) \times T(V) \rightarrow \mathbb{k}$ such that $(1|1) = 1$, and:*

$$(x|yy') = (x_{(1)}|y)(x_{(2)}|y'), \quad \text{for all } x, y, y' \in T(V); \quad (2.7)$$

$$(xx'|y) = (x|y_{(1)})(x'|y_{(2)}), \quad \text{for all } x, x', y \in T(V), \quad (2.8)$$

$$(x_i|x_j) = \delta_{ij}a_i, \quad \text{for all } i, j. \quad (2.9)$$

This form is symmetric and satisfies

$$(x|y) = 0, \quad \text{for all } x \in T(V)_g, y \in T(V)_h, g, h \in \Gamma, g \neq h. \quad (2.10)$$

The radical of this form $\{x \in T(V) : (x|y) = 0, \forall y \in T(V)\}$ is $I(V)$, so $(\cdot | \cdot)$ induces a non degenerate bilinear form on $\mathfrak{B}(V)$ denoted by the same name. \square

In consequence, if (V, c) is of diagonal type, then the ideal $I(V)$ is \mathbb{Z}^θ -homogeneous and $\mathfrak{B}(V)$ is \mathbb{Z}^θ -graded, see [AS2, Prop. 2.10].

2.2 Lyndon words and PBW basis of braided graded Hopf algebras generated in degree zero and one

A word $u \in \mathbb{X}$, $u \neq 1$, is *Lyndon* if u is smaller than any of its proper ends; that is, for any decomposition $u = vw$, $v, w \in \mathbb{X} - \{1\}$, we have $u < w$. We denote by L the set of Lyndon words, see [Lo, Chapter 5]

Note that $X \subset L$, and any Lyndon word begins by its smallest letter. They also satisfy the following properties.

1. Let $u \in \mathbb{X} - X$. Then u is Lyndon if and only if for any decomposition $u = vw$, $v, w \in \mathbb{X} - \{1\}$, it satisfies $vw = u < wv$.
2. If $v, w \in L$, $v < w$, then $vw \in L$.
3. Let $u \in \mathbb{X} - X$. Then $u \in L$ if and only if there exist $v, w \in L$ with $v < w$ such that $u = vw$.

Definition 2.2. Let $u \in L - X$. The *Shirshov decomposition* of u is the decomposition $u = vw$, with $v, w \in L$ such that w is the smallest end among those proper non-empty ends of u , see [Lo]. Following [He], we denote it by $\text{Sh}(u) = (v, w) \in L \times L$. It satisfies that w is the longest end between the ends that are Lyndon words.

Given $u, v, w \in L$ be such that $u = vw$, $u \neq 1$, then $\text{Sh}(u) = (v, w)$ if and only if either $v \in X$, or else $\text{Sh}(v) = (v_1, v_2)$ satisfies $w \leq v_2$.

Lyndon Theorem says that any word $u \in \mathbb{X}$ admits a unique decomposition $u = l_1 l_2 \dots l_r$, as a product of non-increasing Lyndon words: $l_i \in L$, $l_r \leq \dots \leq l_1$; see [Lo, Thm. 5.1.5]. This is called the *Lyndon decomposition* of $u \in \mathbb{X}$; we call *Lyndon letters* of u to any $l_i \in L$ appearing in such decomposition.

We recall the endomorphism $[-]_c$, see [K1], defined inductively on $k\mathbb{X}$ using Shirshov and Lyndon decomposition:

$$[u]_c := \begin{cases} u, & \text{if } u = 1 \text{ or } u \in X; \\ [[v]_c, [w]_c]_c, & \text{if } u \in L, \ell(u) > 1 \text{ and } \text{Sh}(u) = (v, w); \\ [u_1]_c \dots [u_t]_c, & \text{if } u \in \mathbb{X} - L \end{cases}$$

with Lyndon decomposition $u = u_1 \dots u_t$.

Definition 2.3. [K1]. The *hyperletter* corresponding to $l \in L$ is $[l]_c$. A *hyperword* is a word in hyperletters, and a *monotone hyperword* is a hyperword $[u_1]_c^{k_1} \dots [u_m]_c^{k_m}$ such that $u_1 > \dots > u_m$.

Note that for any $u \in L$, $[u]_c$ is a homogeneous polynomial with coefficients in the subring $\mathbb{Z}[q_{ij}]$ and $[u]_c \in u + \mathbb{Z}[q_{ij}]\mathbb{X}_{>u}^{\ell(u)}$.

The hyperletters inherit the order from the Lyndon words; this induces in turn an ordering in the hyperwords (the lexicographical order on the hyperletters). We describe now the braided commutator of hyperwords.

Theorem 2.4. [R2, Thm. 10]. Let $m, n \in L$, with $m < n$. Then $[[m]_c, [n]_c]_c$ is a $\mathbb{Z}[q_{ij}]$ -linear combination of monotone hyperwords $[l_1]_c \dots [l_r]_c$, $l_i \in L$, such that the hyperletters of those hyperwords satisfy $n > l_i \geq mn$.

Moreover, $[mn]_c$ appears in the expansion with non-zero coefficient, and for any hyperword of this decomposition, $\deg(l_1 \dots l_r) = \deg(mn)$. \square

The coproduct of $T(V)$ can be described also in the basis of hyperwords.

Lemma 2.5. [R2]. Let $u \in \mathbb{X}$, and $u = u_1 \dots u_r v^m$, $v, u_i \in L$, $v < u_r \leq \dots \leq u_1$ be the Lyndon decomposition of u . Then

$$\begin{aligned} \Delta([u]_c) &= 1 \otimes [u]_c + \sum_{i=0}^m \binom{m}{i}_{q_{v,v}} [u_1]_c \dots [u_r]_c [v]_c^i \otimes [v]_c^{m-i} \\ &\quad + \sum_{\substack{l_1 \geq \dots \geq l_p > v, l_i \in L \\ 0 \leq j \leq m}} x_{l_1, \dots, l_p}^{(j)} \otimes [l_1]_c \dots [l_p]_c [v]_c^j; \end{aligned}$$

where each $x_{l_1, \dots, l_p}^{(j)}$ is \mathbb{Z}^θ -homogeneous, $\deg(x_{l_1, \dots, l_p}^{(j)} l_1 \dots l_p v^j) = \deg(u)$. \square

We have then the following result from [R2].

Lemma 2.6. *For each $l \in L$ call W_l the subspace of $T(V)$ generated by*

$$[l_1]_c [l_2]_c \cdots [l_k]_c, \quad k \in \mathbb{N}_0, l_i \in L, l_1 \geq \dots \geq l_k \geq l. \quad (2.11)$$

Then W_l is a right coideal subalgebra of $T(V)$.

Proof. It follows from Theorem 2.4 and Lemma 2.5. \square

We consider as in [U] and [K1] another order in \mathbb{X} . Given $u, v \in \mathbb{X}$, we say that $u \succ v$ if and only if either $\ell(u) < \ell(v)$, or else $\ell(u) = \ell(v)$ and $u > v$ for the lexicographical order. We call \succ the *deg-lex order*, which is a total order. The empty word 1 is the maximal element for \succ , and this order is invariant by right and left multiplication.

Let I be a proper ideal of $T(V)$, and set $R = T(V)/I$. Let $\pi : T(V) \rightarrow R$ be the canonical projection. Let us consider the subset of \mathbb{X} :

$$G_I := \{u \in \mathbb{X} : u \notin \mathbb{k}\mathbb{X}_{\succ u} + I\}.$$

Such set satisfies:

- (a) If $u \in G_I$ and $u = vw$, then $v, w \in G_I$.
- (b) Any $u \in G_I$ factorizes uniquely as a non-increasing product of Lyndon words in G_I .

Proposition 2.7. [K1, R2]. *The set $\pi(G_I)$ is a basis of R .* \square

In what follows, we assume that I is a Hopf ideal. Consider now

$$S_I := G_I \cap L. \quad (2.12)$$

We then define the *height* function $h_I : S_I \rightarrow \{2, 3, \dots\} \cup \{\infty\}$ by

$$h_I(u) := \min \{t \in \mathbb{N} : u^t \in \mathbb{k}\mathbb{X}_{\succ u^t} + I\}. \quad (2.13)$$

One can find a PBW-basis by hyperwords of the quotient R of $T(V)$ using the set S_I and the height previously defined.

Theorem 2.8. [K1]. *The following set is a PBW-basis of $R = T(V)/I$:*

$$\{[u_1]_c^{n_1} \cdots [u_k]_c^{n_k} : k \in \mathbb{N}_0, u_1 > \dots > u_k \in S_I, 0 \leq n_i < h_I(u_i)\}. \quad \square$$

Proofs are in [K1], where the next consequences are also considered.

Proposition 2.9. *For any $v \in S_I$ such that $h_I(v) < \infty$, $q_{v,v}$ is a root of unity, whose order coincides with $h_I(v)$.* \square

Corollary 2.10. *A word u does not belong to G_I if and only if the associated hyperletter $[u]_c$ is a linear combination, modulo I , of hyperwords $[w]_c$, $w \succ u$, where all the hyperwords have their hyperletters in S_I .*

Moreover, if $h_I(v) := h < \infty$, then $[v]^h$ is a linear combination of hyperwords $[w]_c$, $w \succ v^h$. \square

3 Root systems and coideal subalgebras

In this section we recall the definition of Weyl groupoid and the associated generalized root system given in [CH1] and [HY]. We recall also some properties of these objects that we shall use in the subsequent sections, and the relation with Nichols algebras of diagonal type. After that, we describe convex orders for subsets of the root systems as a generalization of Papi's results in [P] for Weyl groups. We consider then a family of coideal subalgebras of a Nichols algebra of diagonal type with finite root system in order to prove that the ordering on the Lyndon words of a PBW basis as in Section 2.2 is convex. For the proof of the convexity we use the characterization of coideal subalgebras given in [HS].

3.1 Weyl groupoid and root systems

The notation used here is the same as in [CH1].

Fix a non-empty set \mathcal{X} , a non-empty finite set I and call $\{\alpha_i\}_{i \in I}$ the canonical basis of \mathbb{Z}^I . For each $i \in I$ consider a map $r_i : \mathcal{X} \rightarrow \mathcal{X}$, and for each $X \in \mathcal{X}$ a generalized Cartan matrix $A^X = (a_{ij}^X)_{i,j \in I}$ in the sense of [Ka].

Definition 3.1. [HY, CH1] The quadruple $\mathcal{C} := \mathcal{C}(I, \mathcal{X}, (r_i)_{i \in I}, (A^X)_{X \in \mathcal{C}})$ is a *Cartan scheme* if

- for all $i \in I$, $r_i^2 = id$, and
- for all $X \in \mathcal{X}$ and $i, j \in I$: $a_{ij}^X = a_{ij}^{r_i(X)}$.

For each $i \in I$ and $X \in \mathcal{X}$ denote by s_i^X the automorphism of \mathbb{Z}^I given by

$$s_i^X(\alpha_j) = \alpha_j - a_{ij}^X \alpha_i, \quad j \in I.$$

The *Weyl groupoid* of \mathcal{C} is the groupoid $\mathcal{W}(\mathcal{C})$ whose set of objects is \mathcal{X} and whose morphisms are generated by s_i^X , where we consider $s_i^X \in \text{Hom}(X, r_i(X))$, $i \in I$, $X \in \mathcal{X}$.

In general we shall denote $\mathcal{W}(\mathcal{C})$ simply by \mathcal{W} , and for any $X \in \mathcal{X}$:

$$\text{Hom}(\mathcal{W}, X) := \cup_{Y \in \mathcal{X}} \text{Hom}(Y, X), \quad (3.1)$$

$$\Delta^{X \text{ re}} := \{w(\alpha_i) : i \in I, w \in \text{Hom}(\mathcal{W}, X)\}. \quad (3.2)$$

$\Delta^{X \text{ re}}$ is the set of *real roots* of X . Each $w \in \text{Hom}(\mathcal{W}, X_1)$ can be written as a product $s_{i_1}^{X_1} s_{i_2}^{X_2} \cdots s_{i_m}^{X_m}$, where $X_j = r_{i_{j-1}} \cdots r_{i_1}(X_1)$, $i \geq 2$. We denote $w = id_{X_1} s_{i_1} \cdots s_{i_m}$: this means that $w \in \text{Hom}(\mathcal{W}, X_1)$, because the elements $X_j \in \mathcal{X}$ are univocally determined. The *length* of w is defined by

$$\ell(w) = \min\{n \in \mathbb{N}_0 : \exists i_1, \dots, i_n \in I \text{ such that } w = id_X s_{i_1} \cdots s_{i_n}\}.$$

In what follows we will assume that \mathcal{W} is a connected groupoid:

$$\mathrm{Hom}(Y, X) \neq \emptyset, \quad \forall X, Y \in \mathcal{X}.$$

Definition 3.2. [HY, CH1] Fix a Cartan scheme \mathcal{C} , and for each $X \in \mathcal{X}$ a set $\Delta^X \subset \mathbb{Z}^I$. $\mathcal{R} := \mathcal{R}(\mathcal{C}, (\Delta^X)_{X \in \mathcal{X}})$ is a *root system of type \mathcal{C}* if

1. for all $X \in \mathcal{X}$, $\Delta^X = (\Delta^X \cap \mathbb{N}_0^I) \cup -(\Delta^X \cap \mathbb{N}_0^I)$,
2. for each $i \in I$ and each $X \in \mathcal{X}$, $\Delta^X \cap \mathbb{Z}\alpha_i = \{\pm\alpha_i\}$,
3. for each $i \in I$ and each $X \in \mathcal{X}$, $s_i^X(\Delta^X) = \Delta^{r_i(X)}$,
4. if $m_{ij}^X := |\Delta^X \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)|$, then $(r_i r_j)^{m_{ij}^X}(X) = (X)$ for all $i \neq j \in I$ and all $X \in \mathcal{X}$.

We call $\Delta_+^X := \Delta^X \cap \mathbb{N}_0^I$ the set of *positive roots*, and $\Delta_-^X := -\Delta_+^X$ the set of *negative roots*.

By (3) we have that $w(\Delta^X) = \Delta^Y$ for any $w \in \mathrm{Hom}(Y, X)$.

We say that \mathcal{R} is *finite* if Δ^X is finite for some $X \in \mathcal{X}$. By [CH1, Lemma 2.11], this is equivalent to the fact that the sets Δ^X are finite, for all $X \in \mathcal{X}$, and that \mathcal{W} is finite.

The following result plays a fundamental role for our purposes in the next subsection.

Theorem 3.3. [CH2, Thm. 2.10] Let $\alpha \in \Delta_+^X \setminus \{\alpha_i : i = 1, \dots, \theta\}$. There exist $\beta, \gamma \in \Delta_+^X$ such that $\alpha = \beta + \gamma$. \square

Now we recall some results involving real roots and the length of the elements in \mathcal{W} .

Lemma 3.4. [HY, Cor. 3] Let $m \in \mathbb{N}$, $X, Y \in \mathcal{X}$ and $i_1, \dots, i_m, j \in I$. Call $w = \mathrm{id}_X s_{i_1} \cdots s_{i_m} \in \mathrm{Hom}(Y, X)$, and assume that $\ell(w) = m$. Then,

- $\ell(ws_j) = m + 1$ if and only if $w(\alpha_j) \in \Delta_+^X$,
- $\ell(ws_j) = m - 1$ if and only if $w(\alpha_j) \in \Delta_-^X$.

\square

Proposition 3.5. [CH1, Prop. 2.12] For each $w = \mathrm{id}_X s_{i_1} \cdots s_{i_m}$ such that $\ell(w) = m$, the roots $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}) \in \Delta^X$ are positive and pairwise different. If w is an element of maximal length and \mathcal{R} is finite, then $\{\beta_j\} = \Delta_+^X$. In consequence, all the roots are real: i.e., for each $\alpha \in \Delta_+^X$ there exist $i_1, \dots, i_k, j \in I$ such that $\alpha = s_{i_k} \cdots s_{i_1}(\alpha_j)$. \square

As in [HS], consider for $X \in \mathcal{X}$, $m \in \mathbb{N}$ and $(i_1, \dots, i_m) \in I^m$ the sets:

$$\Lambda^X(i_1, \dots, i_m) := \{\beta_k := \text{id}_X s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) : 1 \leq k \leq m\} \subset \Delta^X, \quad (3.3)$$

$$\Lambda_+^X(i_1, \dots, i_m) := \{\beta \in \Delta_+^X : |\{k \in \{1, \dots, m\} : \beta = \pm \beta_k\}| \text{ is odd}\}. \quad (3.4)$$

By [HS, Prop. 1.9], given other elements $j_1, \dots, j_n \in I$, we have

$$\Lambda_+^X(i_1, \dots, i_m) = \Lambda_+^X(j_1, \dots, j_n) \Leftrightarrow \text{id}_X s_{i_1} \cdots s_{i_m} = \text{id}_X s_{j_1} \cdots s_{j_n},$$

and moreover,

$$|\Lambda_+^X(i_1, \dots, i_m)| = \ell(\text{id}_X s_{i_1} \cdots s_{i_m}). \quad (3.5)$$

In this way, if $w = \text{id}_X s_{i_1} \cdots s_{i_m}$ is any expression of $w \in \text{Hom}(\mathcal{W}, X)$, we can define $\Lambda_+^X(w) := \Lambda_+^X(i_1, \dots, i_m)$.

3.2 Convex orders on root systems

Now we characterize convex orders on subsets of root systems of finite Weyl groupoids, extending the results given in [P] for Weyl groups.

Definition 3.6. Consider a root system Δ_+^X with a fixed total order $<$. We say that it is

- *convex* if for each $\alpha, \beta \in \Delta^+$ such that $\alpha < \beta$ and $\alpha + \beta \in \Delta^+$, then

$$\alpha < \alpha + \beta < \beta.$$

- *sub-convex* if for each $\alpha, \beta \in \Delta^+$ such that $\alpha < \beta$ and $\alpha + \beta \in \Delta^+$, then

$$\alpha < \alpha + \beta.$$

- *strongly convex* if for each ordered subset $\alpha_1 \leq \dots \leq \alpha_k$ of Δ^+ such that $\alpha := \sum \alpha_i \in \Delta^+$ then $\alpha_1 < \alpha < \alpha_k$.

Definition 3.7. Let $L = \{\beta_1, \dots, \beta_m\}$ be an ordered subset of Δ_+^X . We say that L is *associated to* $w \in \text{Hom}(\mathcal{W}, X)$ if there exists a reduced expression $w = \text{id}_X s_{i_1} \cdots s_{i_m}$ such that

$$\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}), \quad \forall 1 \leq j \leq m.$$

Compare this with [P]. For any $w \in \text{Hom}(Y, X)$ define

$$R_w := \{\alpha \in \Delta_+^X : w^{-1}(\alpha) \in \Delta_-^Y\}.$$

Now we generalize some results about Weyl groups to the context of Weyl groupoids. First we consider the analogue of a result in [Bo].

Proposition 3.8. *For any ordered set L associated to w , we have $L = R_w$. In consequence, $|R_w| = \ell(w)$ and two ordered sets associated to the same w differ at most by the ordering.*

Proof. Note that for any $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$,

$$w^{-1}(\beta_j) = -s_{i_m} \cdots s_{i_{j+1}}(\alpha_j).$$

$s_{i_m} \cdots s_{i_{j+1}} s_{i_j}$ is a reduced expression because it is contained in a reduced expression, so we have $w^{-1}(\beta_j) \in \Delta_-^Y$ by Lemma 3.4. Therefore $L \subseteq R_w$.

Reciprocally, let $\alpha \in R_w$. As $w^{-1}(\alpha) \in \Delta_-^Y$ and $s_{i_1} \cdots s_{i_m}(w^{-1}(\alpha)) = \alpha \in \Delta_+^X$, consider the greatest j such that $s_{i_j} \cdots s_{i_m} w^{-1}(\alpha)$ is positive. Therefore $s_{i_{j+1}} \cdots s_{i_m} w^{-1}(\alpha)$ is negative, so $s_{i_j} \cdots s_{i_m} w^{-1}(\alpha) = \alpha_{i_j}$, and then $\alpha_{i_j} = s_{i_j} \cdots s_{i_m} w^{-1}(\alpha)$; that is, $\alpha = s_{i_1} \cdots s_{i_{j-1}}(\alpha_j) \in L$. \square

Second, we relate our sets R_w with the ones in [HS], see (3.4). Although the sets are equal, our definition is more comfortable to prove statements about convexity.

Lemma 3.9. *For each $w \in \text{Hom}(\mathcal{W}, X)$, $R_w = \Lambda_+^X(w)$.*

Proof. Fix a reduced expression $w = \text{id}_X s_{i_1} \cdots s_{i_m}$, so $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$ is a positive root, and $\alpha \in \Delta_+^X$ is equal to $\pm\beta_j$ if and only if $\alpha = \beta_j$. Therefore $\Lambda_+^X(w) = L$. \square

Now we extend another result from [P]. Note that condition (a) in our result is weaker than the one in [P], but the proof is very similar. This weaker condition shall simplify some proofs in what follows.

Theorem 3.10. *Let L be an ordered subset of Δ_+^X . There exists $w \in \text{Hom}(\mathcal{W}, X)$ such that L is associated to w if and only if the following conditions are satisfied:*

- (a) *For each pair $\lambda < \mu \in L$ such that $\lambda + \mu \in \Delta_+^X$, then $\lambda + \mu \in L$ and $\lambda < \lambda + \mu$.*
- (b) *If $\lambda + \mu \in L$ and $\lambda, \mu \in \Delta_+^X$, then at least one of them belongs to L and precedes $\lambda + \mu$.*

Proof. Assume that L is associated to $w = \text{id}_X s_{i_1} \cdots s_{i_m}$ for some $w \in \text{Hom}(Y, X)$. If $\lambda = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$ and $\mu = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$ are such that $1 \leq k < j \leq m$ and $\lambda + \mu \in \Delta_+^X$, we have $\lambda + \mu \in L = R_w$, because

$$w^{-1}(\lambda + \mu) = w^{-1}(\lambda) + w^{-1}(\mu) \in \Delta_-^Y.$$

Suppose that $\lambda + \mu < \lambda$. Then $\lambda + \mu = s_{i_1} \cdots s_{i_{l-1}}(\alpha_{i_l})$ for some $1 \leq l < k$, so $s_{i_l} \cdots s_{i_1}(\lambda + \mu) = -\alpha_l \in \Delta_-^{r_{i_l} \cdots r_{i_1}(X)}$. But as $l < k < j$, we have

$$s_{i_l} \cdots s_{i_1}(\lambda), s_{i_l} \cdots s_{i_1}(\mu) \in \Delta_-^{r_{i_l} \cdots r_{i_1}(X)},$$

which is a contradiction. Therefore $\lambda < \lambda + \mu$, and L satisfies (a).

For (b), suppose that $\lambda + \mu \in L$, but $\lambda, \mu \notin L$: $w^{-1}(\lambda), w^{-1}(\mu) \in \Delta_+^Y$, so $w^{-1}(\lambda + \mu)$ is positive, which is a contradiction to the fact that $\lambda + \mu \in R_w$. If both $\lambda, \mu \in L$, a similar proof to (a) gives that one of them precedes $\lambda + \mu$. In consequence, suppose that $\lambda \in L, \mu \notin L$ and $\lambda + \mu < \lambda$. If $l < k$ is such that $\lambda + \mu = s_{i_1} \cdots s_{i_{l-1}}(\alpha_{i_l})$, we have $s_{i_l} \cdots s_{i_1}(\lambda) \in \Delta^+$ and

$$s_{i_l} \cdots s_{i_1}(\lambda) + s_{i_l} \cdots s_{i_1}(\mu) = s_{i_l} \cdots s_{i_1}(\lambda + \mu) = -\alpha_l \in \Delta_-^{r_{i_l} \cdots r_{i_1}(X)},$$

so $s_{i_l} \cdots s_{i_1}(\mu) \in \Delta_-^{r_{i_l} \cdots r_{i_1}(X)}$, and then $\mu \in R_{\text{id}_X s_{i_1} \cdots s_{i_l}} \subset R_{\text{id}_X s_{i_1} \cdots s_{i_m}} = L$, a contradiction.

Reciprocally, we prove that an ordered set L satisfying (a) and (b) is associated to some w by induction on $m := |L|$. If $m = 1$, let $\alpha \in L$. If we suppose that α is not simple, by Theorem 3.3, $\alpha = \beta + \gamma$ for some positive roots β, γ , and by condition (b) one of them belongs to L , so $m \geq 2$, which is a contradiction. Therefore $L = \{\alpha_j\} = R_{s_j}$ for some $1 \leq j \leq \theta$.

Now assume $m > 1$ and call $\beta_1 < \dots < \beta_m$ the elements of L . Notice that $L' = \{\beta_1, \dots, \beta_{m-1}\}$ verifies conditions (a) and (b), so by inductive hypothesis there exists a reduced expression $v = \text{id}_X s_{i_1} \cdots s_{i_{m-1}}$ such that

$$\beta_1 = \alpha_{i_1}, \quad \beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}), \quad j = 2, \dots, m-1.$$

Let $Z = r_{i_{m-1}} \cdots r_{i_1}(X)$. Then $v^{-1}(\beta_m) \in \Delta_+^Z$ because $\beta_m \notin L' = R_v$. Suppose that $v^{-1}(\beta_m)$ is not simple. Then there exist $\alpha, \beta \in \Delta_+^Z$ such that $\alpha + \beta = v^{-1}(\beta_m)$; i.e. $\beta_m = \alpha' + \beta'$, where $\alpha' = v(\alpha), \beta' = v(\beta) \in \Delta^X$. Therefore $\alpha' \in \Delta_+^X$ or $\beta' \in \Delta_+^X$. On the other hand, if both are positive then one of them is β_k for some $k < m$; assume $\alpha' = \beta_k$, but then $\alpha = v^{-1}(\beta_k) \in \Delta_-^Z$, a contradiction. In consequence, we can consider $\alpha' \in \Delta_+^X$ and $\beta' \in \Delta_-^X$. For this case, $\alpha' \notin R_v = L'$ and $-\beta' \in R_v = L' \subset L$. As $\alpha' = \beta_m + (-\beta')$, hypothesis (a) on the set L implies that $\alpha' \in L$, so $\alpha' = \beta_m \in L - L'$, a contradiction. Therefore, $v^{-1}(\beta_m) = \alpha_{i_m}$ for some $i_m \in I$, $w = v s_{i_m} \in \text{Hom}(r_{i_m}(Z), X)$ is a reduced expression by Lemma 3.4, and $L = R_w$. \square

Theorem 3.11. *Given an order on Δ_+^X , the following statements are equivalent:*

1. *the order is associated with a reduced expression of the longest element,*
2. *the order is strongly convex,*
3. *the order is convex.*

Proof. (1) \Rightarrow (2). Let $\omega = \text{id}_X s_{i_1} \cdots s_{i_m}$ be an element of maximal length in $\text{Hom}(\mathcal{W}, X)$. By Proposition 3.5, $m = |\Delta_+^X|$ and

$$\beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}), \quad k = 1, \dots, m,$$

are all different, so $\{\beta_k\} = \Delta_+^X$. In consequence, it induces an order on Δ_+^X :

$$\beta_1 < \cdots < \beta_m.$$

To prove that this order is strongly convex, consider $\beta, \beta_{k_1}, \dots, \beta_{k_l} \in \Delta_+^X$ such that $k_1 < \cdots < k_l$ and $\beta = \beta_{k_1} + \cdots + \beta_{k_l}$. Suppose that $\beta = \beta_k$ with $k < k_1$. Then $u = \text{id}_X s_{i_1} \cdots s_{i_k}$ satisfies $\ell(u) = k$, $\beta \in R_u$ but $\beta_{k_j} \notin R_u$ for all $j = 1, \dots, l$, which is a contradiction because $u(\beta) \in \Delta_-^{r_{i_k} \cdots r_{i_1} X}$ should be the sum of the positive roots $u(\beta_j)$. We obtain a similar contradiction if we assume $k > k_l$. Therefore $k_1 < k < k_l$.

(2) \Rightarrow (3) is clear.

(3) \Rightarrow (1). Assume that a given order on Δ_+^X is convex; i.e. it satisfies trivially condition (a) of Theorem 3.10 because we consider $L = \Delta_+^X$. Therefore it satisfies also condition (b) by the convexity, so the order is associated to some w . As $\ell(w) = |\Delta_+^X|$ by Proposition 3.8, it should be the element of maximal length. \square

3.3 Coideal subalgebras and convex orders for PBW bases

Now we recall a description of coideal subalgebras of Nichols algebras with finite root system given in [HS]. We will use this result to prove that the lexicographical order on the PBW generators of Kharchenko's basis is convex. Before to prove it, we recall the results about the Weyl groupoid attached to a braided vector space of diagonal type. Given a braided vector space (V, c) of diagonal type, fix a basis $\{x_1, \dots, x_\theta\}$ and scalars $q_{ij} \in \mathbb{k}^\times$ as in (2.1), and the bilinear form as in (2.3). We set as in [H1], Δ_+^V the set of degrees of a PBW basis of $\mathfrak{B}(V)$, counted with their multiplicities. Such set does not depend on the PBW basis, as it is remarked in [H1] and proved in [AA].

In what follows, **we fix a braided vector space (V, c) of diagonal type and assume that the root system Δ_+^V is finite.** In such case we can attach a Cartan scheme \mathcal{C} , a Weyl groupoid \mathcal{W} and the corresponding root system \mathcal{R} , see [HS, Thms. 6.2, 6.9] and the references therein, which coincides with the Weyl groupoid defined in [H1] for braided vector space of diagonal type. Such Weyl groupoid can be built as follows, see [AA]. Set \mathcal{X} the set of ordered bases of \mathbb{Z}^θ , and for each $F = \{f_1, \dots, f_\theta\} \in \mathcal{X}$, set $\tilde{q}_{ij} = \chi(f_i, f_j)$. Define for each $1 \leq i \neq j \leq \theta$,

$$m_{ij}(F) := \min \{n \in \mathbb{N}_0 : (n+1)\tilde{q}_{ii}(1 - \tilde{q}_{ii}^n \tilde{q}_{ij} \tilde{q}_{ji}) = 0\}, \quad (3.6)$$

and set $s_{i,F} \in \text{Aut}(\mathbb{Z}^\theta)$ such that $s_{i,F}(f_j) = f_j + m_{ij}(F)f_i$. Here $m_{ii} = -2$.

Note that $\mathcal{G} = \text{Aut}(\mathbb{Z}^\theta) \times \mathcal{X}$ is a groupoid whose set of objects is \mathcal{X} and whose morphisms are:

$$x \xrightarrow{(g,x)} g(x).$$

The *Weyl Groupoid* $W(\chi)$ of χ is the least subgroupoid of \mathcal{G} such that

- $(id, E) \in W(\chi)$,
- if $(id, F) \in W(\chi)$ and $s_{i,F}$ is defined, then $(s_{i,F}, F) \in W(\chi)$.

Recall that the (right) *Duflo order* on $\text{Hom}(\mathcal{W}, X)$ is defined as follows: if $x \in \text{Hom}(Y, X)$ and $y \in \text{Hom}(Z, Y)$, then $x \leq_D xy$ iff $\ell(xy) = \ell(x) + \ell(y)$; see [HS, Defn. 1.11]. By [HS, Thm. 1.13], given $v, w \in \text{Hom}(\mathcal{W}, X)$ we have $v \leq_D w$ if and only if $\Lambda_+^X(v) \subset \Lambda_+^X(w)$.

Remark 3.12. Let $w_1 \leq_D w_2 \leq_D \cdots \leq_D w_k$ be a maximal chain in $\text{Hom}(\mathcal{W}, X)$. Then there exist a reduced expression $\text{id}_X s_{i_1} \cdots s_{i_k}$ for some $i_1, \dots, i_k \in I$ such that $w_j = \text{id}_X s_{i_1} \cdots s_{i_j}$, for each $1 \leq j \leq k$.

In particular, a chain $w_1 \leq_D w_2 \leq_D \cdots \leq_D w_k$ has maximal length iff it is associated to a reduced expression of the longest element in $\text{Hom}(\mathcal{W}, X)$, and in consequence $k = |\Delta_+^X|$.

We recall now some results from [HS] about the classification of coideal subalgebras of $\mathfrak{B}(V)$. As in loc. cit., we denote by $\mathcal{K}(V)$ the set of all the \mathbb{N}_0^θ -graded left coideal subalgebras of $\mathfrak{B}(V)$. We rewrite these results in the context of diagonal braidings (in [HS] the authors work in a more general context).

First results about the classification of coideal subalgebras were obtained in [K3, KL, Po] for quantized enveloping algebras $U_q(\mathfrak{g})$ of type A_n, B_n and G_2 , respectively, where it was proved that coideal subalgebras admit a PBW basis and these subalgebras were classified.

Given $n = (n_1, \dots, n_\theta) \in \mathbb{N}_0^\theta$, we set $X^n = X_1^{n_1} \cdots X_\theta^{n_\theta}$ in $\mathbf{k}[[x_1, \dots, x_\theta]]$. We also set

$$\mathfrak{q}_h(t) := \frac{t^h - 1}{t - 1} \in \mathbf{k}[t], \quad h \in \mathbb{N}; \quad \mathfrak{q}_\infty(t) := \frac{1}{1 - t} = \sum_{s=0}^{\infty} t^s \in \mathbf{k}[[t]].$$

For each \mathbb{N}_0^θ -graded \mathbf{k} -vector space $W = \bigoplus_{\alpha \in \mathbb{N}_0^\theta} W_\alpha$, we denote its Hilbert series by

$$\mathcal{H}_W := \sum_{\alpha \in \mathbb{N}_0^\theta} (\dim W_\alpha) X^\alpha \in \mathbf{k}[[x_1, \dots, x_\theta]].$$

For any $\alpha \in \mathbb{N}_0^\theta$, we set $q_\alpha = \chi(\alpha, \alpha)$, where χ is the bicharacter over \mathbb{Z}^θ as in (2.3), and $N_\alpha = \text{ord} q_\alpha$, where $N_\alpha = \infty$ if q_α is not a root of unity.

Theorem 3.13. [HS] For each $w \in \text{Hom}(\mathcal{W}, V)$ there exists a unique left coideal subalgebra $F(w) \in \mathcal{K}(V)$ such that its Hilbert series is

$$\mathcal{H}_{F(w)} = \prod_{\beta \in \Lambda_+^V(w)} \mathfrak{q}_{N_\beta}(X^\beta). \tag{3.7}$$

Moreover, the correspondence $w \mapsto F(w)$ gives an order preserving and order reflecting bijection between $\text{Hom}(\mathcal{W}, V)$ and $\mathcal{K}(V)$, where we consider the Duflo order over $\text{Hom}(\mathcal{W}, V)$ and the inclusion order over $\mathcal{K}(V)$; i.e.

$$w_1 \leq_D w_2 \Leftrightarrow F(w_1) \subset F(w_2).$$

Proof. Note that in [HS] the authors classify right coideal subalgebras, but that E is a right coideal subalgebras if and only if $\mathcal{S}(E)$ is a left coideal subalgebra, where \mathcal{S} denotes the antipode of $\mathfrak{B}(V)$. Moreover, if they are \mathbb{N}^θ -graded, then $\mathcal{H}_E = \mathcal{H}_{\mathcal{S}(E)}$, because \mathcal{S} is \mathbb{N}^θ -graded, and the order given by inclusion on the family of left coideal subalgebras corresponds with the one on the family of right coideal subalgebras because \mathcal{S} is bijective. In this context we define $F(w) = \mathcal{S}(E^V(w))$, where $E^V(w)$ is as in [HS, Thm. 6.12].

By [HS, Lemma 6.11], we have an isomorphism of \mathbb{N}_0^θ -graded spaces

$$F(w) \cong \otimes_{\beta \in \Lambda_+^V(w)} \mathfrak{B}(V_\beta),$$

where V_β corresponds to N_β of [HS, Defn. 6.5]. In this way V_β is a 1-dimensional braided vector space of diagonal type generated by a non-zero vector v_β , such that $c(v_\beta \otimes v_\beta) = q_\beta v_\beta \otimes v_\beta$. Therefore, $\mathcal{H}_{\mathfrak{B}(V_\beta)} = \mathfrak{q}_{N_\beta}(X^\beta)$, and equation (3.7) follows.

The uniqueness of a coideal subalgebra with a given Hilbert series follows from [HS, Lemma 6.4]. The map $\text{Hom}(\mathcal{W}, V) \rightarrow \mathcal{K}(V)$ is bijective and preserves the order in both directions by [HS, Thms. 6.12, 6.15] (note that we can apply these Theorems because we assume that V has diagonal braiding and Δ_+^V is finite). \square

Consider the PBW basis of Lyndon words given in Theorem 2.8 for the fixed basis $\{x_1, \dots, x_\theta\}$ of V . We assume that Δ_+^V is finite, so all the roots are real and have multiplicity one. In this way, we can label the PBW generators by the elements $\beta \in \Delta_+^V$: they are $x_\beta = [l_\beta]_c$ for some Lyndon word l_β of degree β . It induces a total order on the roots: if $l_{\beta_1} < l_{\beta_2} < \dots < l_{\beta_M}$ are ordered lexicographically, we consider $\beta_1 < \beta_2 < \dots < \beta_M$, where $M = |\Delta_+^V|$ and in particular $l_{\beta_1} = x_1, l_{\beta_M} = x_\theta$. Call B the basis of $\mathfrak{B}(V)$ consisting of hyperwords as above.

Let $\pi : T(V) \rightarrow \mathfrak{B}(V) = T(V)/I(V)$ be the canonical projection. Recall the definition of the coideal subalgebras W_{l_β} in Lemma 2.6, and call

$$W_\beta := \pi(W_{l_\beta}), \quad \beta \in \Delta_+^V.$$

Remark 3.14. W_β is a left coideal subalgebra of $\mathfrak{B}(V)$, because π is a morphism of braided Hopf algebras and W_{l_β} is a left coideal subalgebra of $T(V)$. Also $W_{\beta_j} \subseteq W_{\beta_i}$ if $i < j$ and

$$W_{\beta_1} = \mathfrak{B}(V), \quad W_{\beta_M} = \mathfrak{k}\langle x_\theta \rangle.$$

Lemma 3.15. *With the notation above, $x_{\beta_i} \notin W_{\beta_j}$ if $i < j$.*

In consequence, $\mathfrak{B}(V) = W_{\beta_1} \supsetneq W_{\beta_2} \supsetneq \cdots \supsetneq W_{\beta_M}$.

Proof. Suppose that $x_{\beta_i} \in W_{\beta_j}$ with $i < j$. Then $x_{\beta_i} \in G_{I(V)}$ is a linear combination of hyperwords greater or equal that x_{β_j} in $\mathfrak{B}(V)$, which is a contradiction to Corollary 2.10. Therefore $x_{\beta_i} \notin W_{\beta_j}$. Then the second statement of the lemma follows from Remark 3.14. \square

We prove now the main result of this section.

Theorem 3.16. *Keep the notation above. The order $\beta_1 < \beta_2 < \cdots < \beta_M$ on Δ_+^V is convex.*

Proof. Each W_{β_i} corresponds with one $F(w_i)$. As we have a chain as in previous lemma, by Theorem 3.13 we have $w_1 \geq_D w_2 \geq_D \cdots \geq_D w_M$.

As the w_i 's are pairwise different, we have a chain of maximal length, and by Remark 3.12 there exists a reduced expression of the longest element $\omega^V = \text{id}_V s_{i_M} \cdots s_{i_1}$ such that $w_k = \text{id}_V s_{i_M} \cdots s_{i_k}$ for each $1 \leq k \leq M$.

We will prove by induction (descending on j) that $\beta_j = s_{i_M} \cdots s_{i_{j+1}}(\alpha_j)$. If so, we conclude the proof because of Theorem 3.11. For the first step, notice that $\mathcal{H}_{w_M} = \mathfrak{q}_{N_{\alpha_\theta}}(x_\theta)$ by Theorem 3.13, and by Remark 3.14 we have $i_m = \theta$, so we have the initial step.

Assume now that $k < M$ and $\beta_j = s_{i_M} \cdots s_{i_{j+1}}(\alpha_j)$ for $j = k+1, \dots, M$. Call $\gamma = s_{i_M} \cdots s_{i_{k+1}}(\alpha_k)$, so by inductive hypothesis we have

$$\mathcal{H}_{W_{\beta_{k+1}}} = \prod_{j=k+1}^M \mathfrak{q}_{N_{\beta_j}}(X^{\beta_j}), \quad \mathcal{H}_{W_{\beta_k}} = \mathfrak{q}_{N_\gamma}(X^\gamma) \left(\prod_{j=k+1}^M \mathfrak{q}_{N_{\beta_j}}(X^{\beta_j}) \right).$$

On the other hand, $\{x_{\beta_M}^{n_M} \cdots x_{\beta_k}^{n_k} : 0 \leq n_j < N_{\beta_j}\}$ is a linearly independent set of W_{β_k} , so

$$\mathcal{H}_{W_{\beta_k}} \geq \prod_{j=k}^M \mathfrak{q}_{N_{\beta_j}}(X^{\beta_j}),$$

where the inequality between the series means that the inequality holds for all the corresponding coefficients. Looking at the coefficient of X^{β_k} we obtain that there exists an expression

$$\beta_k = n\gamma + \sum_{j=k+1}^M n_j \beta_j, \quad n \in \mathbb{N}, n_j \in \mathbb{N}_0.$$

Note that $R_{w_k} = \Lambda_+^V = \{\gamma, \beta_{k+1}, \dots, \beta_M\}$, so if we apply w_k to the last equality, we obtain that $w_k^{-1}(\beta_k) \in \Delta_m^{r_1 \cdots r_M(V)}$. Therefore $\beta_k \in R_{w_k}$, and as $\beta_k \neq \beta_j$ for all $j > k$, we conclude $\beta_k = \gamma$. \square

The next result is analogous to the one for the positive part of quantized enveloping algebras $U_q(\mathfrak{g})$ given in [Le], and gives an inductive way to obtain the words l_β for $\beta \in \Delta_+^V$.

Corollary 3.17. *For each $\beta \in \Delta_+^V$, $\beta \neq \alpha_1, \dots, \alpha_\theta$,*

$$l_\beta = \max\{l_{\delta_1}l_{\delta_2} : \delta_1, \delta_2 \in \Delta_+^V, \delta_1 + \delta_2 = \beta, l_{\delta_1} < l_{\delta_2}\}. \quad (3.8)$$

Proof. Any factor of an element of $G_{I(V)}$ is in $G_{I(V)}$ (see Subsection 2.2). If $l_\beta = uv$ is the Shirshov decomposition of l_β , then there exist $\gamma_1, \gamma_2 \in \Delta_+^V$ such that $u = l_{\gamma_1} < v = l_{\gamma_2}$ and $\beta = \gamma_1 + \gamma_2$.

On the other hand, let $\delta_1, \delta_2 \in \Delta_+^V$ be such that $\delta_1 + \delta_2 = \beta$ and $l_{\delta_1} < l_{\delta_2}$. By the previous theorem, $l_{\delta_1} < l_\beta < l_{\delta_2}$. If l_β does not begin with l_{δ_1} , then $l_{\delta_1}u < l_\beta$ for every word u , so in particular $l_{\delta_1}l_{\delta_2} < l_\beta$. If l_β begins with l_{δ_1} , then $l_\beta = l_{\delta_1}u$, where u has degree δ_2 . Let $u = l_p l_{p-1} \cdots l_1$ be its Lyndon decomposition. Therefore each $l_i \in G_{I(V)}$, so $u = l_{\beta_M}^{n_M} \cdots l_{\beta_1}^{n_1}$ for some $n_i \in \mathbb{N}_0$. Let $k = \max\{j : n_j \neq 0\}$. As the order is strongly convex, $x_{\beta_k} \geq x_{\delta_2}$; i.e. $l_{\beta_k} \geq l_{\delta_2}$, so $u \geq l_{\delta_2}$ and then $l_\beta = l_{\delta_1}u \geq l_{\delta_1}l_{\delta_2}$. In any case, $l_\beta = l_{\delta_1}u \geq l_{\delta_1}l_{\delta_2}$. \square

Another consequence shows that the family of coideal subalgebras W_β (which are in particular left $\mathfrak{B}(V)$ -comodules) behaves as a kind of modules of highest weight.

Theorem 3.18. *The set $B_k = \{x_{\beta_M}^{n_M} \cdots x_{\beta_k}^{n_k} : 0 \leq n_j < N_{\beta_j}\}$ is a basis of W_{β_k} . Moreover, if $W_{\beta_k} = \bigoplus_{\alpha \in \mathbb{N}_0^\theta} W_{\beta_k}(\alpha)$ denotes the decomposition in the \mathbb{N}_0^θ homogeneous components, then $\dim W_{\beta_k}(\beta_k) = 1$.*

Proof. The first statement follows because B_k is included in W_{β_k} , it is linearly independent and the Hilbert series of the k -linear subspace spanned by B_k coincides with the Hilbert series of W_{β_k} .

For the second statement, if $\sum_{i=1}^M n_i \beta_i = \beta_k$ for some $n_i \in \mathbb{N}_0$, then $n_i = \delta_{i,k}$ or there exists $i < k$ such that $n_i > 0$, by Theorem 3.11. \square

The first consequence of the description of coideal subalgebras W_α as in previous theorem is a new expression of the coproduct of hyperwords which we will use in next section. We set

$$C_k := \{x_{\beta_k}^{n_k} x_{\beta_{k-1}}^{n_{k-1}} \cdots x_{\beta_1}^{n_1} : 0 \leq n_j < N_{\beta_j}\}, \quad (3.9)$$

$$D_k := \{x_{\beta_M}^{n_M} \cdots x_{\beta_1}^{n_1} : 0 \leq n_j < N_{\beta_j}, \exists j \geq k \text{ such that } n_j \neq 0\}. \quad (3.10)$$

Lemma 3.19. *Let $a \in B_k - \{1\}$, $b \in B_l$, $l \leq k$. Then $ab = 0$ or ab is spanned by elements of $B_l \cap D_k$.*

Proof. If $l = k$, it follows directly. Assume then $l < k$ and write $b = b_1 b_2$ with $b_1 \in B_k$ and $b_2 \in C_{k-1} \cap B_l$ (possibly $b_1 = 1$). Then $ab_1 \in W_{\beta_k}$, because W_{β_k} is a subalgebra, so it is spanned by B_k . To end, just note that if $c \in B_k$, then $cb_2 \in B_l \cap D_k$. \square

We set also $ht(u) := \sum n_i$, if $u = x_{\beta_M}^{n_M} x_{\beta_{k-1}}^{n_{k-1}} \cdots x_{\beta_1}^{n_1}$.

Lemma 3.20. *Let $u = x_{\beta_k}^{n_k} \cdots x_{\beta_l}^{n_l} \in B_l - D_{k+1}$, $l \leq k$, be such that $n_k, n_l \neq 0$. Then,*

$$\Delta(u) \in \left(\bigoplus_{v \in B, w \in D_k \cap B_l} \mathbf{k} v \otimes w \right) \oplus \left(\bigoplus_{v \in D_k, w \in B_l - D_k} \mathbf{k} v \otimes w \right).$$

Proof. We prove it by induction on the height. If $ht(u) = 1$, $u = x_{\beta_i}$ for some i . Then, $\Delta(u) \in u \otimes 1 + 1 \otimes u + \mathfrak{B}(V) \otimes W_{\beta_i}$, so the result follows.

Assume it holds for $ht(w) < n$, and $u = x_{\beta_k}^{n_k} \cdots x_{\beta_l}^{n_l}$ is such that $ht(u) = n$. Write $u = x_{\beta_k} w$, so by inductive hypothesis,

$$\Delta(u) \in \left(\bigoplus_{v \in B, w \in D_s \cap B_l} \mathbf{k} v \otimes w \right) \oplus \left(\bigoplus_{v \in D_s, w \in B_l - D_s} \mathbf{k} v \otimes w \right),$$

where $s = k - 1$ if $n_k = 1$, or $s = k$ if $n_k > 1$. We calculate $\Delta(u) = \Delta(x_{\beta_k}) \Delta(w)$. Using that the braiding is diagonal and Lemma 3.19 we conclude that

$$(\Delta(x_{\beta_k}) - x_{\beta_k} \otimes 1) \Delta(w) \in \bigoplus_{v \in B, w \in D_k \cap B_l} \mathbf{k} v \otimes w.$$

Also, for any $v \in B$ we have $x_{\beta_k} v \in D_k$, because if $v \in B_k$ then $x_{\beta_k} v \in W_{\beta_k}$ and if $v \in B_i$ for $i < k$ then we apply Lemma 3.19 again, and we conclude the proof. \square

4 Presentation by generators and relations of Nichols algebras of diagonal type

In this section we use the convex order of a PBW basis of hyperletters to prove that, when the diagonal braiding is symmetric, such PBW basis is orthogonal with respect to the bilinear form of Proposition 2.1. This fact gives a way to obtain relations which holds in the Nichols algebras, even when the braiding is not symmetric. We obtain then a presentation by generators and relations for any Nichols algebras of diagonal type whose root system is finite considering a suitable set of relations.

4.1 A general presentation

We continue with the setting fixed in Subsection 3.3. To begin with, we prove the orthogonality of the PBW basis with respect to the bilinear form in Proposition 2.1. This result extends [Ang, Prop. 5.1], and the proof is very similar; anyway we rewrite it in this general setting.

Proposition 4.1. *Consider a PBW basis of $\mathfrak{B}(V)$ as above given by Lyndon words, and assume that the braiding matrix is symmetric. Then the PBW basis is orthogonal with respect to the bilinear form in Proposition 2.1.*

Proof. We prove by induction on $k = \ell(u) + \ell(v)$ that $(u|v) = 0$, where $u \neq v$ are ordered products of PBW generators. If $k = 1$, then $u = 1, v = x_j$ or $u = x_i, v = 1$, for some $i, j \in \{1, \dots, \theta\}$, and $(1|x_j) = (x_i|1) = 0$.

Suppose the statement is valid when $\ell(u) + \ell(v) < k$, and let $u \neq v$ be hyperwords such that $\ell(u) + \ell(v) = k$. If both are hyperletters, they have different degrees $\alpha \neq \beta \in \mathbb{Z}^\theta$, so $u = x_\alpha, v = x_\beta$, and $(x_\alpha|x_\beta) = 0$, since the homogeneous components are orthogonal for $(\cdot|\cdot)$.

Suppose that $u = x_\alpha$ and $v = x_{\beta_k}^{h_k} x_{\beta_{k-1}}^{h_{k-1}} \dots x_{\beta_i}^{h_i}$, for some $1 \leq i \leq k \leq M$ (we consider $h_k, h_l \neq 0$). If they have different \mathbb{Z}^θ -degree, they are orthogonal. Then, we assume that $\alpha = \sum_{j=i}^k h_j \beta_j$, so $\beta_i < \alpha$ because the ordered root system is strongly convex by Theorem 3.16. Using Lemma 2.5 and (2.1),

$$\begin{aligned} (u|v) &= (x_\alpha|w)(1|x_{\beta_i}) + (1|w)(x_\alpha|x_{\beta_i}) \\ &\quad + \sum_{l_1 \geq \dots \geq l_p > l_\alpha, l_i \in L} (x_{l_1, \dots, l_p}|w)([l_1]_c \cdots [l_p]_c|x_{\beta_i}) \end{aligned}$$

where $v = wx_{\beta_i}$. Note that $(1|x_{\beta_i}) = (1|w) = 0$. Also, $[l_1]_c \cdots [l_p]_c$ is a linear combination of greater hyperwords of the same degree and an element of $I(V)$, so by inductive hypothesis and the fact that $I(V)$ is the radical of the bilinear form, we conclude $([l_1]_c \cdots [l_p]_c|x_{\beta_i}) = 0$. Therefore $(u|v) = 0$.

For the final case, we consider

$$u = x_{\beta_k}^{h_k} \dots x_{\beta_i}^{h_i}, \quad 1 \leq i \leq k \leq M, \quad v = x_{\beta_q}^{f_q} \dots x_{\beta_p}^{f_p}, \quad 1 \leq p \leq q \leq M.$$

The bilinear form is symmetric, so we can assume $i \leq p$. By Lemma 2.5 and (2.7),

$$\begin{aligned} (u|v) &= (w|1)(x_{\beta_i}|v) + \sum_{j=0}^{f_p} \binom{f_p}{j}_{q_{\beta_p}} (w|x_{\beta_q}^{f_q} \dots x_{\beta_{p-1}}^{f_{p-1}} x_{\beta_p}^j)(x_{\beta_i}|x_{\beta_p}^{f_p-j}) \\ &\quad + \sum_{\substack{l_1 \geq \dots \geq l_t > l_{\beta_p}, l_s \in L \\ 0 \leq j \leq f_p}} (w|x_{l_1, \dots, l_t}^{(j)})(x_{\beta_i}|[l_1]_c \cdots [l_t]_c x_{\beta_p}^j) \end{aligned}$$

where $u = wx_{\beta_i}$. Note that $(w|1) = 0$, and $[l_1]_c \dots [l_p]_c x_{\beta_p}^j$ is a combination of hyperwords of the PBW basis greater or equal than it and an element of $I(V)$. Using induction hypothesis and the fact that $I(V)$ is the radical of this bilinear form, we conclude that $(x_{\beta_i} | [l_1]_c \dots [l_p]_c x_{\beta_p}^j) = 0$. As also $x_{\beta_i}, x_{\beta_p}^{f_p-j}$ are different elements of the PBW basis for $f_p - j \neq 1$, we have that

$$(u|v) = (f_p)_{q_{\beta_p}}(w|x_{\beta_q}^{f_q} \dots x_{\beta_{p-1}}^{f_{p-1}} x_{\beta_p}^{f_p-1})(x_{\beta_i}|x_{\beta_p}). \quad (4.1)$$

Then it is zero if $i < p$, but also if $i = p$, because in that case $w \neq x_{\beta_q}^{f_q} \dots x_{\beta_{p-1}}^{f_{p-1}} x_{\beta_p}^{f_p-1}$ and we use induction hypothesis. \square

Corollary 4.2. *If $u = x_{\beta_M}^{n_M} \dots x_{\beta_1}^{n_1}$, where $0 \leq n_j < N_{\beta_j}$, then*

$$c_u := (u|u) = \prod_{j=1}^M n_j!_{q_{\beta_j}} c_{x_{\beta_j}^{n_j}} \neq 0. \quad (4.2)$$

Proof. We check the equality by induction on $ht(w)$. If $ht(w) = 1$, w is an hyperletter. If we assume it holds for $ht(w) < k$, and $ht(u) = k$, we use the orthogonality of the PBW basis and a calculation as (4.1) for $v = u$ to deduce (4.2) from the inductive hypothesis.

Such scalar is not zero because $u \notin I(V)$ and the PBW basis generates a k -linear complement to $I(V)$, the radical of this bilinear form. \square

Remark 4.3. Note that:

$$(x_{\beta_i} x_{\beta_j} | u) = (x_{\beta_i} | u_{(1)})(x_{\beta_j} | u_{(2)}) = d_{i,j} c_{x_{\beta_i}} c_{x_{\beta_j}},$$

where $d_{i,j}$ is the coefficient of $x_{\beta_i} \otimes x_{\beta_j}$ in the expression of $\Delta(u)$ as a linear combination of elements of the PBW basis in both sides of the tensor product.

We return to the general case where the braiding matrix is not necessarily symmetric. We obtain some relations and prove then the presentation of Nichols algebras by generators and relations. To obtain these relations is the key step to find the presentation in Theorem 4.9. Note that $B_i \cap C_j$ is the set of monotone hyperwords whose hyperletters are between x_{β_i} and x_{β_j} , see Theorem 3.18 and the definition of C_j in Subsection 3.3.

Let (W, d) be a braided vector space of diagonal type, $\hat{x}_1, \dots, \hat{x}_\theta$ a basis of W and $\hat{q}_{ij} \in k^\times$ such that $d(\hat{x}_i \otimes \hat{x}_j) = \hat{q}_{ij} \hat{x}_j \otimes \hat{x}_i$. Assume that $\hat{q}_{ij} = \hat{q}_{ji}$ for all $1 \leq i, j \leq \theta$, and that (V, c) and (W, d) are twist equivalent:

$$q_{ij} q_{ji} = \hat{q}_{ij} \hat{q}_{ji}, \quad q_{ii} = \hat{q}_{ii}, \quad 1 \leq i \neq j \leq \theta.$$

We define $\hat{x}_\beta = [l_\beta]_d$: that is, the corresponding hyperletter to l_β , but where we change the braiding c by d . By Corollary 3.17 and the invariance of the root system under twist

equivalence, the set of all the \hat{x}_β , $\beta \in \Delta_+^V = \Delta_+^W$, is a set of generators of a PBW basis as in Kharchenko's Theorem. If $u = x_{\beta_M}^{n_M} \cdots x_{\beta_1}^{n_1}$, then we denote $\hat{u} = \hat{x}_{\beta_M}^{n_M} \cdots \hat{x}_{\beta_1}^{n_1}$.

Let $\sigma : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \rightarrow \mathbb{k}^\times$ the bilinear form given by

$$\sigma(g_i, g_j) = \begin{cases} \hat{q}_{ij} q_{ij}^{-1}, & i \leq j \\ 1, & i > j \end{cases} \quad (4.3)$$

By [AS2, Prop. 3.9, Rem. 3.10] there exists a linear isomorphism $\Psi : \mathfrak{B}(W) \rightarrow \mathfrak{B}(V)$ such that $\Psi(\hat{x}_i) = x_i$ and for any $x \in \mathfrak{B}(W)_\alpha$, $y \in \mathfrak{B}(W)_\beta$, $\alpha, \beta \in \mathbb{N}_0^\theta$,

$$\Psi(xy) = \sigma(\alpha, \beta) \Psi(x) \Psi(y), \quad (4.4)$$

$$\Psi([x, y]_d) = \sigma(\alpha, \beta) [\Psi(x), \Psi(y)]_d. \quad (4.5)$$

Define $t_{\alpha_i} = 1$ for all $1 \leq i \leq \theta$, and inductively

$$t_\beta = \sigma(\beta_1, \beta_2) t_{\beta_1} t_{\beta_2}, \quad \text{Sh}(l_\beta) = (l_{\beta_1}, l_{\beta_2}).$$

Also for any $u = x_{\beta_M}^{n_M} \cdots x_{\beta_1}^{n_1}$ define

$$f(u) := \prod_{1 \leq i < j \leq M} \sigma(\beta_j, \beta_i)^{n_i n_j} \prod_{1 \leq i \leq M} \sigma(\beta_i, \beta_i)^{\binom{n_i}{2}} t_{\beta_i}^{n_i}. \quad (4.6)$$

Lemma 4.4. For any $u = x_{\beta_M}^{n_M} \cdots x_{\beta_1}^{n_1}$, $\Psi(\hat{u}) = f(u)u$.

Proof. We prove first by induction on $\ell(l_\beta)$, $\beta \in \Delta_+^V$, that $\Psi(\hat{x}_\beta) = t_\beta x_\beta$. It follows by definition when $\ell(l_\beta) = 1$, i.e. when $\beta = \alpha_i$ for some $1 \leq i \leq \theta$. Now assume it holds for $\ell(l_\gamma) < k$, and consider $\beta \in \Delta_+^V$ such that $\ell(l_\beta) = k$. Let $\text{Sh}(l_\beta) = (\beta_1, \beta_2)$. Then,

$$\begin{aligned} \Psi(\hat{x}_\beta) &= \Psi([\hat{x}_{\beta_1}, \hat{x}_{\beta_2}]_d) = \sigma(\beta_1, \beta_2) [\Psi(\hat{x}_{\beta_1}), \Psi(\hat{x}_{\beta_2})]_c \\ &= \sigma(\beta_1, \beta_2) t_{\beta_1} t_{\beta_2} [x_{\beta_1}, x_{\beta_2}]_c = t_\beta x_\beta, \end{aligned}$$

by (4.5) and inductive hypothesis.

Now we prove that $\Psi(\hat{u}) = f(u)u$ by induction on $ht(u)$. Note that if $ht(u) = 1$ it reduces to $\Psi(\hat{x}_\beta) = t_\beta x_\beta$. Assume now that it holds for $ht(v) < N$, and consider $u = x_{\beta_M}^{n_M} \cdots x_{\beta_k}^{n_k}$ such that $ht(u) = N$ and $n_k > 0$. Call $v = x_{\beta_M}^{n_M} \cdots x_{\beta_k}^{n_k-1}$. Then,

$$\begin{aligned} \Psi(\hat{u}) &= \sigma \left((n_k - 1)\beta_k + \sum_{i=k+1}^M n_i \beta_i, \beta_k \right) \Psi(\hat{v}) \Psi(\hat{x}_{\beta_k}) \\ &= \left(\prod_{i=k+1}^M \sigma(\beta_i, \beta_k)^{n_i} \right) \sigma(\beta_k, \beta_k)^{n_k-1} f(v) v t_{\beta_k} x_{\beta_k} = f(u)u, \end{aligned}$$

by (4.4) and inductive hypothesis. \square

We define for $1 \leq i < j \leq \theta$ and $u = x_{\beta_M}^{n_M} \cdots x_{\beta_1}^{n_1}$,

$$c_{i,j}^u := \frac{f(u) (\hat{x}_{\beta_i} \hat{x}_{\beta_j} | \hat{u})}{\sigma(\beta_i, \beta_j) t_{\beta_i} t_{\beta_j} c_{\hat{u}}}, \quad (4.7)$$

where $(\cdot | \cdot)$ denotes the bilinear form associated to (W, d) , and $c_{\hat{u}}$ is computed as in Corollary 4.2. Note that if (q_{ij}) is symmetric and we consider $q_{ij} = \hat{q}_{ij}$, then $\sigma(\alpha, \beta) = 1$ for all $\alpha, \beta \in \mathbb{Z}^\theta$ and then $f(u) = 1$ for any u . In consequence, $c_{i,j}^u = (x_{\beta_i} x_{\beta_j} | u) c_u^{-1}$.

We obtain a first set of relations for our presentation.

Lemma 4.5. *Let $1 \leq i < j \leq M$ be such that $l_{\beta_i} l_{\beta_j} \neq l_{\beta_k}$ for all k , and $\text{Sh}(l_{\beta_i} l_{\beta_j}) = (l_{\beta_i}, l_{\beta_j})$, and $c_{i,j}^u \in \mathfrak{k}$ as above. Then,*

$$[x_{\beta_i}, x_{\beta_j}]_c = \sum_{u \in B_i \cap C_j - \{x_{\beta_j} x_{\beta_i}\}: \deg u = \beta_i + \beta_j} c_{i,j}^u u. \quad (4.8)$$

Proof. Assume that the braiding is symmetric. As $l_{\beta_i} l_{\beta_j} \neq l_{\beta_k}$ for all k , and $\text{Sh}(l_{\beta_i} l_{\beta_j}) = (l_{\beta_i}, l_{\beta_j})$, $[l_{\beta_i} l_{\beta_j}]_c = [x_{\beta_i}, x_{\beta_j}]_c = x_{\beta_i} x_{\beta_j} - \chi(\beta_i, \beta_j) x_{\beta_j} x_{\beta_i}$ is a linear combination of greater monotone hyperwords by Corollary 2.10.

As $x_{\beta_i} x_{\beta_j} \in W_{\beta_i}$, it is a linear combination of elements in B_i by Theorem 3.18. Also, $\mathfrak{B}(V)$ is \mathbb{N}_0^θ -graded, so this linear combination is over elements of B_i of the degree $\beta_i + \beta_j$. Moreover, if $c_{i,j}^u \neq 0$ for $u = x_{\beta_k}^{n_k} \cdots x_{\beta_l}^{n_l}$, $l \leq k$, such that $n_k, n_l \neq 0$, then $x_{\beta_i} \otimes x_{\beta_j}$ appears in the expression of $\Delta(u)$ by Remark 4.3. Note that $x_{\beta_i} \otimes x_{\beta_j} \notin D_k \otimes (B_l - D_k)$, because $i < j$. By Lemma 3.20, we have $x_{\beta_j} \in B_k$, so $j \geq k$, and $u \in C_j$.

The explicit formula of the coefficients comes from Proposition 4.1.

If we want to compute $c_{i,j}^{x_{\beta_j} x_{\beta_i}}$, we have to calculate the coefficient of $x_{\beta_i} \otimes x_{\beta_j}$ in $\Delta(x_{\beta_j} x_{\beta_i})$, because of Remark 4.3 and the formula $c_{x_{\alpha_j} x_{\alpha_i}} = c_{x_{\alpha_i}} c_{x_{\alpha_j}}$. This coefficient is $\chi(\beta_j, \beta_i)$, but as the braiding matrix is symmetric, $\chi(\beta_j, \beta_i) = \chi(\beta_i, \beta_j)$. Therefore we conclude the proof when the matrix braiding is symmetric.

When the braiding is not symmetric, we use the linear isomorphism Ψ considered previously to reduce the computation to the symmetric case. Then,

$$\begin{aligned} 0 &= \Psi \left([x_{\beta_i}, x_{\beta_j}]_d - \sum (\hat{x}_{\beta_i} \hat{x}_{\beta_j} | \hat{u}) c_{\hat{u}}^{-1} \hat{u} \right) \\ &= \sigma(\beta_i, \beta_j) t_{\beta_i} t_{\beta_j} [x_{\beta_i}, x_{\beta_j}]_c - \sum (\hat{x}_{\beta_i} \hat{x}_{\beta_j} | \hat{u}) c_{\hat{u}}^{-1} f(u) u, \end{aligned}$$

by (4.5) and Lemma 4.4, so (4.8) holds in $\mathfrak{B}(V)$. \square

Corollary 4.6. *Assume that i, j are as in Lemma 4.5, and $\beta_i + \beta_j = \sum_{k=i}^j n_k \beta_k$, $n_k \in \mathbb{N}_0$ if and only if $n_i = n_j = 1$, $n_k = 0$ for $k \neq i, j$. Then,*

$$[x_{\beta_i}, x_{\beta_j}]_c = 0. \quad (4.9)$$

Proof. It follows from the previous proposition. \square

Now we extend [Ang, Cor. 5.2]. Recall that $N_\beta = \text{ord}(q_\beta) = h(x_\beta)$.

Lemma 4.7. *If $\beta \in \Delta_+^V$ and N_β is finite, then*

$$x_\beta^{N_\beta} = 0, \quad \text{in } \mathfrak{B}(V). \quad (4.10)$$

Proof. Assume first that (q_{ij}) is symmetric. Consider $w = \tilde{w}x_\beta^m$, where $\beta \in \Delta^+$ and \tilde{w} is a non-increasing product of hyperletters x_γ , $\gamma \in \Delta^+$, $\gamma > \beta$ or $\tilde{w} = 1$. If $\beta > \alpha$,

$$\begin{aligned} (x_\alpha^{N_\alpha} | w) &= (x_\alpha^{N_\alpha-1} | 1)(x_\alpha | w) + \sum_{i=0}^m \binom{m}{i}_{q_\beta} (x_\alpha^{N_\alpha-1} | \tilde{w}x_\beta^i)(x_\alpha | x_\beta^{m-i}) \\ &+ \sum_{l_1 \geq \dots \geq l_p > x_\beta, 0 \leq j \leq m} (x_\alpha^{N_\alpha-1} | x_{l_1, \dots, l_p}^{(j)})(x_\alpha | [l_1]_c \dots [l_p]_c x_\beta^j) = 0, \end{aligned}$$

where we use that $(x_\alpha^{N_\alpha-1} | 1) = (x_\alpha | x_\beta^{m-i}) = (x_\alpha | [l_1]_c \dots [l_p]_c x_\beta^j) = 0$ by the orthogonality of the PBW basis.

If $\beta \leq \alpha$, then

$$\begin{aligned} (x_\alpha^{N_\alpha} | w) &= (1 | \tilde{w}x_\beta^{m-1})(x_\alpha^{N_\alpha} | x_\beta) + \sum_{i=1}^{N_\alpha} \binom{N_\alpha}{i}_{q_\alpha} (x_\alpha^i | \tilde{w}x_\beta^{m-1})(x_\alpha^{N_\alpha-i} | x_\beta) \\ &+ \sum_{l_1 \geq \dots \geq l_p > x_\alpha, 0 \leq j \leq N_\alpha} (x_{l_1, \dots, l_p}^{(j)} | \tilde{w}x_\beta^{m-1})([l_1]_c \dots [l_p]_c x_\alpha^j | x_\beta) \end{aligned}$$

where we use that $q_\alpha \in \mathbb{G}_{N_\alpha}$, the orthogonality of the PBW basis and the fact that $N_\beta \notin \Delta^+$ if $N > 1$ (so $(x_\alpha^{N_\alpha} | x_\beta) = 0$).

Therefore $(x_\alpha^{N_\alpha} | v) = 0$ for all v in the PBW basis. Also $(I(V) | x_\alpha^{N_\alpha}) = 0$, because it is the radical of this bilinear form, so $(T(V) | x_\alpha^{N_\alpha}) = 0$, and then $x_\alpha^{N_\alpha} \in I(V)$. That is, we have $x_\alpha^{N_\alpha} = 0$ in $\mathfrak{B}(V)$.

For the general case, we recall that a diagonal braiding is twist equivalent to a braiding with a symmetric matrix, see [AS2, Theorem 4.5]. Also, there exists a linear isomorphism between the corresponding Nichols algebras. The corresponding x_α are related by a non-zero scalar, because they are an iteration of braided commutators between the hyperwords. \square

Before proving the main result of this section, we need another technical lemma.

Lemma 4.8. *Let \mathfrak{B} be a quotient of $T(V)$ such that relations (4.8) hold. Then for any $i < j$, $x_{\beta_i}x_{\beta_j}$ can be written as a linear combination of monotone hyperwords greater than x_{β_i} , whose hyperletters are x_{β_k} , $i \leq k \leq j$.*

Proof. It is similar to the proof of Theorem 2.4, see [R2, Thm. 10]. Set for each $n \geq 2$,

$$L_n := \{(x_{\beta_i}, x_{\beta_j}) : i < j, \ell(l_{\beta_i}) + \ell(l_{\beta_j}) = n\}.$$

We order L_k as follows: $(x_{\beta_i}, x_{\beta_j}) < (x_{\beta_k}, x_{\beta_m})$ if $l_{\beta_i}l_{\beta_j} < l_{\beta_k}l_{\beta_m}$, or $l_{\beta_i}l_{\beta_j} = l_{\beta_k}l_{\beta_m}$ and $l_{\beta_i} < l_{\beta_k}$.

We prove the statement by induction on $n = \ell(x_{\beta_i}) + \ell(x_{\beta_j})$, and then by induction on the previous order on L_n . When $n = 2$, then β_i, β_j are simple, and $[x_i, x_j]_c = x_{\alpha_i + \alpha_j}$ or $[x_i, x_j]_c = 0$ in \mathfrak{B} .

Fix then a pair $(x_{\beta_i}, x_{\beta_j}) \in L_n$ and assume that the statement holds for $(x_{\beta_k}, x_{\beta_m}) \in L_n$, $(x_{\beta_i}, x_{\beta_j}) > (x_{\beta_k}, x_{\beta_m})$, and for $(x_{\beta_k}, x_{\beta_m}) \in L_{n'}$, $n' < n$. If $\text{Sh}(l_{\beta_i}l_{\beta_j}) = (l_{\beta_i}, l_{\beta_j})$ then the assertion holds because

- if $l_{\beta_i}l_{\beta_j} = l_{\beta_k}$ for some k , necessarily (by the definition of the order) $i < k < j$ and $[x_{\beta_i}, x_{\beta_j}]_c = x_{\beta_k}$,
- if not, it holds because we assume (4.8).

If $\text{Sh}(l_{\beta_i}l_{\beta_j}) \neq (l_{\beta_i}, l_{\beta_j})$, let $\text{Sh}(l_{\beta_i}) = (l_{\beta_p}, l_{\beta_q})$, so $x_{\beta_i} = [x_{\beta_p}, x_{\beta_q}]_c$. Therefore $l_{\beta_q} < l_{\beta_j}$ (see Subsection 2.2). By (2.4),

$$[x_{\beta_i}, x_{\beta_j}]_c = [x_{\beta_p}, [x_{\beta_q}, x_{\beta_j}]_c]_c - \chi(\beta_p, \beta_q)x_{\beta_q}[x_{\beta_p}, x_{\beta_j}]_c + \chi(\beta_q, \beta_j)[x_{\beta_p}, x_{\beta_j}]_cx_{\beta_q}.$$

We apply induction hypothesis and express $[x_{\beta_q}, x_{\beta_j}]_c$ as a linear combination of monotone hyperwords whose hyperletters are between x_{β_q} and x_{β_j} . By (2.5) and inductive hypothesis, we express $[x_{\beta_p}, [x_{\beta_q}, x_{\beta_j}]_c]_c$ as a linear combination of monotone hyperwords whose letters are between x_{β_i} and x_{β_j} . It is important here the order in L_n , because in such linear combination can appear a single hyperletter x_{β_k} , which by hypothesis is between x_{β_q} and x_{β_j} , and so $(l_{\beta_i}, l_{\beta_j}) > (l_{\beta_p}, l_{\beta_k})$.

We use also inductive hypothesis to express $[x_{\beta_p}, x_{\beta_j}]_c$ as a linear combination of hyperwords whose hyperletters are between x_{β_p} and x_{β_j} . As in the previous step we can reorder the hyperletters in order to find the desired expression by inductive hypothesis. \square

Now we are ready to prove the main result of this work.

Theorem 4.9. *Let (V, c) be a finite-dimensional braided vector space of diagonal type such that Δ_+^V is finite. Let x_1, \dots, x_θ be a basis of V such that $c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$, where $(q_{ij}) \in (\mathbf{k}^\times)^{\theta \times \theta}$ is the braiding matrix, and let $\{x_{\beta_k}\}_{\beta_k \in \Delta_+^V}$ be the associated set of hyperletters.*

Then $\mathfrak{B}(V)$ is presented by generators x_1, \dots, x_θ , and relations

$$x_\beta^{N_\beta} = 0, \quad \beta \in \Delta_+^V, \text{ord}(q_\beta) = N_\beta < \infty, \quad (4.11)$$

$$[x_{\beta_i}, x_{\beta_j}]_c = \sum_{u \in B_i \cap C_j - \{x_{\beta_j}, x_{\beta_i}\}: \deg u = \beta_i + \beta_j} c_{i,j}^u u, \quad (4.12)$$

$$1 \leq i < j \leq M, \text{Sh}(l_{\beta_i}l_{\beta_j}) = (l_{\beta_i}, l_{\beta_j}), l_{\beta_i}l_{\beta_j} \neq l_{\beta_k}, \forall k,$$

where $c_{i,j}^u$ are as in (4.7). Moreover, $\{x_{\beta_M}^{n_M} \cdots x_{\beta_1}^{n_1} : 0 \leq n_j < N_{\beta_j}\}$ is a basis of $\mathfrak{B}(V)$.

Proof. The statement about the basis follows by Kharchenko's theory on PBW bases (see Subsection 2.2) and the definition of Δ_+^V (see Subsection 3.1), where the hyperletters x_{β_k} are univocally determined by Corollary 3.17.

Let $\mathfrak{B} := T(V)/I$, where I is the ideal of $T(V)$ generated by (4.11), (4.12): by Lemmata 4.5 and 4.7, $I \subseteq I(V)$, so the projection $\pi : T(V) \rightarrow \mathfrak{B}(V)$ induces canonically a projection $\pi' : \mathfrak{B} \rightarrow \mathfrak{B}(V)$. Let W be the subspace of \mathfrak{B} spanned by B , where B is the PBW basis of $\mathfrak{B}(V)$; $1 \in W$. For each pair $1 \leq i \leq j \leq M$, we set $W_{i,j}$ the subspace of W spanned by $B_i \cap C_j$.

We assert that

$$x_{\beta_k} W_{i,j} \subset W_{\min\{i,k\}, \max\{j,k\}}. \quad (4.13)$$

We shall prove it by induction on k . When $k = M$, fix $i \leq j$. For each $w \in B_i \cap C_j$, we have that $x_{\beta_M} w \in B_i \cap C_M = B_i$, or $x_{\beta_M} w = 0$ if $j = M$, $N_M < \infty$ and w begins with $x_{\beta_M}^{N_M-1}$, so $x_{\beta_M} W_{i,j} \subset W_{i,M}$.

Now assume that (4.13) holds for all $l > k$ and all $i \leq j$. We argue by induction on j . If $i \leq j \leq k$, for each $w \in B_i \cap C_j$, we have that $x_{\beta_k} w \in B_i \cap C_k$ or $x_{\beta_k} w = 0$ as in the initial step, so $x_{\beta_k} W_{i,j} \subset W_{i,k}$. Now assume $j > k$, and consider $w \in B_i \cap C_j$; it is enough to prove that $x_{\beta_k} w \in W_{\min\{i,k\}, j}$. Moreover, we can assume $w = x_{\beta_j} w'$ for some monotone hyperword w' in $W_{i,j}$ (if w begins with another hyperletter x_{β_l} , $l < j$, we consider $w \in W_{i,l} \subset W_{i,j}$). By Lemma 4.8, we can write $x_{\beta_k} x_{\beta_j}$ as a linear combination of monotone hyperwords whose hyperletters belong to $B_k \cap C_j$. Therefore the result follows by the inductive hypothesis: any of these hyperwords has no letters x_{β_k} 's and we use the first inductive hypothesis (it holds for all $l > k$), or it ends with hyperletters x_{β_k} 's and we write $x_{\beta_k} w'$ as a linear combination of hyperwords in $B_{\min\{i,k\}} \cap C_j$ by the second inductive hypothesis.

In this way we prove that W is a left ideal which contains 1, so $W = \mathfrak{B}$. But then the projection π' is an isomorphism, and $\mathfrak{B} = \mathfrak{B}(V)$. \square

Remark 4.10. Recall that we have defined for $i, j \in \{1, \dots, \theta\}$,

$$m_{ij} := \max\{m : (\text{ad}_c x_i)^m x_j \neq 0\},$$

see (3.6), and then $m\alpha_i + \alpha_j \in \Delta_+^V$ iff $0 \leq m \leq m_{ij}$. Moreover assume $i < j$. Then $x_{m\alpha_i + \alpha_j} = (\text{ad}_c x_i)^m x_j$, and a pair as in Corollary 4.6 is $(x_i, x_i^{m_{ij}} x_j)$, so such corollary implies the well-known quantum Serre relation in $\mathfrak{B}(V)$: $(\text{ad}_c x_i)^{m_{ij}+1} x_j = 0$. If $i > j$, then the pair changes to $(x_j x_i^{m_{ij}}, x_i)$, but then $0 = [x_{m\alpha_i + \alpha_j}, x_i]_c = a(\text{ad}_c x_i)^{m_{ij}+1} x_j$ for some $a \in k^\times$. In any case we have $(\text{ad}_c x_i)^{m_{ij}+1} x_j = 0$.

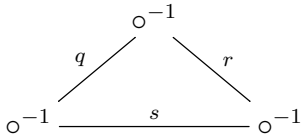
This shows that the set of relations (4.8), (4.10) is not minimal: if $\text{ord} q_{ii} = m_{ij} + 1$, then $x_i^{m_{ij}+1}$ is one of the relations (4.10), and then $(\text{ad}_c x_i)^{m_{ij}+1} x_j$ belongs to the ideal generated by $x_i^{m_{ij}+1}$.

5 Explicit presentations by generators and relations of some Nichols algebras of diagonal type

We shall apply the previous theory about how to obtain a PBW basis (Corollary 3.17) and a presentation of the corresponding Nichols algebra (Theorem 4.9) in some concrete examples.

5.1 Examples when $\dim V = 3$

We consider the Weyl equivalence classes 9, 10, 11 in [H2, Table 2]. We fix the following notation: let $q, r, s \in k^\times$ be such that $qrs = 1$. Set $M, N, P \in \mathbb{N}$ as the orders of these scalars, if they are finite. Such Weyl equivalence class includes the following generalized Dynkin diagrams:

- $\circ^q \xrightarrow{q^{-1}} \circ^{-1} \xrightarrow{r^{-1}} \circ^r$,
- $\circ^q \xrightarrow{q^{-1}} \circ^{-1} \xrightarrow{s^{-1}} \circ^s$,
- $\circ^r \xrightarrow{r^{-1}} \circ^{-1} \xrightarrow{s^{-1}} \circ^s$,
- 

Notice that 10, 11 are particular cases of 9 when $q = r$, $q = r = s \in \mathbb{G}_3$, respectively. Also the second and the third diagrams are analogous to the first one, so it is enough to obtain the presentation for the first and the last braidings.

If $i < j$, $l_{\alpha_i + \alpha_j} = x_i x_j$, so $x_{\alpha_i + \alpha_j} = [x_i, x_j]_c = (\text{ad}_c x_i)x_j$. Also,

$$l_{\alpha_1 + \alpha_2 + \alpha_3} = \begin{cases} x_1 x_2 x_3 & \text{if } (\text{ad}_c x_1)x_3 = 0 : x_{\alpha_1 + \alpha_2 + \alpha_3} = [x_1, x_{\alpha_2 + \alpha_3}]_c; \\ x_1 x_3 x_2 & \text{if } (\text{ad}_c x_1)x_3 \neq 0 : x_{\alpha_1 + \alpha_2 + \alpha_3} = [x_{\alpha_1 + \alpha_3}, x_2]_c; \end{cases}$$

When $(\text{ad}_c x_1)x_3 = 0$, we have also

$$l_{\alpha_1 + 2\alpha_2 + \alpha_3} = x_1 x_2 x_3 x_2 : x_{\alpha_1 + 2\alpha_2 + \alpha_3} = [x_{\alpha_1 + \alpha_2 + \alpha_3}, x_2]_c$$

Proposition 5.1. *Let (V, c) be a braided vector space such that $\dim V = 3$, and the corresponding generalized Dynkin diagram is*

$$\circ^q \xrightarrow{q^{-1}} \circ^{-1} \xrightarrow{r^{-1}} \circ^r .$$

Then $\mathfrak{B}(V)$ is presented by generators x_1, x_2, x_3 , and relations

$$x_1^M = x_2^2 = x_3^N = x_{\alpha_1+2\alpha_2+\alpha_3}^P = 0, \quad (5.1)$$

$$(\text{ad}_c x_1)^2 x_2 = (\text{ad}_c x_3)^2 x_2 = (\text{ad}_c x_1) x_3 = 0, \quad (5.2)$$

$$[x_{\alpha_1+\alpha_2}, x_{\alpha_1+\alpha_2+\alpha_3}]_c = [x_{\alpha_1+\alpha_2+\alpha_3}, x_{\alpha_2+\alpha_3}]_c = 0. \quad (5.3)$$

Moreover, $\mathfrak{B}(V)$ has a PBW basis as follows:

$$\{x_3^{n_3} x_{\alpha_2+\alpha_3}^{n_{23}} x_2^{n_2} x_{\alpha_1+2\alpha_2+\alpha_3}^{n_{1232}} x_{\alpha_1+\alpha_2+\alpha_3}^{n_{123}} x_{\alpha_1+\alpha_2}^{n_{12}} x_1^{n_1} : \\ 0 \leq n_1 < M, 0 \leq n_2 < N, 0 \leq n_{1232} < P, n_{12}, n_{123}, n_2, n_{23} \in \{0, 1\}\}.$$

If $M, N, P < \infty$, then $\dim \mathfrak{B}(V) = 16MNP$.

Proof. For this case,

$$\Delta_+^V = \{\alpha_3, \alpha_2 + \alpha_3, \alpha_2, \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_1\}.$$

Therefore we obtain $l_\beta, \beta \in \Delta_+^V$, easily from Corollary 3.17.

By Remark 4.10, we consider the relations

$$(\text{ad}_c x_1)^2 x_2 = (\text{ad}_c x_3)^2 x_2 = (\text{ad}_c x_1) x_3 = 0,$$

because $(\text{ad}_c x_2)^2 x_1, (\text{ad}_c x_2)^2 x_3$ follows from $x_2^2 = 0$.

We have the following decompositions:

$$\text{Sh}(l_{\alpha_1+\alpha_2} l_{\alpha_1+\alpha_2+\alpha_3}) = (l_{\alpha_1+\alpha_2}, l_{\alpha_1+\alpha_2+\alpha_3}), \\ \text{Sh}(l_{\alpha_1+\alpha_2+\alpha_3} l_{\alpha_2+\alpha_3}) = (l_{\alpha_1+\alpha_2+\alpha_3}, l_{\alpha_2+\alpha_3}).$$

Relations (5.3) then follows by Corollary 4.6.

Also $\text{Sh}(l_{\alpha_1} l_{\alpha_1+\alpha_2+\alpha_3}) = (l_{\alpha_1}, l_{\alpha_1+\alpha_2+\alpha_3})$, so

$$[x_1, x_{\alpha_1+\alpha_2+\alpha_3}]_c = 0.$$

Note that $x_{\alpha_1+\alpha_2+\alpha_3} = [x_{\alpha_1+\alpha_2}, x_3]_c$ by $(\text{ad}_c x_1) x_3 = 0$ and the identity (2.4). Therefore, this relation is redundant because of (2.4), $x_1^2 = 0$. The same holds for relation $[x_{\alpha_1+\alpha_2+\alpha_3}, x_3]_c = 0$, coming from the decomposition $\text{Sh}(l_{\alpha_1+\alpha_2+\alpha_3} l_{\alpha_3}) = (l_{\alpha_1+\alpha_2+\alpha_3}, l_{\alpha_3})$.

Also $\text{Sh}(l_{\alpha_1} l_{\alpha_1+2\alpha_2+\alpha_3}) = (l_{\alpha_1}, l_{\alpha_1+2\alpha_2+\alpha_3})$, so by Lemma 4.5 there exists $a \in \mathfrak{k}$ such that:

$$[x_1, x_{\alpha_1+2\alpha_2+\alpha_3}]_c = a x_{\alpha_1+\alpha_2+\alpha_3} x_{\alpha_1+\alpha_2}.$$

This relation is also redundant:

$$[x_1, x_{\alpha_1+2\alpha_2+\alpha_3}]_c = [[x_1, x_{\alpha_1+\alpha_2+\alpha_3}]_c, x_2]_c + q_{11} q_{12} q_{13} x_{\alpha_1+\alpha_2+\alpha_3} x_{\alpha_1+\alpha_2} \\ - q_{12} q_{22} q_{32} x_{\alpha_1+\alpha_2} x_{\alpha_1+\alpha_2+\alpha_3} \\ = q_{11} q_{12} q_{13} (1 - s) x_{\alpha_1+\alpha_2+\alpha_3} x_{\alpha_1+\alpha_2}.$$

where we use (2.4) and the previous relations.

We have finally

$$\begin{aligned} \text{Sh}(l_{\alpha_1+\alpha_2}l_{\alpha_1+2\alpha_2+\alpha_3}) &= (l_{\alpha_1+\alpha_2}, l_{\alpha_1+2\alpha_2+\alpha_3}), \\ \text{Sh}(l_{\alpha_1+\alpha_2+\alpha_3}l_{\alpha_1+2\alpha_2+\alpha_3}) &= (l_{\alpha_1+\alpha_2+\alpha_3}, l_{\alpha_1+2\alpha_2+\alpha_3}), \end{aligned}$$

which give place to the following relations:

$$[x_{\alpha_1+\alpha_2}, x_{\alpha_1+2\alpha_2+\alpha_3}]_c = [x_{\alpha_1+\alpha_2+\alpha_3}, x_{\alpha_1+2\alpha_2+\alpha_3}]_c = 0$$

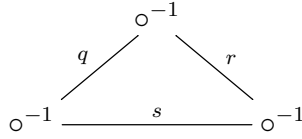
These relations also follow by the previous ones using (2.4).

We can prove in the same way that

$$x_{\alpha_1+\alpha_2}^2, x_{\alpha_2+\alpha_3}^2, x_{\alpha_1+\alpha_2+\alpha_3}^2$$

are redundant relations too. The proposition follows then by Theorem 4.9, where we omit some redundant relations. \square

Proposition 5.2. *Let (V, c) be a braided vector space such that $\dim V = 3$, and the corresponding generalized Dynkin diagram is*



Then $\mathfrak{B}(V)$ is presented by generators x_1, x_2, x_3 , and relations

$$x_1^2 = x_2^2 = x_3^2 = x_{\alpha_1+\alpha_2+\alpha_3}^2 = 0, \tag{5.4}$$

$$x_{\alpha_1+\alpha_2}^M = x_{\alpha_2+\alpha_3}^N = x_{\alpha_1+\alpha_3}^P = 0, \tag{5.5}$$

$$[x_{\alpha_i+\alpha_j}, x_{\alpha_i+\alpha_k}]_c = 0, \quad \{i, j, k\} = \{1, 2, 3\}. \tag{5.6}$$

$$[x_1, x_{\alpha_2+\alpha_3}]_c = \frac{1-s}{q_{23}(1-r)}x_{\alpha_1+\alpha_2+\alpha_3} + q_{12}(1-s)x_2x_{\alpha_1+\alpha_3}. \tag{5.7}$$

Moreover, $\mathfrak{B}(V)$ has a PBW basis as follows:

$$\begin{aligned} &\{x_3^{n_3} x_{\alpha_2+\alpha_3}^{n_{23}} x_2^{n_2} x_{\alpha_1+\alpha_3}^{n_{13}} x_{\alpha_1+\alpha_2+\alpha_3}^{n_{123}} x_{\alpha_1+\alpha_2}^{n_{12}} x_1^{n_1} : \\ &0 \leq n_{12} < M, 0 \leq n_{23} < N, 0 \leq n_{13} < P, n_1, n_{123}, n_2, n_3 \in \{0, 1\}\}. \end{aligned}$$

If $M, N, P < \infty$, then $\dim \mathfrak{B}(V) = 16MNP$.

Proof. Again we obtain $l_\beta, \beta \in \Delta_+^V$, easily from Corollary 3.17, because

$$\Delta_+^V = \{\alpha_3, \alpha_2 + \alpha_3, \alpha_2, \alpha_1 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_1\}.$$

By Remark 4.10, all the quantum Serre relations $(\text{ad}_c x_i)^2 x_j = 0$, $i \neq j$, follow from $x_i^2 = 0$, $i = 1, 2, 3$.

We have the decompositions:

$$\begin{aligned} \text{Sh}(l_{\alpha_1+\alpha_2} l_{\alpha_1+\alpha_3}) &= (l_{\alpha_1+\alpha_2}, l_{\alpha_1+\alpha_3}), \\ \text{Sh}(l_{\alpha_1+\alpha_2} l_{\alpha_2+\alpha_3}) &= (l_{\alpha_1+\alpha_2}, l_{\alpha_2+\alpha_3}), \\ \text{Sh}(l_{\alpha_1+\alpha_3} l_{\alpha_2+\alpha_3}) &= (l_{\alpha_1+\alpha_3}, l_{\alpha_2+\alpha_3}), \end{aligned}$$

which give place to relations (5.6) by Corollary 4.6.

The decomposition $\text{Sh}(l_{\alpha_1} l_{\alpha_2+\alpha_3}) = (l_{\alpha_1}, l_{\alpha_2+\alpha_3})$ tell us that $[x_1, x_{\alpha_2+\alpha_3}]_c$ is a linear combination of $x_{\alpha_1+\alpha_2+\alpha_3}$ and $x_2 x_{\alpha_1+\alpha_3}$ by Lemma 4.5, and we calculate the corresponding coefficients using Lemma 4.4.

Also $\text{Sh}(l_{\alpha_1} l_{\alpha_1+\alpha_2+\alpha_3}) = (l_{\alpha_1}, l_{\alpha_1+\alpha_2+\alpha_3})$, so

$$[x_1, x_{\alpha_1+\alpha_2+\alpha_3}]_c = 0.$$

This relation is again redundant because of (2.4), $x_1^2 = 0$ and the first relation in (5.6). The same holds for the relation $[x_{\alpha_1+\alpha_2+\alpha_3}, x_2]_c = 0$, coming from the decomposition $\text{Sh}(l_{\alpha_1+\alpha_2+\alpha_3} l_{\alpha_2}) = (l_{\alpha_1+\alpha_2+\alpha_3}, l_{\alpha_2})$.

Also $\text{Sh}(l_{\alpha_1+\alpha_2} l_{\alpha_1+\alpha_2+\alpha_3}) = (l_{\alpha_1+\alpha_2}, l_{\alpha_1+\alpha_2+\alpha_3})$, so

$$[x_{\alpha_1+\alpha_2}, x_{\alpha_1+\alpha_2+\alpha_3}]_c = 0.$$

This relation is also redundant by the previous relations and (2.4). In the same way, $[x_{\alpha_1+\alpha_2+\alpha_3}, x_{\alpha_1+\alpha_3}]_c = [x_{\alpha_1+\alpha_2+\alpha_3}, x_{\alpha_2+\alpha_3}]_c = 0$ are redundant. The proposition follows by Theorem 4.9. \square

Remark 5.3. We can prove that if (V, c) is a braided vector space as in Proposition 5.1 or Proposition 5.2, and $R = \bigoplus_{n \geq 0} R_n$ is a finite-dimensional graded braided Hopf algebra such that $R_0 = \mathbf{k}1$ and $R_1 \cong V$ as braided vector spaces, then R is generated by R_1 as an algebra. The proof is exactly as in [AnGa, Thm. 2.7], using the corresponding presentation by generators and relations.

Remark 5.4. When the braiding is of standard type, we obtain the presentation by generators and relations given in [Ang, Section 5]. In fact, Corollary 3.17 gives the set of Lyndon words obtained in [Ang, Section 4B]. Then we obtain a set of relations as in Theorem 4.9, where the set of relations (4.12) includes the ones [Ang, Theorems 5.14, 5.19, 5.22, 5.25] which are not root vectors powers. Then we can reduce this set of relations because of (2.4) as in such paper, in order to obtain a minimal set of relations.

Acknowledgements

I want to thank my advisor Nicolás Andruskiewitsch for his excellent guidance, his suggestions and his reading of this work.

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