DEFORMATION BY COCYCLES OF POINTED HOPF ALGEBRAS OVER NON-ABELIAN GROUPS

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Abstract. We introduce a method to construct multiplicative 2-cocycles for bosonizations of Nichols algebras over Hopf algebras with bijective antipode. These cocycles arise as liftings of invariant ε-biderivations defined on the Nichols algebras. Using this construction, we show that all known finite dimensional pointed Hopf algebras over the dihedral groups $\mathbb{D}_m$ with $m = 4t \geq 12$, over the symmetric group $\mathbb{S}_3$ and some families over $\mathbb{S}_4$ are cocycle deformations of bosonizations of Nichols algebras, by constructing explicitly the 2-cocycles.

Introduction

Let $\Bbbk$ be an algebraically closed field of characteristic zero. A Hopf algebra A is said to be pointed if all simple subcoalgebras are one dimensional; in particular, its coradical A_0 coincides with the group algebra $\Bbbk G(A)$ over the group of group-like elements. The best method for classifying finite dimensional pointed Hopf algebras over $\Bbbk$ was developed by Andruskiewitsch and Schneider, see [AS]. Shortly, the method consists on finding first all braided vector spaces V in ${}^{A_0}_{A_0}\mathcal{YD}$ such that its Nichols algebra $\mathfrak{B}(V)$ is finite dimensional, then find all pointed Hopf algebras H such that the graded algebra $\operatorname{gr} H$ induced by the coradical filtration is isomorphic to the bosonization $\mathfrak{B}(V)\#A_0$, and finally prove the generation in degree one.

Using this method, they were able to classify in [AS2] all finite dimensional pointed Hopf algebras A such that $G(A)$ is abelian and whose order is relative prime to 210. When the group of group-likes is not abelian, the problem is far from being completed. Some hope is present in the lack of examples: in this situation, Nichols algebras tend to be infinite dimensional, see for example [AFGV]. Nevertheless, examples on which the Nichols algebras are finite dimensional do exist. Over the symmetric groups $\mathbb{S}_3$ and $\mathbb{S}_4$ these algebras were determined in [AHS]. All of them arise from racks associated to a cocycle, and in *loc. cit.* and [GG] the classification of all finite dimensional pointed Hopf algebras over $\mathbb{S}_3$ and $\mathbb{S}_4$ is completed, respectively. Over the dihedral groups $\mathbb{D}_m$ with $m = 4t \geq 12$, the classification of finite dimensional Nichols algebras and finite dimensional pointed Hopf algebras was done in [FG]. In this case, it turns out that all Nichols algebras are isomorphic to exterior algebras.

Among others, useful tools for constructing new Hopf algebras are the deformation of the multiplication using multiplicative 2-cocycles and its dual notion of deforming the coproduct using twists. With this in mind, it is interesting to ask when two new non-isomorphic Hopf algebras are cocycle deformations of each other. It was proved that the family of finite dimensional pointed Hopf algebras over abelian groups Γ appearing in [AS2] such that the braided vector space V is a quantum linear space, can be constructed by deforming the multiplication in $\mathfrak{B}(V)\#\Bbbk\Gamma$, see [Mk], [GM]. Also, García Iglesias and Mombelli [GIM] proved using module category theory the same result for all known finite dimensional pointed Hopf algebras over the symmetric groups.

In these notes, we prove in Theorem 3.6 and Theorem 3.11 that all finite dimensional pointed Hopf algebras over the dihedral groups $\mathbb{D}_m$ with $m = 4t \geq 12$ are cocycle deformations of

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bosonizations of finite dimensional Nichols algebras, by giving explicitly the 2-cocycles. Moreover, using these techniques, we construct the cocycles that give the deformation of the bosonizations of Nichols algebras over $\mathbb{S}_3$ and some families over $\mathbb{S}_4$, see Theorem 4.10.

To construct the 2-cocycles we generalized to non-abelian groups the theory developed in [GM]. Specifically, we first introduce the notion of ε-biderivations in braided graded algebras and then apply them in the case of Nichols algebras $\mathfrak{B}(V)$ over Hopf algebras H with bijective antipode. In the case an ε-biderivation η is H-invariant, one can lift it to an H-invariant Hochschild 2-cocycle $\tilde{\eta}$ on $\mathfrak{B}(V)\#H$. Applying [GM, Lemma 4.1], it turns out that the exponentiation $\sigma = e^{\tilde{\eta}}$ of this cocycle is indeed a multiplicative 2-cocycle, provided two equations in the convolution algebra hold. A consequence of using H-invariant ε-biderivations is that σ coincides with η on the elements of V, simplifying in this way the computation of the deformation, see Lemma 2.3.

Before we apply the construction for the cases of the dihedral and symmetric groups, we first prove in Lemma 1.1 that these equations are equivalent on bosonizations of Nichols algebras $\mathfrak{B}(V)\#H$ and then we show in Lemmas 2.5 and 2.6 that they are always satisfied whenever H is semisimple and the braiding of V is symmetric. As a consequence, the exponentiation of the $\mathbb{D}_m$-invariant ε-biderivations on the Nichols algebras over the dihedral groups are multiplicative 2-cocycles and these are the ones we use to prove our first main theorems, Theorem 3.6 and Theorem 3.11.

The paper is organized as follows: in Section 1 we fix the notation, make some definitions and recall some facts that are used along the paper. In Section 2 we discuss deformations of graded algebras, define ε-biderivations and prove the preliminary results on invariant ε-biderivations and lifting of them. In Section 3 we recall the classification of finite dimensional pointed Hopf algebras over the dihedral groups $\mathbb{D}_m$ with $m = 4t \geq 12$ and prove that they are cocycle deformation of bosonizations. Finally in Section 4, we first recall the notion of racks and the classification of finite dimensional pointed Hopf algebras over $\mathbb{S}_3$ and $\mathbb{S}_4$, and then prove our last main result, Theorem 4.10 about cocycle deformations of bosonizations of finite dimensional Nichols algebras over the symmetric groups.

1. Preliminaries

1.1. **Conventions.** We work over an algebraically closed field $\Bbbk$ of characteristic zero. Good references for Hopf algebra theory are [M] and [Sw1].

If H is a Hopf algebra over $\Bbbk$ then Δ, ε and $\mathcal{S}$ denote respectively the comultiplication, the counit and the antipode. Comultiplication and coactions are written using the Sweedler-Heynemann notation with summation sign suppressed, *e.g.*, $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for $h \in H$.

The *coradical* H_0 of H is the sum of all simple sub-coalgebras of H. In particular, if $G(H)$ denotes the group of *group-like elements* of H, we have $\Bbbk G(H) \subseteq H_0$. A Hopf algebra is *pointed* if $H_0 = \Bbbk G(H)$.

Let $n, m \in \mathbb{N}$. We denote by $\mathbb{S}_n$ the symmetric group on n letters and by $\mathbb{D}_m$ the dihedral group of order $2m$. The later can be presented by generators and relations as follows

$$\mathbb{D}_m := \langle g, h |\ g^2 = 1 = h^m\ ,\ gh = h^{-1}g\rangle. \tag{1}$$

Throughout the paper derivations and derivation-like maps play a very important role. Recall that if A is an algebra and M is an A-bimodule, then a map $\delta\colon A \to M$ is said to be a derivation if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in A$. In our setting, (A, ε) is an augmented algebra acting on $M = \Bbbk$ via ε. To emphasize this distinction, we will refer to such derivations as ε-derivations.

1.2. **Yetter-Drinfeld modules and Nichols algebras.** Let H be a Hopf algebra. A *left Yetter-Drinfeld module* M over H is a left H-module $(M, \cdot)$ and a left H-comodule (M, δ) satisfying the compatibility condition $\delta(h \cdot m) = h_{(1)} m_{(-1)} \mathcal{S}(h_{(3)}) \otimes h_{(2)} \cdot m_{(0)}$ for all $m \in M, h \in H$. We denote by ${}^H_H\mathcal{YD}$ the category of left Yetter-Drinfeld modules over H. It is a braided monoidal category. If $H = \Bbbk\Gamma$ with Γ a finite group, we denote this category simply by ${}^\Gamma_\Gamma\mathcal{YD}$.

It is a semisimple category and the irreducible Yetter-Drinfeld modules are parameterized by pairs $(\mathcal{O},\rho)$, where $\mathcal{O}$ is a conjugacy class and (ρ,V) is a simple representation of the centralizer $C_\Gamma(\sigma)$ of a fixed point $\sigma\in\mathcal{O}$. We denote the corresponding Yetter-Drinfeld module by $M(\mathcal{O},\rho)$.

The notion of Nichols algebras appeared first in the work of Nichols [N] and then it was later rediscovered by several authors. We follow [AS], see also [N, AG, H]. Let (V,c) be a braided vector space. Since c satisfies the braid equation, it induces a representation of the braid group $\mathbb{B}_n$ for each $n\geq 2$. Consider the morphisms

$$Q_n=\sum_{\sigma\in\mathbb{S}_n}(M(\sigma))\in\operatorname{End}(V^{\otimes n}),$$

where $M:\mathbb{S}_n\to\mathbb{B}_n$ is the Matsumoto section corresponding to the canonical projection $\mathbb{B}_n\twoheadrightarrow\mathbb{S}_n$. Then the Nichols algebra $\mathfrak{B}(V)$ is the quotient of the tensor algebra $T(V)$ by the two-sided ideal $\mathcal{J}=\bigoplus_{n\geq2}\operatorname{Ker}Q_n$.

Let Γ be a finite group. We denote by $\mathfrak{B}(\mathcal{O},\rho)$ the Nichols algebra associated to the irreducible Yetter-Drinfeld module $M(\mathcal{O},\rho)\in{}^{\Gamma}_{\Gamma}\mathcal{YD}$.

1.3. **Deforming cocycles.** In this subsection we follow [GM]. Let A be a Hopf algebra and B an algebra with a twisted action of A, *i.e.* there is a $\Bbbk$-linear map $\cdot\;:A\otimes B\to B$ such that $a\cdot 1=\varepsilon(a)1$ and $a\cdot(xy)=(a_{(1)}\cdot x)(a_{(2)}\cdot y)$ for all $a\in A,\ x,y\in B$. Recall that a convolution invertible linear map σ in $\operatorname{Hom}_\Bbbk(A\otimes A,B)$ is a *normalized multiplicative 2-cocycle* if

$$a_{(1)}\cdot\sigma(b_{(1)},c_{(1)})\sigma(a_{(2)},b_{(2)}c_{(2)})=\sigma(a_{(1)},b_{(1)})\sigma(a_{(2)}b_{(2)},c)$$

and $\sigma(a,1)=\varepsilon(a)=\sigma(1,a)$ for all $a,b,c\in A$, see [M, Sec. 7.1]. In particular, if $B=\Bbbk$ with the trivial action of A, a normalized multiplicative 2-cocycle $\sigma:A\otimes A\to\Bbbk$ is a convolution invertible linear map such that

$$(\varepsilon\otimes\sigma)*\sigma(\mathrm{id}_A\otimes m)=(\sigma\otimes\varepsilon)*\sigma(m\otimes\mathrm{id}_A)\tag{2}$$

and $\sigma(\mathrm{id}_A,1)=\varepsilon=\sigma(1,\mathrm{id}_A)$. The deformed multiplication $m_\sigma=\sigma*m*\sigma^{-1}:A\otimes A\to A$ and antipode $\mathcal{S}_\sigma=\sigma*\mathcal{S}*\sigma^{-1}:A\to A$ on A, together with the original unit, counit and comultiplication define a new Hopf algebra structure on A which we denote by A_σ.

1.3.1. *Deforming cocycles for graded Hopf algebras.* If $A=\oplus_{n\geq0}A_n$ is a graded bialgebra and $f\colon A\to\Bbbk$ is a linear map such that $f|_{A_0}=0$, then

$$e^f=\sum_{i=0}^{\infty}\frac{f^{*i}}{i!}\colon A\to\Bbbk$$

is a well-defined convolution invertible map with convolution inverse e^{-f}. When $f\colon A\otimes A\to\Bbbk$ is a Hochschild 2-cocycle on A, that is $\varepsilon(a)f(b,c)+f(a,bc)=f(a,b)\varepsilon(c)+f(ab,c)$ for all $a,b,c\in A$, such that $f|_{A\otimes A_0+A_0\otimes A}=0$, then *often* $e^f\colon A\otimes A\to\Bbbk$ will be a multiplicative 2-cocycle. For instance, this happens whenever $f(\mathrm{id}\otimes m)$ and $f(m\otimes\mathrm{id})$ commute (with respect to the convolution product) with $\varepsilon\otimes f$ and $f\otimes\varepsilon$, respectively. Also note that if $f*f=0$, then $e^f=\varepsilon+f$.

The following result is well-known in the cocommutative setting [Sw2]. It is a generalization of a result from [GZ], where the cocycle in question is a tensor product of ε-derivations.

Lemma 1.1. [GM, Lemma 4.1] *If $f\colon A\otimes A\to\Bbbk$ is a Hochschild 2-cocycle such that $f(\mathrm{id}\otimes m)$ commutes with $\varepsilon\otimes f$ and $f(m\otimes\mathrm{id})$ commutes with $f\otimes\varepsilon$ in the convolution algebra $\operatorname{Hom}_\Bbbk(A\otimes A\otimes A,\Bbbk)$, then e^f is a multiplicative 2-cocycle with graded infinitesimal part equal to f.* □

2. Cocycles on bosonizations of Nichols algebras

In this section we discuss necessary and sufficient conditions for a lifting of an ε-biderivation (defined below) on a Nichols algebra to satisfy the conditions of Lemma 1.1; and thus give rise to multiplicative cocycles via the exponential map.

Definition 2.1. Let B be an augmented algebra. An *ε-biderivation* is a map $\eta\colon B\otimes B\to \Bbbk$ that is an ε-derivation in each variable; that is, we have $\eta(1,-)=0=\eta(-,1)$ and $\eta(xy,-)=0=\eta(-,xy)$ for all $x,y\in B$ such that $\varepsilon(x)=0=\varepsilon(y)$.

From now on we assume that A is a bosonization $\mathfrak{B}(V)\#H$, where H is Hopf algebra with bijective antipode, V is a left Yetter-Drinfeld module over H and $\mathfrak{B}(V)\in {}^H_H\mathcal{YD}$ is the Nichols algebra of V. Since $\mathfrak{B}(V)$ is graded, we have that A is also graded with the gradation given by $A_0=H$ and $A_n=\mathfrak{B}(V)_n\#H$.

Remark 2.2. (a) Let $B=\mathfrak{B}(V)$ and $\eta\in \operatorname{Hom}_{\Bbbk}(B\otimes B,\Bbbk)$ be an ε-biderivation on B. Then by definition we have $\eta|_{B_0\otimes B+B\otimes B_0}=0$.

(b) Let $B=\mathfrak{B}(V)$ with V finite dimensional. Let $(x_i)_{i\in I}$ be a basis for V and $(d_i)_{i\in I}$ be the dual basis on V^*. Then each d_i induces an ε-derivation given by $d_i(x_j)=\delta_{ij}$ for all $i,j\in I$. Then any linear combination of the tensor products $d_i\otimes d_j$ is an ε-biderivation on $B\otimes B$ which is a Hochschild 2-cocycle. Conversely, if η be an ε-biderivation on $B\otimes B$ then there exists $a_{ij}\in\Bbbk$ such that $\eta=\sum_{i,j}a_{ij}d_i\otimes d_j$.

Let $M\in {}^H_H\mathcal{YD}$. Since H acts on M, we have that H acts on the set of all linear maps $\eta : M\otimes M\to\Bbbk$ by $\eta^h(x,y)=\eta(h_{(1)}\cdot x,h_{(2)}\cdot y)$ for all $h\in H, x,y\in M$. We say that η is *H-invariant* if $\eta^h=\eta$ for all $h\in H$.

Let η be an H-invariant linear map on V. Then it induces an H-invariant ε-biderivation on A, $\tilde{\eta}\colon A\otimes A\to\Bbbk$ by letting $\tilde{\eta}(A_m\otimes A_n)=0$ if $(m,n)\neq(1,1)$ and

$$\tilde{\eta}(x\#h,y\#h')=\eta(x,h\cdot y)\varepsilon(h') \text{ for all } x,y\in V \text{ and } h,h'\in H.$$

Clearly, by definition we have that $\tilde{\eta}|_{A_0\otimes A+A\otimes A_0}=0$.

The following lemma concerning H-invariant Hochschild 2-cocycles on $\mathfrak{B}(V)$ will be very useful in finding liftings.

Lemma 2.3. *Let η be an H-invariant Hochschild 2-cocycle on $\mathfrak{B}(V)$ such that $\tilde{\eta}|_{A_0\otimes A+A\otimes A_0}=0$ and $\sigma=e^{\tilde{\eta}}$ is a multiplicative 2-cocycle. Let $x_1,x_2\in V$ be homogeneous elements with $\delta(x_1)=h_1\otimes x_1$ and $\delta(x_1)=h_2\otimes x_2$, $h_1,h_2\in G(H)$, and denote $z_1=x_1\#1$, $z_2=x_2\#1\in A$. Then $\sigma(z_1,z_2)=\eta(x_1,x_2)$. In particular, in A_σ it holds that*

$$z_1\cdot_\sigma z_2=\eta(x_1,x_2)(1-h_1h_2)+z_1z_2.$$

Proof. First we show that $\tilde{\eta}^2(z_1,z_2)=0$. Since x_1, x_2 are homogeneous we have that $\Delta(z_i)=z_i\otimes 1+h_i\otimes z_i$, that is, z_i is $(1,h_i)$-primitive for $i=1,2$. Since by assumption $\tilde{\eta}|_{A_0\otimes A+A\otimes A_0}=0$ it follows that

$$\begin{aligned}\tilde{\eta}^2(z_1,z_2)&=\tilde{\eta}([z_1]_{(1)},[z_2]_{(1)})\tilde{\eta}([z_1]_{(2)},[z_2]_{(2)})=\\&=\tilde{\eta}(z_1,z_2)\tilde{\eta}(1,1)+\tilde{\eta}(z_1,h_2)\tilde{\eta}(1,z_2)+\tilde{\eta}(h_1,z_2)\tilde{\eta}(z_1,1)+\tilde{\eta}(h_1,h_2)\tilde{\eta}(z_1,z_2)=0.\end{aligned}$$

Thus, $\sigma(z_1,z_2)=\varepsilon(z_1)\varepsilon(z_2)+\tilde{\eta}(z_1,z_2)=\eta(x_1,x_2)$; in particular, $\sigma^{-1}(z_1,z_2)=e^{-\tilde{\eta}}(z_1,z_2)=-\eta(x_1,x_2)$. Finally,

$$\begin{aligned}
z_1\cdot_\sigma z_2 &= \sigma([z_1]_{(1)},[z_2]_{(1)})[z_1]_{(2)}[z_2]_{(2)}\sigma^{-1}([z_1]_{(3)},[z_2]_{(3)})\\
&= \sigma(z_1,z_2)\sigma^{-1}(1,1)+\sigma(z_1,h_2)z_2\sigma^{-1}(1,1)+\sigma(z_1,h_2)h_2\sigma^{-1}(1,z_2)+\sigma(h_1,z_2)z_1\sigma^{-1}(1,1)+\\
&\qquad+\sigma(h_1,h_2)z_1z_2\sigma^{-1}(1,1)+\sigma(h_1,h_2)z_1h_2\sigma^{-1}(1,z_2)+\sigma(h_1,z_2)h_1\sigma^{-1}(z_1,1)+\\
&\qquad+\sigma(h_1,h_2)h_1z_2\sigma^{-1}(z_1,1)+\sigma(h_1,h_2)h_1h_2\sigma^{-1}(z_1,z_2)=\\
&= \sigma(z_1,z_2)\sigma^{-1}(1,1)+\sigma(h_1,h_2)z_1z_2\sigma^{-1}(1,1)+\sigma(h_1,h_2)h_1h_2\sigma^{-1}(z_1,z_2)=\\
&= \eta(x_1,x_2)+z_1z_2-h_1h_2\eta(x_1,x_2)=\eta(x_1,x_2)(1-h_1h_2)+z_1z_2,
\end{aligned}$$

which finish the proof. □

If $x\in\mathfrak{B}(V)$ and $h\in H$, then we identify $x=x\#1\in A$ and $h=1\#h\in A$; in particular we have $xh=x\#h$ and $hx=h_{(1)}\cdot x\#h_{(2)}$.

Denote by c the braiding of V. Then c induces an action of the braid group $\mathbb{B}_n$ on $V^{\otimes n}$. If $\pi\in\mathbb{B}_n$, we denote by $c_\pi: V_1\otimes\ldots\otimes V_n\to V_{\pi(1)}\otimes\ldots\otimes V_{\pi(n)}$ the map induced by this action. In particular we write

$$c_{231}=(\mathrm{id}\otimes c)(c\otimes\mathrm{id}),\quad c_{1324}=(\mathrm{id}\otimes c\otimes\mathrm{id}),\quad c_{2413}=(\mathrm{id}\otimes c\otimes\mathrm{id})(c\otimes c),$$
$$c_{1423}=(\mathrm{id}\otimes c\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes c)\text{ and }c_{2314}=(\mathrm{id}\otimes c\otimes\mathrm{id})(c\otimes\mathrm{id}\otimes\mathrm{id}).$$

Let η be an H-invariant ε-biderivation on $\mathfrak{B}(V)$, then we have that

$$c(\mathrm{id}\otimes\eta)=(\eta\otimes\mathrm{id})(1\otimes c)(c\otimes 1)=(\eta\otimes\mathrm{id})c_{231},\tag{3}$$

that is, for all $a,b,c\in V$ we have:

$$\eta(b,c)\otimes a=\eta(a_{(-1)}b,c)\otimes a_{(0)}=(\eta\otimes\mathrm{id})c_{231}(a\otimes b\otimes c).\tag{4}$$

Indeed, $(\eta\otimes\mathrm{id})c_{231}(a\otimes b\otimes c)=\eta(a_{(-2)}\cdot b,a_{(-1)}\cdot c)\otimes a_{(0)}=\eta^{a_{(-1)}}(b,c)\otimes a_{(0)}=\eta(b,c)\otimes\varepsilon(a_{(-1)})a_{(0)}=c(\mathrm{id}\otimes\eta)(a\otimes b\otimes c)$ for all $a,b,c\in V$.

Using the definition of $\tilde{\eta}$ we also have for all $a,b\in A$ and $c\in V$ that

$$\tilde{\eta}(a,b)\otimes c=\tilde{\eta}(a,bc_{(-1)})\otimes c_{(0)}.\tag{5}$$

The following lemma shows that both conditions on the commutativity of the maps in Lemma 1.1, (b) and (c) below, are equivalent in the case when $A=\mathfrak{B}(V)\#H$ is a bosonization. Note that, as a consequence, we need only to verify equalities on $V^{\otimes 4}$.

Lemma 2.4. *Let η be an H-invariant ε-biderivation on $\mathfrak{B}(V)$. The following are equivalent:*

(a) *The following conditions hold on $V^{\otimes 4}$:*

$$(\eta\otimes\eta)c_{1324}=(\eta\otimes\eta)c_{2413}\text{ and}\tag{6}$$
$$(\eta\otimes\eta)c_{1423}=(\eta\otimes\eta)c_{2314}.\tag{7}$$

(b) *The following condition holds on $A^{\otimes 3}$: $(\varepsilon\otimes\tilde{\eta})*\tilde{\eta}(\mathrm{id}\otimes m)=\tilde{\eta}(\mathrm{id}\otimes m)*(\varepsilon\otimes\tilde{\eta})$.*

(c) *The following condition holds on $A^{\otimes 3}$: $(\tilde{\eta}\otimes\varepsilon)*\tilde{\eta}(m\otimes\mathrm{id})=\tilde{\eta}(m\otimes\mathrm{id})*(\tilde{\eta}\otimes\varepsilon)$.*

Proof. Throughout the proof we use that fact that for $v\in V\subseteq A$ we have $\Delta(v)=v_{(1)}\otimes v_{(2)}=v_{(-1)}\otimes v_{(0)}+v\otimes 1$. We first show that (a) is equivalent to (b). Note that it is both necessary and sufficient to verify (b) by evaluating at $\mathfrak{B}(V)_m\otimes\mathfrak{B}(V)_n\otimes\mathfrak{B}(V)_p$ for all (m,n,p). Unless $(m,n,p)=(1,2,1)$ or $(m,n,p)=(1,1,2)$ both sides trivially give 0. Now evaluation at $a\otimes b\otimes cd$ for $a,b,c,d\in V$ yields that $\tilde{\eta}(b,(cd)_{(1)})\tilde{\eta}(a,(cd)_{(2)})=\tilde{\eta}(b_{(0)},(cd)_{(2)})\tilde{\eta}(a,b_{(-1)}(cd)_{(1)})$. After expanding $\Delta(cd)$ and using the fact that η is H-invariant we get

$$\eta(b,c)\eta(a,d)+\eta(b,c_{(-1)}\cdot d)\eta(a,c_{(0)})=\eta(b_{(0)},d)\eta(a,b_{(-1)}\cdot c)+\eta(b_{(0)},c_{(0)})\eta(a,(b_{(-1)}c_{(-1)})\cdot d).$$

Since by (b) we have that $(\eta\otimes\eta)(a,d,b,c)=(\eta\otimes\eta)c_{2314}(a,b,c,d)$, this is equivalent to

$$(\eta\otimes\eta)(c_{2314}+c_{2413}-c_{1324}-c_{1423})=0. \tag{8}$$

We now examine evaluation of (b) at $a\otimes bc\otimes d$ for $a,b,c,d\in V$. Using (5) we get:

$$\tilde{\eta}((bc)_{(1)},d)\tilde{\eta}(a,(bc)_{(2)})=\tilde{\eta}((bc)_{(2)},d)\tilde{\eta}(a,(bc)_{(1)}). \tag{9}$$

Expanding $\Delta(bc)$ and using equations (4) and (5) as well as the definition of $\tilde{\eta}$ we get

$$\eta(b,c_{(-1)}\cdot d)\eta(a,c_{(0)})+\eta(c,d)\eta(a,b)=\eta(c,d)\eta(a,b)+\eta(b_{(0)},d)\eta(a,b_{(-1)}\cdot c),$$

which simplifies to $\eta(b,c_{(-1)}\cdot d)\eta(a,c_{(0)})=\eta(a,b_{(-1)}\cdot c)\eta(b_{(0)},d)$, or equivalently, $(\eta\otimes\eta)c_{2413}=(\eta\otimes\eta)c_{1324}$, which is exactly (6). Hence we have that (b) is equivalent to (8) and (6), and consequently to (a), since (8) and (6) give (7).

The equivalence of (a) and (c) works almost exactly the same way. For the sake of completeness we provide some intermediate steps. We verify (c) by evaluating it at $a\otimes bc\otimes d$ and $ab\otimes c\otimes d$ for $a,b,c,d\in V$. Evaluation at $a\otimes bc\otimes d$ yields (9) which is equivalent to (6). Evaluation at $ab\otimes c\otimes d$ and using that $\Delta(c)=c\otimes 1+c_{(-1)}\otimes c_{(0)}$ yields

$$\tilde{\eta}((ab)_{(1)},c)\tilde{\eta}((ab)_{(2)},d)=\tilde{\eta}((ab)_{(2)},c_{(0)})\tilde{\eta}((ab)_{(1)},c_{(-1)}\cdot d).$$

This simplifies to

$$\eta(a,b_{(-1)}\cdot c)\eta(b_{(0)},d)+\eta(b,c)\eta(a,d)=\eta(b_{(0)},c_{(0)})\eta(a,b_{(-1)}c_{(-1)}\cdot d)+\eta(a,c_{(0)})\eta(b,c_{(-1)}\cdot d),$$

which by (4) can be written as $(\eta\otimes\eta)(c_{1324}+c_{2314}-c_{1423}-c_{2413})=0$. We conclude the proof by noting that this equation together with (6) are equivalent to (a). □

For the following lemma, observe that using $c_{1324}=\mathrm{id}\otimes c\otimes\mathrm{id}$ and $c_{2413}=(\mathrm{id}\otimes c\otimes\mathrm{id})(c\otimes c)$, we get that (6) is equivalent to $(\eta\otimes\eta)(\mathrm{id}\otimes c\otimes\mathrm{id})=(\eta\otimes\eta)(\mathrm{id}\otimes c\otimes\mathrm{id})(c\otimes c)$.

The next two results state that the conditions in Lemma 2.4(a) are always fulfilled when H is semisimple and the braiding is symmetric.

Lemma 2.5. *If $S_H^2=\mathrm{id}_H$, then* (6) *is equivalent to either of the following equations on $V^{\otimes 4}$:*

$$(\eta\otimes\eta)(\mathrm{id}\otimes c\otimes\mathrm{id}) = (\eta\otimes\eta)(\mathrm{id}\otimes c^{-1}\otimes\mathrm{id}), \tag{10}$$

$$(\eta\otimes\eta)(\mathrm{id}\otimes c^{2}\otimes\mathrm{id}) = \eta\otimes\eta. \tag{11}$$

In particular, (6) *is always satisfied when $c^2=\mathrm{id}_V$ and H is semisimple.*

Proof. Note that in the case S is an involution, and therefore for $h\in H$ we have $\varepsilon(h)=S(h_{(2)})h_{(1)}$ (this is used for going from line 8 to line 9 and for going to the last line from the line above it), we get for all $a,b,c,d\in$:

$$\begin{aligned}
&(\eta\otimes\eta)c_{2413}(a,b,c,d)=(\eta\otimes\eta)(\mathrm{id}\otimes c\otimes\mathrm{id})(a_{(-1)}\cdot b,a_{(0)},c_{(-1)}\cdot d,c_{(0)})\\
&=\eta(a_{(-2)}\cdot b,a_{(-1)}c_{(-1)}\cdot d)\eta(a_{(0)},c_{(0)})=\eta^{a_{(-1)}}(b,c_{(-1)}\cdot d)\eta(a_{(0)},c_{(0)})=\eta(b,c_{(-1)}\cdot d)\eta(a,c_{(0)})\\
&=\eta^{S(c_{(-1)})}(b,c_{(-2)}\cdot d)\eta(a,c_{(0)})=\eta(S(c_{(-1)})_{(1)}\cdot b,(S(c_{(-1)})_{(2)}c_{(-2)})\cdot d)\eta(a,c_{(0)})\\
&=\eta(S(c_{(-1)_{(2)}})\cdot b,(S(c_{(-1)_{(1)}})c_{(-2)})\cdot d)\eta(a,c_{(0)})=\eta(S(c_{(-1)})\cdot b,(S(c_{(-2)})c_{(-3)})\cdot d)\eta(a,c_{(0)})\\
&=\eta(S(c_{(-1)})\cdot b,d)\eta(a,c_{(0)})=\eta(a,c_{(0)})\eta(S(c_{(-1)})\cdot b,d)=(\eta\otimes\eta)(\mathrm{id}\otimes c^{-1}\otimes\mathrm{id})(a,b,c,d).
\end{aligned}$$

□

Lemma 2.6. *Let $\eta = \eta_1 \otimes \eta_2$. Then (7) is equivalent to the following equations on $V^{\otimes 4}$:*

$$(\eta \otimes \eta)(\mathrm{id} \otimes c \otimes \mathrm{id})(c \otimes \mathrm{id} \otimes \mathrm{id}) = (\eta \otimes \eta)(\mathrm{id} \otimes c \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes c), \tag{12}$$

$$\eta_1 \otimes \eta \otimes \eta_2 = (\eta \otimes \eta)(\mathrm{id} \otimes c_{312}). \tag{13}$$

Moreover, if $S_H^2 = \mathrm{id}_H$ and $c^2 = \mathrm{id}_V$, then (7) is always satisfied.

Proof. The first equation is just a translation of (7). The second equation is obtained directly from the first. The left hand side is obtained by invoking (3); the right hand side by using $c_{312} = c \otimes \mathrm{id}(\mathrm{id} \otimes c)$. Now if S_H and c are involutions, then by Lemma 2.5 we have that

$$[(\eta \otimes \eta)(\mathrm{id} \otimes c \otimes \mathrm{id})(c \otimes \mathrm{id} \otimes \mathrm{id})](\mathrm{id} \otimes \mathrm{id} \otimes c) = (\eta \otimes \eta)(\mathrm{id} \otimes c \otimes \mathrm{id}).$$

On the other hand

$$[(\eta \otimes \eta)(\mathrm{id} \otimes c \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes c)](\mathrm{id} \otimes \mathrm{id} \otimes c) = (\eta \otimes \eta)(\mathrm{id} \otimes c \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes c^2) = (\eta \otimes \eta)(\mathrm{id} \otimes c \otimes \mathrm{id}).$$

□

2.1. **Cocycles on bosonizations over groups.** Let Γ be a finite group and $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ such that $\mathfrak{B}(V)$ is finite dimensional. Let $\{x_i\}_{i \in I}$ be homogeneous primitive elements that span linearly V with $\delta(x_i) = g_i \otimes x_i$ and $g_i \in \Gamma$ for all $i \in I$. Since for all $h \in \Gamma$, $h \cdot x_i$ is again a homogeneous primitive element, from now on we assume that

$$h \cdot x_i = \chi_i(h) x_{\sigma(h)(i)} \text{ for all } i \in I,\ h \in \Gamma, \tag{14}$$

where $\sigma : \Gamma \to \mathbb{S}_I$ and $\chi_i : \Gamma \to \Bbbk$ is a character, see [AG2, Ex. 5.9]. This condition holds for all finite dimensional pointed Hopf algebras over the symmetric groups and over the dihedral groups, see Sections 3 and 4. We write $\sigma(g_i)(j) = i \triangleright j$ for all $i, j \in I$, see Remark 2.7(c).

Remark 2.7. (a) If V is irreducible, then $V \simeq M(\mathcal{O}, \rho)$ with $\mathcal{O}$ a conjugacy class of Γ. In such a case, I can be identified with $\mathcal{O}$ and $\sigma(h)$ is just the conjugation by h.

(b) If Γ is abelian, then V is a braided space of diagonal type, *i. e.* $h \cdot x_i = \chi_i(h) x_i$ for all $i \in I$ and $\sigma(h) = \mathrm{id}$ for all $h \in \Gamma$.

(c) Let $\mathfrak{B}(V)$ with V finite dimensional, $(x_i)_{i \in I}$ be a basis for V and $(d_i)_{i \in I}$ the dual basis on V^*. Then by Remark2.2(b), any linear combination of the tensor products $d_i \otimes d_j$ induces an ε-biderivation on $B \otimes B$ which is a Hochschild 2-cocycle. Consider then the ε-biderivation $\eta = \sum_{i,j \in I} a_{ij} d_i \otimes d_j$ on $\mathfrak{B}(V)$. It is Γ-invariant if

$$a_{k,\ell} = \chi_k(g)\chi_\ell(g) a_{\sigma(g)(k), \sigma(g)(\ell)} \text{ for all } g \in \Gamma, k, \ell \in I.$$

(d) By Remark 2.2(a) and (b), the result above holds for $\eta = \sum_{i,j} a_{ij} d_i \otimes d_j$ a Γ-invariant ε-biderivation on $\mathfrak{B}(V)$, with V finite dimensional.

3. On pointed Hopf algebras over dihedral groups

All pointed Hopf algebras with group of group-likes isomorphic to $\mathbb{D}_m$ with $m = 4t \geq 12$ were classified in [FG]. To give the complete list we need first to introduce some terminology. From now on we assume that $m = 4t \geq 12$, $n = \frac{m}{2} = 2t$ and we fix ω an m-th primitive root of unity.

3.1. **Yetter-Drinfeld modules and Nichols algebras over $\mathbb{D}_m$.** Let $M_\ell = M(\mathcal{O}_{h^n}, \rho_\ell)$ with ℓ odd be the irreducible Yetter-Drinfeld module associated to a simple two-dimensional representation of $\mathbb{D}_m$. It is spanned linearly by the elements $x_1^{(\ell)}, x_2^{(\ell)}$ and its structure is given by

$$\begin{aligned} g\cdot x_1^{(\ell)} &= x_2^{(\ell)}, & h\cdot x_1^{(\ell)} &= \omega^{\ell} x_1^{(\ell)}, & \delta(x_1^{(\ell)}) &= h^n\otimes x_1^{(\ell)},\\ g\cdot x_2^{(\ell)} &= x_1^{(\ell)}, & h\cdot x_2^{(\ell)} &= \omega^{-\ell} x_2^{(\ell)}, & \delta(x_2^{(\ell)}) &= h^n\otimes x_2^{(\ell)}. \end{aligned}$$

Consider now the set

$$\mathcal{L} = \left\{ L = \coprod_{s=1}^{r}\{\ell_s\} : \ell_i \text{ is odd } \forall 1\leq \ell_1,\dots,\ell_r < n \right\}.$$

Then for $L\in\mathcal{L}$ we define $M_L = \bigoplus_{\ell\in L} M_\ell$. By [FG, Prop. 2.8], we have that $\mathfrak{B}(M_L)\simeq \bigwedge M_L$ and $\dim \mathfrak{B}(M_L) = 4^{|L|}$.

Let $M_{(i,k)} = M(\mathcal{O}_{h^i}, \chi_{(k)})$ with $1\leq i<n$, $0\leq k<m$ and $\chi_{(k)}$ the simple representation of $C_{h^i} = \langle h\rangle \simeq \mathbb{Z}/(m)$ given by $\chi_{(k)}(h) = \omega^k$. $M_{(i,k)}$ is spanned linearly by the elements $y_1^{(i,k)}, y_2^{(i,k)}$ and the Yetter-Drinfeld module structure is given by

$$\begin{aligned} g\cdot y_1^{(i,k)} &= y_2^{(i,k)}, & h\cdot y_1^{(i,k)} &= \omega^{k} y_1^{(i,k)}, & \delta(y_1^{(i,k)}) &= h^i\otimes y_1^{(i,k)},\\ g\cdot y_2^{(i,k)} &= y_1^{(i,k)}, & h\cdot y_2^{i,k} &= \omega^{-k} y_2^{(i,k)}, & \delta(y_2^{(i,k)}) &= h^{-i}\otimes y_2^{(i,k)}. \end{aligned} \tag{15}$$

Let

$$\mathcal{I} = \left\{ I = \coprod_{s=1}^{r}\{(i_s,k_s)\} : \omega^{i_sk_s} = -1, \omega^{i_sk_t+i_tk_s} = 1, 1\leq i_s<n, 1\leq k_s<m \right\}.$$

For $I\in\mathcal{I}$, we define $M_I = \bigoplus_{(i,k)\in I} M_{(i,k)}$. By [FG, Prop. 2.5], we have that $\mathfrak{B}(M_I)\simeq\bigwedge M_I$ and $\dim\mathfrak{B}(M_I) = 4^{|I|}$.

Finally, consider the set

$$\mathcal{K} = \left\{ (I,L):\ I\in\mathcal{I}, L\in\mathcal{L} \text{ and } \omega^{i\ell} = -1 \text{ for all } \ell\in L, (i,k)\in I \text{ with } k \text{ odd } \right\}.$$

As before, for $(I,L)\in\mathcal{K}$ we define $M_{I,L} = \left(\bigoplus_{(i,k)\in I} M_{i,k}\right)\oplus\left(\bigoplus_{\ell\in L} M_\ell\right)$. By [FG, Prop. 2.12], we have that $\mathfrak{B}(M_{I,L})\simeq\bigwedge M_{I,L}$ and $\dim\mathfrak{B}(M_{I,L}) = 4^{|I|+|L|}$.

Theorem 3.1. [FG, Thm. A] *Let $\mathfrak{B}(M)$ be a finite dimensional Nichols algebra in ${}^{\mathbb{D}_m}_{\mathbb{D}_m}\mathcal{YD}$. Then $\mathfrak{B}(M)\simeq\bigwedge M$, with M isomorphic either to M_I, or to M_L, or to $M_{I,L}$, with $I\in\mathcal{I}$, $L\in\mathcal{L}$ and $(I,L)\in\mathcal{K}$, respectively.* □

3.2. **Classification of finite dimensional pointed Hopf algebras over $\mathbb{D}_m$.** Let $I\in\mathcal{I}$ and $L\in\mathcal{L}$ be as Subsection 3.1 and let $\lambda = (\lambda_{p,q,i,k})_{(p,q),(i,k)\in I}$, $\gamma = (\gamma_{p,q,i,k})_{(p,q),(i,k)\in I}$, $\theta = (\theta_{p,q,\ell})_{(p,q)\in I,\ell\in L}$ and $\mu = (\mu_{p,q,\ell})_{(p,q)\in I,\ell\in L}$ be families of elements in $\Bbbk$ satisfying:

$$\lambda_{p,m-k,i,k} = \lambda_{i,k,p,m-k} \quad\text{and}\quad \gamma_{p,k,i,k} = \gamma_{i,k,p,k}. \tag{16}$$

In particular, θ and μ are families of free parameters in $\Bbbk$.

Definition 3.2. For $I\in\mathcal{I}$, denote by $A_I(\lambda,\gamma)$ the algebra generated by $g, h, a_1^{(p,q)}, a_2^{(p,q)}$ with $(p,q)\in I$ satisfying the relations:

$$g^2 = 1 = h^m,\quad ghg = h^{m-1},\quad ga_1^{(p,q)} = a_2^{(p,q)}g,\quad ha_1^{(p,q)} = \omega^q a_1^{(p,q)}h,\quad ha_2^{(p,q)} = \omega^{-q}a_1^{(p,q)}h,$$

$$a_1^{(p,q)}a_1^{(i,k)} + a_1^{(i,k)}a_1^{(p,q)} = \delta_{q,m-k}\lambda_{p,q,i,k}(1-h^{p+i}),\ a_1^{(p,q)}a_2^{(i,k)} + a_2^{(i,k)}a_1^{(p,q)} = \delta_{q,k}\gamma_{p,q,i,k}(1-h^{p-i}).$$

It is a Hopf algebra with its structure determined by g, h being group-likes and

$$\Delta(a_1^{(p,q)}) = a_1^{(p,q)} \otimes 1 + h^p \otimes a_1^{(p,q)}, \qquad \Delta(a_2^{(p,q)}) = a_2^{(p,q)} \otimes 1 + h^{-p} \otimes a_2^{(p,q)}, \text{ for all } (p,q) \in I.$$

It turns out that the diagram of $A_I(\lambda, \gamma)$ is exactly $\mathfrak{B}(M_I)$, thus we call the pair (λ, γ) a *lifting datum* for $\mathfrak{B}(M_I)$. Set $\gamma = 0$ if $|I| = 1$.

Definition 3.3. For $(I, L) \in \mathcal{K}$, denote by $B_{I,L}(\lambda, \gamma, \theta, \mu)$ the algebra generated by $g, h, a_1^{(p,q)}$, $a_2^{(p,q)}, b_1^{(\ell)}, b_2^{(\ell)}$ satisfying the relations:

$$g^2 = 1 = h^m, \quad ghg = h^{m-1}, \quad ga_1^{(p,q)} = a_2^{(p,q)} g,$$
$$ha_1^{(p,q)} = \omega^q a_1^{(p,q)} h, \quad gb_1^{(\ell)} = b_2^{(\ell)} g, \qquad hb_1^{(\ell)} = \omega^\ell b_1^{(\ell)} h,$$
$$[a_1^{(p,q)}]^2 = 0 = [a_2^{(p,q)}]^2, \qquad b_1^{(\ell)} b_2^{(\ell')} + b_2^{(\ell')} b_1^{(\ell)} = 0, \qquad b_1^{(\ell)} b_1^{(\ell')} + b_1^{(\ell')} b_1^{(\ell)} = 0,$$
$$a_1^{(p,q)} a_1^{(i,k)} + a_1^{(i,k)} a_1^{(p,q)} = \delta_{q,m-k} \lambda_{p,q,i,k} (1 - h^{p+i}), \; a_1^{(p,q)} a_2^{(i,k)} + a_2^{(i,k)} a_1^{(p,q)} = \delta_{q,k} \gamma_{p,q,i,k} (1 - h^{p-i}),$$
$$a_1^{(p,q)} b_1^{(\ell)} + b_1^{(\ell)} a_1^{(p,q)} = \delta_{q,m-\ell} \theta_{p,q,\ell} (1 - h^{n+p}), \; a_1^{(p,q)} b_2^{(\ell)} + b_2^{(\ell)} a_1^{(p,q)} = \delta_{q,\ell} \mu_{p,q,\ell} (1 - h^{n+p}).$$

It is a Hopf algebra with its structure determined by g, h being group-likes and

$$\Delta(a_1^{(p,q)}) = a_1^{(p,q)} \otimes 1 + h^p \otimes a_1^{(p,q)}, \qquad \Delta(a_2^{(p,q)}) = a_2^{(p,q)} \otimes 1 + h^{-p} \otimes a_2^{(p,q)},$$
$$\Delta(b_1^{(\ell)}) = b_1^{(\ell)} \otimes 1 + h^n \otimes b_1^{(\ell)}, \qquad \Delta(b_2^{(\ell)}) = b_2^{(\ell)} \otimes 1 + h^n \otimes b_2^{(\ell)},$$

for all $(p,q) \in I, \ell \in L$. It turns out that the diagram of $B_{I,L}(\lambda, \gamma, \theta, \mu)$ is $\mathfrak{B}(M_{I,L})$, thus we call the 4-tuple $(\lambda, \gamma, \theta, \mu)$ a lifting datum for $\mathfrak{B}(M_{I,L})$.

3.3. **Cocycle deformations and finite dimensional pointed Hopf algebras over $\mathbb{D}_m$.** In this subsection we prove that all pointed Hopf algebras $A_I(\lambda, \gamma)$ and $B_{I,L}(\lambda, \gamma, \theta, \mu)$ can be obtained by deforming the multiplication of a bosonization of a Nichols algebra using a multiplicative 2-cocycle.

3.3.1. *Cocycle deformations and the algebras $A_I(\lambda, \gamma)$.* Let $I \in \mathcal{I}$ and consider the Nichols algebra $\mathfrak{B}(M_I)$. Define the ε-derivations $d_1^{(p,q)}, d_2^{(p,q)}$ for all $(p,q) \in I$ by the rule $d_r^{(p,q)}(y_s^{(i,k)}) = \delta_{r,s} \delta_{p,i} \delta_{q,k}$ for all $r, s = 1, 2$, $(p,q), (i,k) \in I$ and consider the ε-biderivation

$$\eta = \sum_{\substack{(p,q),(i,k) \in I, \\ 1 \leq r,s \leq 2}} \alpha_{p,q,i,k}^{r,s} d_r^{(p,q)} \otimes d_s^{(i,k)}.$$

Lemma 3.4. *η is $\mathbb{D}_m$-invariant if and only if the following conditions hold:*

$$\alpha_{p,q,i,k}^{r,s} = \alpha_{p,q,i,k}^{s,r} \qquad \forall (p,q), (i,k) \in I, r, s = 1, 2, \tag{17}$$
$$\alpha_{p,q,i,k}^{1,1} = \alpha_{p,q,i,k}^{2,2} \qquad \forall (p,q), (i,k) \in I, \tag{18}$$
$$\alpha_{p,q,i,k}^{r,r} = \delta_{q,m-k} \alpha_{p,m-k,i,k}^{r,r} \qquad \forall (p,q), (i,k) \in I, r = 1, 2, \tag{19}$$
$$\alpha_{p,q,i,k}^{r,s} = \delta_{q,k} \alpha_{p,k,i,k}^{r,s} \qquad \forall (p,q), (i,k) \in I, 1 \leq r \neq s \leq 2. \tag{20}$$

Proof. To prove that η is $\mathbb{D}_m$-invariant it is enough to show that $\eta^g = \eta^h = \eta$. Since $[d_1^{(p,q)}]^g = d_2^{(p,q)}$ and $[d_2^{(p,q)}]^g = d_1^{(p,q)}$ for all $(p,q) \in I$, and η is a linear combination of tensor products of ε-derivations, we have that $\eta^g = \eta$ if and only if (17) and (18) hold. Analogously, since $[d_i^{(p,q)}]^h = \omega^{(-1)^{i-1} q} d_i^{(p,q)}$ for all $(p,q) \in I$ and $i = 1, 2$ we have that $\eta^h = \eta$ if and only if

$$\eta = \sum_{\substack{(p,q),(i,k) \in I, \\ 1 \leq r,s \leq 2}} \alpha_{p,q,i,k}^{r,s} \omega^{(-1)^{r-1} q + (-1)^{s-1} k} d_r^{(p,q)} \otimes d_s^{(i,k)},$$

which holds if and only if $\alpha^{r,s}_{p,q,i,k} = \alpha^{r,s}_{p,q,i,k}\omega^{(-1)^{r-1}q+(-1)^{s-1}k}$ for all $(p,q), (i,k) \in I$ and $r,s = 1,2$. Thus, if $r = s$ we must have that $\alpha^{r,r}_{p,q,i,k} = 0$ or $q \equiv -k \mod m$ which gives (19) and if $r \neq s$ then $\alpha^{r,s}_{p,q,i,k} = 0$ or $q \equiv k \mod m$ which gives (20). □

Lemma 3.5. *Assume η satisfies conditions (17) – (20). Then $\sigma = e^{\tilde{\eta}}$ is a multiplicative 2-cocycle for $\mathfrak{B}(M_I)\#\Bbbk\mathbb{D}_m$.*

Proof. By assumption, we know that η is $\mathbb{D}_m$-invariant. Since by Theorem 3.1, the braiding in ${}^{\mathbb{D}_m}_{\mathbb{D}_m}\mathcal{YD}$ is symmetric, then by Lemmata 2.4, 2.5 and 2.6, we get that $\tilde{\eta}$ fulfills the conditions in Lemma 1.1, and consequently $\sigma = e^{\tilde{\eta}}$ is a multiplicative 2-cocycle for $\mathfrak{B}(M_I)\#\Bbbk\mathbb{D}_m$. □

Theorem 3.6. *Let $H = \mathfrak{B}(M_I)\#\Bbbk\mathbb{D}_m$ and $\sigma = e^{\tilde{\eta}}$ be the multiplicative $2-$cocycle given by* Lemma 3.5. *Then $H_\sigma \simeq A_I(\lambda,\gamma)$ with $\lambda_{p,q,i,k} = \alpha^{r,r}_{p,q,i,k}+\alpha^{r,r}_{i,k,p,q}$ and $\gamma_{p,q,i,k} = \alpha^{r,s}_{p,q,i,k}+\alpha^{r,s}_{i,k,p,q}$ for all $(p,q),(i,k) \in I$. In particular, $A_I(\lambda,\gamma)$ is a cocycle deformation of H for all lifting datum.*

Proof. To show that H_σ is isomorphic to $A_I(\lambda,\gamma)$ it suffices to prove that the generators of H_σ satisfy the relations given in Definition 3.2, for this would imply that there exists a Hopf algebra surjection $H_\sigma \twoheadrightarrow A_I$ and since both algebras have the same dimension they must be isomorphic.

For $(p,q) \in I$ and $1 \leq r \leq 2$, denote $a^{(p,q)}_r = y^{(p,q)}_r\#1 \in \mathfrak{B}(M_I)\#\Bbbk\mathbb{D}_m$. Then by Lemma 2.3 we have for all $(p,q),(i,k) \in I$ and $r,s = 1,2$ that

$$\begin{aligned} a^{(p,q)}_r \cdot_\sigma a^{(i,k)}_s &= \eta(y^{(p,q)}_r, y^{(i,k)}_s)(1 - h^{p(-1)^{r-1}}h^{i(-1)^{s-1}}) + a^{(p,q)}_r a^{(i,k)}_s \\ &= \alpha^{r,s}_{p,q,i,k}(1 - h^{p(-1)^{r-1}+i(-1)^{s-1}}) + a^{(p,q)}_r a^{(i,k)}_s. \end{aligned}$$

Using Lemma 3.4 we obtain that

$$\begin{aligned} a^{(p,q)}_1 \cdot_\sigma a^{(i,k)}_1 + a^{(i,k)}_1 \cdot_\sigma a^{(p,q)}_1 &= a^{(p,q)}_1 a^{(i,k)}_1 + a^{(i,k)}_1 a^{(p,q)}_1 + \delta_{q,m-k}(\alpha^{1,1}_{p,q,i,k} + \alpha^{1,1}_{i,k,p,q})(1 - h^{p+i}) \\ &= \delta_{q,m-k}(\alpha^{1,1}_{p,q,i,k} + \alpha^{1,1}_{i,k,p,q})(1 - h^{p+i}) \qquad \text{and} \\ a^{(p,q)}_1 \cdot_\sigma a^{(i,k)}_2 + a^{(i,k)}_2 \cdot_\sigma a^{(p,q)}_1 &= a^{(p,q)}_1 a^{(i,k)}_2 + a^{(i,k)}_2 a^{(p,q)}_1 + \delta_{q,k}(\alpha^{1,2}_{p,q,i,k} + \alpha^{2,1}_{i,k,p,q})(1 - h^{p-i}) \\ &= \delta_{q,k}(\alpha^{1,2}_{p,q,i,k} + \alpha^{1,2}_{i,k,p,q})(1 - h^{p-i}). \end{aligned}$$

Thus, defining $\lambda_{p,q,i,k} = \alpha^{r,r}_{p,q,i,k} + \alpha^{r,r}_{i,k,p,q}$ and $\gamma_{p,q,i,k} = \alpha^{r,s}_{p,q,i,k} + \alpha^{r,s}_{i,k,p,q}$ with $1 \leq r \neq s \leq 2$ we get that condition (16) is satisfied. Since the other relations follows from the Yetter-Drinfeld structure of M_I, the theorem is proved. □

Remark 3.7. Note that given a lifting datum (λ,γ), using Lemma 3.4 and Theorem 3.6 one is able to construct a multiplicative 2-cocycle that gives the desired deformation of $\mathfrak{B}(M_I)\#\Bbbk\mathbb{D}_m$.

3.3.2. *Cocycle deformations and the algebras $B_{I,L}(\lambda,\gamma,\theta,\mu)$.* Let $(I,L) \in \mathcal{K}$ and consider the Nichols algebra $\mathfrak{B}(M_{I,L})$. Define the ε-derivations $d^{(p,q)}_1, d^{(p,q)}_2$ and $d^{(\ell)}_1, d^{(\ell)}_2$ for all $(p,q) \in I, \ell \in L$ by the rules

$$d^{(p,q)}_r(y^{(i,k)}_s) = \delta_{r,s}\delta_{p,i}\delta_{q,k}, \quad d^{(p,q)}_r(x^{(\ell)}_s) = 0, \quad d^{(\ell)}_r(x^{(\ell')}_s) = \delta_{r,s}\delta_{\ell,\ell'}, \quad d^{(\ell)}_r(y^{(i,k)}_s) = 0.$$

for all $r,s = 1,2,\ (p,q),(i,k) \in I,\ \ell \in L$, and consider the ε-biderivation

$$\begin{aligned} \eta = &\sum_{\substack{(p,q),(i,k)\in I,\\ 1\leq r,s\leq 2}} \alpha^{r,s}_{p,q,i,k} d^{(p,q)}_r \otimes d^{(i,k)}_s + \sum_{\substack{(p,q)\in I,\ell\in L\\ 1\leq r,s\leq 2}} [\beta^{r,s}_{p,q,\ell} d^{(p,q)}_r \otimes d^{(\ell)}_s + \zeta^{r,s}_{p,q,\ell} d^{(\ell)}_s \otimes d^{(p,q)}_r] + \\ &+ \sum_{\substack{\ell,\ell'\in L\\ 1\leq r,s\leq 2}} \xi^{r,s}_{\ell,\ell'} d^{(\ell)}_r \otimes d^{(\ell')}_s. \end{aligned}$$

Lemma 3.8. *η is $\mathbb{D}_m$-invariant if and only if the following conditions hold:* (17)–(20) *from Lemma 3.4,*

$$\beta^{r,s}_{p,q,\ell} = \beta^{s,r}_{p,q,\ell} \quad \forall (p,q) \in I, \ell \in L, r,s = 1,2, \tag{21}$$
$$\beta^{1,1}_{p,q,\ell} = \beta^{2,2}_{p,q,\ell} \quad \forall (p,q) \in I, \ell \in L, \tag{22}$$
$$\beta^{r,r}_{p,q,\ell} = \delta_{q,m-\ell}\beta^{r,r}_{p,m-\ell,\ell} \quad \forall (p,q) \in I, \ell \in L, r = 1,2, \tag{23}$$
$$\beta^{r,s}_{p,q,\ell} = \delta_{q,\ell}\beta^{r,s}_{p,\ell,\ell} \quad \forall (p,q) \in I, \ell \in L, 1 \leq r \neq s \leq 2, \tag{24}$$
$$\xi^{r,s}_{\ell,\ell'} = \xi^{s,r}_{\ell,\ell'} \quad \forall \ell, \ell' \in L, r,s = 1,2, \tag{25}$$
$$\xi^{r,r}_{\ell,\ell'} = 0 \quad \forall \ell, \ell' \in L, r = 1,2, \tag{26}$$
$$\xi^{r,s}_{\ell,\ell'} = \delta_{\ell,\ell'}\xi^{r,s}_{\ell,\ell'} \quad \forall \ell, \ell' \in L, 1 \leq r \neq s \leq 2, \tag{27}$$

and the coefficients $\zeta^{r,s}_{p,q,\ell}$ satisfy the same conditions as the coefficients $\beta^{r,s}_{p,q,\ell}$, for all $(p,q) \in I,\ \ell \in L,\ r,s = 1,2$.

Remark 3.9. Note that in this case, equation (19) implies that $\alpha^{r,r}_{p,q,p,q} = 0$ for all $(p,q),(i,k) \in I$, since $m = 4t$, q is odd for all $(p,q) \in I, (I,L) \in \mathcal{K}$ and $m - q \equiv q \mod m$ if and only if $m = 2q$.

Proof. To prove that η is $\mathbb{D}_m$-invariant it is enough to show that $\eta^g = \eta^h = \eta$. Thus the first four conditions follows directly from Lemma 3.4. The proof of the remaining conditions goes along the same lines. Only note that condition (26) is different because it never holds that $\ell' \equiv m - \ell \mod m$ since $1 \leq \ell, \ell' < n$ and $m = 2n$. □

The proof of the following lemma is completely analogous to the proof of Lemma 3.5.

Lemma 3.10. *Assume η satisfies conditions* (17) – (27)*. Then $\sigma = e^{\tilde{\eta}}$ is a multiplicative* 2*-cocycle for $\mathfrak{B}(M_{I,L})\#\Bbbk\mathbb{D}_m$.* □

Theorem 3.11. *Let $H = \mathfrak{B}(M_{I,L})\#\Bbbk\mathbb{D}_m$ and $\sigma = e^{\tilde{\eta}}$ be the multiplicative $2-$cocycle given by* Lemma 3.10*. Then $H_\sigma \simeq B_{I,L}(\lambda,\gamma,\theta,\mu)$ with $\lambda_{p,q,i,k} = \alpha^{r,r}_{p,q,i,k} + \alpha^{r,r}_{i,k,p,q}$, $\gamma_{p,q,i,k} = \alpha^{r,s}_{p,q,i,k} + \alpha^{r,s}_{i,k,p,q}$, $\theta_{p,q,\ell} = \beta^{1,1}_{p,q,\ell} + \zeta^{1,1}_{p,q,\ell}$, and $\mu_{p,q,\ell} = \beta^{1,2}_{p,q,\ell} + \zeta^{1,2}_{p,q,\ell}$, for all $(p,q) \in I, \ell \in L$. In particular, $B_{I,L}(\lambda,\gamma,\theta,\mu)$ is a cocycle deformation of H for all lifting datum.*

Proof. As in the proof of Theorem 3.6, it suffices to show that the generators of H_σ satisfy the relations given in Definition 3.3. For $(p,q) \in I, \ell \in L$ and $1 \leq r \leq 2$, denote $a^{(p,q)}_r = y^{(p,q)}_r\#1$ and $b^{(\ell)}_r = x^{(\ell)}_r\#1 \in \mathfrak{B}(M_{I,L})\#\Bbbk\mathbb{D}_m$.

Since $\tilde{\eta}$ coincides with the multiplicative cocycle given by Lemma 3.5 when it takes values in $\{a^{(p,q)}_r : (p,q) \in I, r = 1,2\}$, by the proof of Theorem 3.6 we have that the equations involving the generators $a^{(p,q)}_r$ are satisfied. In particular, since q is odd for all (p,q) we have that $q \not\equiv m-q \mod m$ for all $(p,q) \in I$ and by Lemma 2.3

$$a^{(p,q)}_r \cdot_\sigma a^{(p,q)}_r = [a^{(p,q)}_r]^2 + \delta_{q,m-q}\alpha^{r,r}_{p,q,p,q}(1 - h^{2p(-1)^{r-1}}) = 0.$$

Moreover, again by Lemma 2.3 we get that

$$b^{(\ell)}_r \cdot_\sigma b^{(\ell')}_s = \eta(x^{(\ell)}_r, x^{(\ell')}_s)(1 - h^n h^n) + b^{(\ell)}_r b^{(\ell')}_s = b^{(\ell)}_r b^{(\ell')}_s \quad \text{for all } \ell, \ell' \in L, r,s = 1,2.$$

Hence, using the relations of the Nichols algebra $\mathfrak{B}(M_{I,L})$ we have that

$$b^{(\ell)}_r \cdot_\sigma b^{(\ell')}_s + b^{(\ell')}_s \cdot_\sigma b^{(\ell)}_r = b^{(\ell)}_r b^{(\ell')}_s + b^{(\ell')}_s b^{(\ell)}_r = 0 \quad \text{for all } \ell, \ell' \in L, r,s = 1,2.$$

Besides, by (23) we get

$$a_1^{(p,q)} \cdot_\sigma b_1^{(\ell)} = \eta(y_1^{(p,q)}, x_1^{(\ell)})(1 - h^p h^n) + a_1^{(p,q)} b_1^{(\ell)} = \delta_{q,m-\ell}\beta^{1,1}_{p,q,\ell}(1 - h^{p+n}) + a_1^{(p,q)} b_1^{(\ell)} \quad \text{and}$$
$$b_1^{(\ell)} \cdot_\sigma a_1^{(p,q)} = \eta(x_1^{(\ell)}, y_1^{(p,q)})(1 - h^n h^p) + b_1^{(\ell)} a_1^{(p,q)} = \delta_{q,m-\ell}\zeta^{1,1}_{p,q,\ell}(1 - h^{p+n}) + b_1^{(\ell)} a_1^{(p,q)},$$

for all $(p,q) \in I, \ell \in L$. Hence, using again the relations of the Nichols algebra $\mathfrak{B}(M_{I,L})$ we have

$$a_1^{(p,q)} \cdot_\sigma b_1^{(\ell)} + b_1^{(\ell)} \cdot_\sigma a_1^{(p,q)} = \delta_{q,m-\ell}(\beta^{1,1}_{p,q,\ell} + \zeta^{1,1}_{p,q,\ell})(1 - h^{p+n}).$$

If we set $\theta_{p,q,\ell} = \beta^{1,1}_{p,q,\ell} + \zeta^{1,1}_{p,q,\ell}$ with $(p,q) \in I, \ell \in L$, then the condition involving the generators $a_1^{(p,q)}, b_1^{(\ell)}$ is satisfied. Finally, by (24) we have that

$$a_1^{(p,q)} \cdot_\sigma b_2^{(\ell)} = \eta(y_1^{(p,q)}, x_2^{(\ell)})(1 - h^p h^n) + a_1^{(p,q)} b_2^{(\ell)} = \delta_{q,\ell}\beta^{1,2}_{p,q,\ell}(1 - h^{p+n}) + a_1^{(p,q)} b_2^{(\ell)} \quad \text{and}$$
$$b_2^{(\ell)} \cdot_\sigma a_1^{(p,q)} = \eta(x_2^{(\ell)}, y_1^{(p,q)})(1 - h^n h^p) + b_2^{(\ell)} a_1^{(p,q)} = \delta_{q,\ell}\zeta^{1,2}_{p,q,\ell}(1 - h^{p+n}) + b_2^{(\ell)} a_1^{(p,q)},$$

for all $(p,q) \in I, \ell \in L$. Thus

$$a_1^{(p,q)} \cdot_\sigma b_2^{(\ell)} + b_2^{(\ell)} \cdot_\sigma a_1^{(p,q)} = \delta_{q,\ell}(\beta^{1,2}_{p,q,\ell} + \zeta^{1,2}_{p,q,\ell})(1 - h^{p+n}).$$

Defining $\mu_{p,q,\ell} = \beta^{1,2}_{p,q,\ell} + \zeta^{1,2}_{p,q,\ell}$ with $(p,q) \in I, \ell \in L$, it follows that the condition involving the generators $a_1^{(p,q)}, b_2^{(\ell)}$ is satisfied. Since the other relations follows from the Yetter-Drinfeld structure of $M_{I,L}$, the theorem is proved. □

Remark 3.12. Note that given a lifting datum $(\lambda, \gamma, \theta, \mu)$, using Lemma 3.8 and Theorem 3.11 one is able to construct a multiplicative 2-cocycle that give the desired deformation.

4. On pointed Hopf algebras over symmetric groups

Finite dimensional pointed Hopf algebras whose coradical is the group algebra of the groups $\mathbb{S}_3$ and $\mathbb{S}_4$ were classified in [AHS] and [GG], respectively. In this section, we prove that some of them are cocycle deformations by giving, as in Section 3.2, explicitly the cocycles.

4.1. **Racks, Yetter-Drinfeld modules and Nichols algebras over $\mathbb{S}_n$.** To present finite dimensional Nichols algebras over $\mathbb{S}_n$ we need first to introduce the notion of racks, see [AG2, Def. 1.1] for more details.

A *rack* is a pair $(X, \triangleright)$, where X is a non-empty set and $\triangleright : X \times X \to X$ is a function, such that $\phi_i = i \triangleright (\cdot) : X \to X$ is a bijection for all $i \in X$ satisfying that $i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k)$ for all $i,j,k \in X$. A group G is a rack with $x \triangleright y = xyx^{-1}$ for all $x, y \in G$. If $G = \mathbb{S}_n$, then we denote by $\mathcal{O}_j^n$ the conjugacy class of all j-cycles in $\mathbb{S}_n$.

Let $(X, \triangleright)$ be a rack. A *rack 2-cocycle* $q : X \times X \to \Bbbk^\times$, $(i,j) \mapsto q_{ij}$ is a function such that $q_{i,j\triangleright k}\, q_{j,k} = q_{i\triangleright j, i\triangleright k}\, q_{i,k}$, for all $i,j,k \in X$. It determines a braiding c^q on the vector space $\Bbbk X$ with basis $\{x_i\}_{i\in X}$ by $c^q(x_i \otimes x_j) = q_{ij} x_{i\triangleright j} \otimes x_i$ for all $i,j \in X$. We denote by $\mathfrak{B}(X,q)$ the Nichols algebra of this braided vector space $(\Bbbk X, c^q)$.

Let $X = \mathcal{O}_2^n$ with $n \geq 3$ or $X = \mathcal{O}_4^4$ and consider the cocycles:

$$-1 : X \times X \to \Bbbk^\times, \qquad (j,i) \mapsto \mathrm{sg}(j) = -1, \qquad i,j \in X;$$
$$\chi : \mathcal{O}_2^n \times \mathcal{O}_2^n \to \Bbbk^\times, \qquad (j,i) \mapsto \chi_i(j) = \begin{cases} 1, & \text{if } i = (a,b) \text{ and } j(a) < j(b), \\ -1, & \text{if } i = (a,b) \text{ and } j(a) > j(b). \end{cases} \qquad i,j \in \mathcal{O}_2^n.$$

By [MS, Ex. 6.4], [Gr], [AG2, Thm. 6.12], [GG, Prop. 2.5], the Nichols algebras are given by

(*a*) $\mathfrak{B}(\mathcal{O}_2^n, -1)$; generated by the elements $\{x_{(\ell m)}\}_{1\leq \ell < m \leq n}$ satisfying for all $1 \leq a < b < c \leq n, 1 \leq e < f \leq n, \{a,b\} \cap \{e,f\} = \emptyset$ that

$$0 = x_{(ab)}^2 = x_{(ab)}x_{(ef)} + x_{(ef)}x_{(ab)} = x_{(ab)}x_{(bc)} + x_{(bc)}x_{(ac)} + x_{(ac)}x_{(ab)}.$$

(*b*) $\mathfrak{B}(\mathcal{O}_2^n,\chi)$; generated by the elements $\{x_{(\ell m)}\}_{1\le \ell<m\le n}$ satisfying for all $1\le a<b<c\le n, 1\le e<f\le n, \{a,b\}\cap\{e,f\}=\emptyset$ that

$$0 = x_{(ab)}^2 = x_{(ab)}x_{(ef)} - x_{(ef)}x_{(ab)} = x_{(ab)}x_{(bc)} - x_{(bc)}x_{(ac)} - x_{(ac)}x_{(ab)},$$
$$0 = x_{(bc)}x_{(ab)} - x_{(ac)}x_{(bc)} - x_{(ab)}x_{(ac)}.$$

(*c*) $\mathfrak{B}(\mathcal{O}_4^4,-1)$; generated by the elements $x_i, i\in\mathcal{O}_4^4$ satisfying for $ij=ki$ and $j\neq i\neq k\in\mathcal{O}_4^4$ that

$$0 = x_i^2 = x_ix_{i^{-1}} + x_{i^{-1}}x_i = x_ix_j + x_kx_i + x_jx_k.$$

Remark 4.1. These Nichols algebras can be seen as Nichols algebras over $\mathbb{S}_n$ by a principal YD-realization (see [AG2, Def. 3.2], [MS, Sec. 5]) of $(\mathcal{O}_2^n,-1)$, $(\mathcal{O}_2^n,\chi)$ over $\mathbb{S}_n$ or $(X,q)=(\mathcal{O}_4^4,-1)$ over $\mathbb{S}_4$. In particular, if we denote the elements of $\mathbb{S}_n$ by h_τ and the elements of X by x_σ with $\sigma\in\mathcal{O}_k^n$, $k=2,4$, then the action and coaction of $\mathbb{S}_n$ are determined by:

$$\delta(x_\tau) = h_\tau\otimes x_\tau,\quad h_\theta\cdot x_\tau = \chi_\tau(h_\theta)\,x_{\theta\triangleright\tau} \text{ for all } \tau\in X,\ \theta\in\mathbb{S}_n. \tag{28}$$

4.2. **Classification of finite dimensional pointed Hopf algebras over $\mathbb{S}_3$ and $\mathbb{S}_4$.** We follow [AG2, Def. 3.7] and [GG, 3.9, 3.10]. Let $\Lambda,\Gamma,\lambda\in\Bbbk$ and $t=(\Lambda,\Gamma)$. For $\theta,\tau\in\mathbb{S}_n$ denote $\theta\triangleright\tau=\theta\tau\theta^{-1}$ the conjugation in $\mathbb{S}_n$.

Definition 4.2. $\mathcal{H}(\mathcal{Q}_n^{-1}[t])$ is the algebra generated by $\{a_i,h_r : i\in\mathcal{O}_2^n, r\in\mathbb{S}_n\}$ satisfying the following relations for $r,s,j\in\mathbb{S}_n$ and $i\in\mathcal{O}_2^n$:

$$h_e=1,\quad h_rh_s=h_{rs},\qquad h_ja_i=-a_{j\triangleright i}h_j,\quad a_{(12)}^2=0,$$
$$a_{(12)}a_{(34)}+a_{(34)}a_{(12)}=\Lambda(1-h_{(12)}h_{(34)}),$$
$$a_{(12)}a_{(23)}+a_{(23)}a_{(13)}+a_{(13)}a_{(12)}=\Gamma(1-h_{(12)}h_{(23)}).$$

Definition 4.3. $\mathcal{H}(\mathcal{Q}_n^{\chi}[\lambda])$ is the algebra generated by $\{a_i,h_r : i\in\mathcal{O}_2^n, r\in\mathbb{S}_n\}$ satisfying the following relations for $r,s,j\in\mathbb{S}_n$ and $i\in\mathcal{O}_2^n$:

$$h_e=1,\quad h_rh_s=h_{rs},\quad h_ja_i=\chi_i(j)\,a_{j\triangleright i}h_j,\quad a_{(12)}^2=0,$$
$$a_{(12)}a_{(34)}-a_{(34)}a_{(12)}=0$$
$$a_{(12)}a_{(23)}-a_{(23)}a_{(13)}-a_{(13)}a_{(12)}=\lambda(1-h_{(12)}h_{(23)}).$$

Definition 4.4. $\mathcal{H}(\mathcal{D}[t])$ is the algebra generated by $\{a_i,h_r : i\in\mathcal{O}_4^4, r\in\mathbb{S}_4\}$ satisfying the following relations for $r,s,j\in\mathbb{S}_n$ and $i\in\mathcal{O}_4^4$:

$$h_e=1,\quad h_rh_s=h_{rs},\quad h_ja_i=-a_{j\triangleright i}h_j,\quad a_{(1234)}^2=\Lambda(1-h_{(13)}h_{(24)}),$$
$$a_{(1234)}a_{(1432)}+a_{(1432)}a_{(1234)}=0,$$
$$a_{(1234)}a_{(1243)}+a_{(1243)}a_{(1423)}+a_{(1423)}a_{(1234)}=\Gamma(1-h_{(12)}h_{(13)}).$$

Remark 4.5. For each quadratic lifting datum $\mathcal{Q}=\mathcal{Q}_n^{-1}[t],\mathcal{Q}_n^{\chi}[\lambda],\mathcal{D}[t]$, the algebra $\mathcal{H}(\mathcal{Q})$ has a structure of a pointed Hopf algebra setting

$$\Delta(h_t)=h_t\otimes h_t \text{ and } \Delta(a_i)=a_i\otimes 1+h_i\otimes a_i,\quad t\in\mathbb{S}_n, i\in X. \tag{29}$$

Moreover, they satisfy that $\operatorname{gr}\mathcal{H}(\mathcal{Q})=\mathfrak{B}(X,q)\#\Bbbk\mathbb{S}_n$, with n as appropiate see [GG].

The following theorem summarizes the classification of finite dimensional pointed Hopf algebras over $\mathbb{S}_3$ and $\mathbb{S}_4$, see [AHS], [GG].

Theorem 4.6. *Let H be a nontrivial finite dimensional pointed Hopf algebra with $G(H)=\mathbb{S}_n$.*

(*i*) *If* $n=3$, *then either* $H \simeq \mathfrak{B}(\mathcal{O}_2^3, -1)\#\Bbbk\mathbb{S}_3$ *or* $H \simeq \mathcal{H}(\mathcal{Q}_3^{-1}[(0,1)])$.
(*ii*) *If* $n=4$, *then either* $H \simeq \mathfrak{B}(X,q)\#\Bbbk\mathbb{S}_4$ *with* $(X,q) = (\mathcal{O}_2^4, -1)$, $(\mathcal{O}_4^4, -1)$ *or* $(\mathcal{O}_2^4, \chi)$, *or* $H \simeq \mathcal{H}(\mathcal{Q}_4^{-1}[t])$, *or* $H \simeq \mathcal{H}(\mathcal{Q}_4^{\chi}[1])$, *or* $H \simeq \mathcal{H}(\mathcal{D}[t])$ *with* $t \in \mathbb{P}^1_{\Bbbk}$. □

4.3. **Cocycle deformations and pointed Hopf algebras over $\mathbb{S}_n$.** In the following we construct multiplicative 2-cocycles and show that some families of the pointed Hopf algebras $\mathcal{H}(\mathcal{Q}_n^{-1}[t])$ and $\mathcal{H}(\mathcal{D}[t])$ are cocycle deformations of bosonizations of Nichols algebras in ${}^{\mathbb{S}_n}_{\mathbb{S}_n}\mathcal{YD}$. As a consequence, we provide the family of cocycles needed to construct all finite dimensional pointed Hopf algebras over $\mathbb{S}_3$ up to isomorphism.

Let $X = \mathcal{O}_2^n$ or $\mathcal{O}_4^4$ and denote the generators of $\mathfrak{B}(X,-1)$ by x_τ with $\tau \in X$. Define the ε-derivations d_τ by $d_\tau(x_\mu) = \delta_{\tau,\mu}$ for all $\sigma,\tau \in X$ and consider the ε-biderivation

$$\eta = \sum_{\mu,\tau \in X} \alpha_{\tau,\mu} d_\tau \otimes d_\mu.$$

The proof of the following lemma follows by a direct computation.

Lemma 4.7. *η is $\mathbb{S}_n$-invariant if and only if $\alpha_{\tau,\mu} = \alpha_{\theta\triangleright\tau,\theta\triangleright\mu}$ for all $\tau,\mu \in X$ and $\theta \in \mathbb{S}_n$.* □

Remark 4.8. Consider the set $\mathcal{T} = X \times X$. Then $\mathbb{S}_n$, and in particular X, acts by conjugation on $\mathcal{T}$ by $\theta \cdot (\tau,\mu) = (\theta \triangleright \tau, \theta \triangleright \mu)$. If we set $\alpha : \mathcal{T} \to \Bbbk$ with $\alpha(\tau,\mu) = \alpha_{\tau,\mu}$, then the coefficients of η are given by the function α and by Lemma 4.7, η is $\mathbb{S}_n$-invariant if and only if α is a class function, *i.e.* it is constant on each conjugacy class. Since (τ,μ) is conjugate to (τ',μ') if and only if $\tau\mu$ is conjugate to $\tau'\mu'$ in $\mathbb{S}_n$, if η is $\mathbb{S}_n$-invariant, we may write in the case $X = \mathcal{O}_2^n$

$$\eta = \beta_{\mathrm{id}} \sum_{\tau \in \mathcal{O}_2^n} d_\tau \otimes d_\tau + \beta_{(123)} \sum_{\substack{\tau,\mu \in \mathcal{O}_2^n \\ \tau\mu \in \mathcal{O}_3^n}} d_\tau \otimes d_\mu + \beta_{(12)(34)} \sum_{\substack{\tau,\mu \in \mathcal{O}_2^n \\ \tau\mu \in \mathcal{O}_{2,2}^n}} d_\tau \otimes d_\mu, \tag{30}$$

with $\beta_{\mathrm{id}}, \beta_{(123)}, \beta_{(12)(34)} \in \Bbbk$, and in the case $X = \mathcal{O}_4^4$

$$\eta = \gamma_{\mathrm{id}} \sum_{\tau \in \mathcal{O}_4^4} d_\tau \otimes d_{\tau^{-1}} + \gamma_{(123)} \sum_{\substack{\tau,\mu \in \mathcal{O}_4^4 \\ \tau\mu \in \mathcal{O}_3^4}} d_\tau \otimes d_\mu + \gamma_{(12)(34)} \sum_{\tau \in \mathcal{O}_4^4} d_\tau \otimes d_\tau, \tag{31}$$

with $\gamma_{\mathrm{id}}, \gamma_{(123)}, \gamma_{(12)(34)} \in \Bbbk$.

Assume η satisfies (30) or (31). The next lemma states that the exponentiation of the lifting of η is a multiplicative 2-cocycle if all coefficients β or γ are equal.

Lemma 4.9. *Assume $\eta = \sum_{\mu,\tau \in X} \alpha_{\tau,\mu} d_\tau \otimes d_\mu$ is an $\mathbb{S}_n$-invariant ε-biderivation. Then η satisfies equations* (6) *and* (7) *if and only if $\alpha_{\tau,\mu} = \alpha_{\tau',\mu'}$ for all $\tau,\tau',\mu,\mu' \in X$. In such a case, $\sigma = e^{\tilde{\eta}}$ is a multiplicative* 2*-cocycle for $\mathfrak{B}(X,-1)\#\Bbbk\mathbb{S}_n$.*

Proof. By Lemma 2.4, we need only to verify equations (6) and (7) on $V = \Bbbk X$. Since $\chi_\tau(h_\mu) = \mathrm{sg}(\mu) = -1$ for all $\tau,\mu \in \mathcal{O}_2^n$ and $\chi_\tau(h_\mu) = 1$ for all $\tau,\mu \in \mathcal{O}_4^4$ these equations on x_r, x_s, x_t, x_u with $r,s,t,u \in X$ equal:

$$\eta(x_r, x_{s\triangleright t})\eta(x_s, x_u) = \eta(x_{r\triangleright s}, x_{r\triangleright(t\triangleright u)})\eta(x_r, x_t), \tag{6}$$

$$\eta(x_r, x_{s\triangleright(t\triangleright u)})\eta(x_s, x_t) = \eta(x_{r\triangleright s}, x_{r\triangleright t})\eta(x_r, x_u). \tag{7}$$

It is clear that if $\eta = \lambda \sum_{\mu,\tau \in X} d_\tau \otimes d_\mu$ for some $\lambda \in \Bbbk$, then both equations are satisfied. Conversely, assume η satisfies (6) and (7). Since $t \triangleright -$ is a bijection for all $t \in X$, by (7) we have that $\alpha_{r,s\triangleright(t\triangleright u)}\alpha_{s,t} = \alpha_{s,t}\alpha_{r,u}$ for all $r,s,t,u \in X$. If $\alpha_{s,t} \neq 0$ for some $s,t \in X$, then $\alpha_{r,u} = \alpha_{r,s\triangleright u}$ for all $r,s,u \in X$. Since η must satisfy (30) or (31), it follows that $\eta = \lambda \sum_{\mu,\tau \in X} d_\tau \otimes d_\mu$ for some $\lambda \in \Bbbk$. The rest of the claim follows now by Lemma 2.4. □

Theorem 4.10. *Let $H = \mathfrak{B}(X,-1)\#\Bbbk\mathbb{S}_n$ and $\sigma = e^{\tilde{\eta}}$ be the multiplicative $2-$cocycle given by* Lemma 4.9 *with $\eta = \frac{\lambda}{3}\sum_{\mu,\tau\in\mathcal{O}_2^n} d_\tau \otimes d_\mu$ and $\lambda \in \Bbbk$.*

(i) If $X = \mathcal{O}_2^n$ then $H_\sigma \simeq \mathcal{H}(\mathcal{Q}_3^{-1}[(0,\lambda)])$ for $n=3$ and $H_\sigma \simeq \mathcal{H}(\mathcal{Q}_n^{-1}[(2\lambda,3\lambda)])$ for $n \geq 4$.

(ii) If $X = \mathcal{O}_4^4$ then $H_\sigma \simeq \mathcal{H}(\mathcal{D}[(\lambda,3\lambda)])$.

In particular, $\mathcal{H}(\mathcal{Q}_3^{-1}[(0,\lambda)])$ is a cocycle deformation of H for all $\lambda \in \Bbbk$.

Proof. As in the proof of Theorems 3.6 and 3.11, it suffices to show that the generators of H_σ satisfy the relations given in Definitions 4.2 and 4.4, respectively. For $\tau \in X$, let $a_\tau = x_\tau\#1 \in H$. Then by Lemma 2.3 we have for all $\tau,\mu \in \mathcal{O}_2^n$ that

$$a_\tau \cdot_\sigma a_\mu = \eta(x_\tau,x_\mu)(1-h_\tau h_\mu) + a_\tau a_\mu = \lambda(1-h_{\tau\mu}) + a_\tau a_\mu.$$

Hence, if $X = \mathcal{O}_2^n$ we get that $a_{(12)} \cdot_\sigma a_{(12)} = a_{(12)}^2 + \frac{\lambda}{3}(1-h_{(12)(12)}) = \frac{\lambda}{3}(1-h_e) = 0$ and

$$\begin{aligned}
&a_{(12)} \cdot_\sigma a_{(23)} + a_{(23)} \cdot_\sigma a_{(13)} + a_{(13)} \cdot_\sigma a_{(12)} = \\
&\quad = a_{(12)}a_{(23)} + a_{(23)}a_{(13)} + a_{(13)}a_{(12)} + \frac{\lambda}{3}(1-h_{(12)(23)}) + \frac{\lambda}{3}(1-h_{(23)(13)}) + \frac{\lambda}{3}(1-h_{(13)(12)}) \\
&\quad = \lambda(1-h_{(123)}) = \lambda(1-h_{(12)(23)}).
\end{aligned}$$

Taking $\Gamma = \lambda$, this implies that $H_\sigma \simeq \mathcal{H}(\mathcal{Q}_3^{-1}[(0,\lambda)])$ if $n=3$, since both algebras have the same dimension. For $n \geq 4$ we need to verify the extra relation involving the product of two disjoint transpositions:

$$\begin{aligned}
&a_{(12)} \cdot_\sigma a_{(34)} + a_{(34)} \cdot_\sigma a_{(12)} = a_{(12)}a_{(34)} + a_{(34)}a_{(12)} + \frac{\lambda}{3}(1-h_{(12)(34)}) + \frac{\lambda}{3}(1-h_{(34)(12)}) = \\
&\quad = \frac{2\lambda}{3}(1-h_{(12)(34)}).
\end{aligned}$$

Thus taking $t = (\Lambda,\Gamma) = (\frac{2\lambda}{3},\lambda)$, we have that $H_\sigma \simeq \mathcal{H}(\mathcal{Q}_n^{-1}[(2\lambda,3\lambda)])$. Assume $X = \mathcal{O}_4^4$, then

$$\begin{aligned}
&a_{(1234)} \cdot_\sigma a_{(1234)} = a_{(1234)}^2 + \frac{\lambda}{3}(1-h_{(1234)(1234)}) = \frac{\lambda}{3}(1-h_{(13)(24)}), \\
&a_{(1234)} \cdot_\sigma a_{(1432)} + a_{(1432)} \cdot_\sigma a_{(1234)} = a_{(1234)}a_{(1432)} + a_{(1432)}a_{(1234)} + \\
&\qquad + \frac{\lambda}{3}(1-h_{(1234)(1432)}) + \frac{\lambda}{3}(1-h_{(1432)(1234)}) = \frac{2\lambda}{3}(1-h_e) = 0, \\
&a_{(1234)} \cdot_\sigma a_{(1243)} + a_{(1243)} \cdot_\sigma a_{(1423)} + a_{(1423)} \cdot_\sigma a_{(1234)} = \\
&\quad = a_{(1234)}a_{(1243)} + a_{(1243)}a_{(1423)} + a_{(1423)}a_{(1234)} + \\
&\qquad + \frac{\lambda}{3}(1-h_{(1234)(1243)}) + \frac{\lambda}{3}(1-h_{(1243)(1423)}) + \frac{\lambda}{3}(1-h_{(1423)(1234)}) = \lambda(1-h_{(12)(13)}).
\end{aligned}$$

Therefore, taking $t = (\Lambda,\Gamma) = (\frac{\lambda}{3},\lambda)$, we have that $H_\sigma \simeq \mathcal{H}(\mathcal{D}[(\lambda,3\lambda)]))$. □

Remark 4.11. *Cocycle deformations and the algebras $\mathcal{H}(\mathcal{Q}_n^\chi[\lambda])$.* As shown in [GIM], the pointed Hopf algebras $\mathcal{H}(\mathcal{Q}_n^\chi[\lambda])$ are cocycle deformations of the bosonizations $\mathfrak{B}(\mathcal{O}_2^n,\chi)\#\Bbbk\mathbb{S}_n$. Regrettably, our construction using $\mathbb{S}_n$-invariant ε-bideriations in $\mathfrak{B}(\mathcal{O}_2^n,\chi)$ only provides the trivial deformation.

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