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AN ASYMPTOTIC FORMULA FOR REPRESENTATIONS OF
INTEGERS BY INDEFINITE HERMITIAN FORMS

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ABSTRACT. We fix a maximal order \mathcal{O} in $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} , and an \mathbb{F} -hermitian form Q of signature $(n, 1)$ with coefficients in \mathcal{O} . Let $k \in \mathbb{N}$. By applying a lattice point theorem on n -dimensional \mathbb{F} -hyperbolic space, we give an asymptotic formula with an error term, as $t \rightarrow +\infty$, for the number $N_t(Q, -k)$ of integral solutions $x \in \mathcal{O}^{n+1}$ of the equation $Q[x] = -k$ satisfying $|x_{n+1}| \leq t$.

1. INTRODUCTION

The representation theory by quadratic forms has a long history. It starts with the qualitative problem of determining which integers are represented by a given quadratic form. For example, Fermat, Legendre and Lagrange dealt with the problem of representation as a sum of two, three and four squares respectively. After that, the quantitative problem was considered by Jacobi, Kloosterman and Liouville among others. For instance, Jacobi proved that the number of ways a positive integer k can be written as a sum of four squares is

$$8 \sum_{m|k, 4 \nmid m} m,$$

by determining the Fourier coefficients of the theta function associated to the form $x_1^2 + \cdots + x_4^2$.

These examples are for positive definite quadratic forms. For indefinite forms, the literature is much less abundant than for the definite case. In the present work, we study the quantitative problem for quadratic and hermitian forms of signature $(n, 1)$.

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} , and let \mathcal{O} denote a maximal order in \mathbb{F} . Thus \mathcal{O} is the ring of integer numbers \mathbb{Z} in the real case, the ring of integers of an imaginary quadratic extension of \mathbb{Q} in the complex case (e.g. the Gaussian integers $\mathbb{Z}[\sqrt{-1}]$), and, for instance, the ring of Hurwitz integers if $\mathbb{F} = \mathbb{H}$, though there are many other choices. We consider in \mathbb{H} (and then by restriction in \mathbb{C}) the canonical involution $\alpha \mapsto \bar{\alpha}$. An \mathbb{F} -hermitian matrix is a square matrix with coefficients in \mathbb{F} that is equal to its own conjugate transpose.

We consider an \mathbb{F} -hermitian matrix

$$(1.1) \quad Q = \begin{pmatrix} A & \\ & -a \end{pmatrix},$$

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with $a \in \mathbb{N}$ and $A \in M_n(\mathcal{O})$ a positive definite \mathbb{F} -hermitian matrix. We also denote by Q the induced \mathbb{F} -hermitian form of signature $(n, 1)$,

$$Q[x] := x^* Q x = A[\hat{x}] - a |x_{n+1}|^2, \quad x \in \mathbb{F}^{n+1},$$

where $\hat{x} = (x_1, \dots, x_n)^t \in \mathbb{F}^n$.

We consider, for $k \in \mathbb{N}$, the solution vectors $x \in \mathcal{O}^{n+1}$ of the equation

$$(1.2) \quad Q[x] = -k.$$

Put

$$(1.3) \quad \mathcal{R}(Q, -k) = \{x \in \mathcal{O}^{n+1} : Q[x] = -k\}.$$

Since Q is an indefinite form, this set is either empty or infinite. From now on, we will assume it is not empty, i.e. that $-k$ is represented by Q . Consequently, in order to study the quantitative behavior, one needs to impose additional restrictions to the integral solutions of (1.2).

A natural condition is to intersect the set in (1.3) with Euclidean balls $\{x \in \mathbb{F}^{n+1} : \|x\| \leq s\}$ in \mathbb{F}^{n+1} , with $\|x\|^2 := A[\hat{x}] + a |x_{n+1}|^2$. Note that this norm is induced by the positive definite \mathbb{F} -hermitian matrix $\begin{pmatrix} A & \\ & a \end{pmatrix}$.

A related condition consists of requiring that the solutions of (1.2) satisfy the bound $|x_{n+1}| \leq t$. We have, for $x \in \mathcal{R}(Q, -k)$, that

$$(1.4) \quad \|x\| \leq s \quad \text{if and only if} \quad |x_{n+1}| \leq \sqrt{\frac{s^2 + k}{2}}.$$

In this paper we establish an asymptotic formula, for large t , for

$$(1.5) \quad N_t(Q, -k) := \#\{x \in \mathcal{R}(Q, -k) : |x_{n+1}| \leq t\},$$

the number of integral solutions of the equation $Q[x] = -k$ satisfying the additional restriction $|x_{n+1}| \leq t$.

Our main result, Theorem 5.1, is the asymptotic formula

$$(1.6) \quad N_t(Q, -k) = \frac{2^{(r-1)(n+1)} a^\rho \text{vol}(S^{nr-1})}{|d_{\mathcal{O}}|^{\frac{n+1}{2}}} \frac{\pi^{\frac{r}{2}}}{2\rho |\det Q|^{\frac{r}{2}} \Gamma(\frac{r}{2})} \delta(Q, -k) t^{2\rho} + O(t^\tau),$$

as $t \rightarrow +\infty$, for $\mathbb{F} = \mathbb{R}$ or \mathbb{C} . Here $r = \dim_{\mathbb{R}}(\mathbb{F})$, $\rho = (n+1)r/2 - 1$, $d_{\mathcal{O}}$ is the discriminant of the quotient field of \mathcal{O} and $\delta(Q, -k)$ is the local density of the representation (1.2) (see (4.6)). The number τ is defined in (3.4). It depends only on the form Q , or more precisely, on the first nonzero eigenvalue of the Laplace-Beltrami operator on $\Gamma_Q^0 \backslash \mathbb{H}_{\mathbb{F}}^n$, where Γ_Q^0 is a subgroup of the group of unimodular matrices (see 3.7) and $\mathbb{H}_{\mathbb{F}}^n$ is the n -dimensional \mathbb{F} -hyperbolic space. When $\mathbb{F} = \mathbb{R}$ and $n \geq 3$, formula (1.6) holds with $\tau = n - 3/2$ (note that $2\rho = n - 1$).

In the particular case when $\mathbb{F} = \mathbb{R}$ and $Q = I_{n,1} = \begin{pmatrix} I_n & \\ & -1 \end{pmatrix}$, the main theorem asserts that the number $N_t(I_{n,1}, -k)$ of vectors $x \in \mathbb{Z}^{n+1}$ such that

$$x_1^2 + \dots + x_n^2 - x_{n+1}^2 = -k \quad \text{and} \quad |x_{n+1}| \leq t,$$

satisfies the following asymptotic estimate, as $t \rightarrow +\infty$,

$$N_t(I_{n,1}, -k) = \frac{\text{vol}(S^{n-1})}{n-1} \delta(I_{n,1}, -k) t^{n-1} + O(t^{n-3/2}).$$

This formula is due to J. Ratcliffe and S. Tschantz [RT97]. The present article was inspired by this work. They also considered $k < 0$ obtaining the same formula for the leading coefficient without an error term. Although a similar result should hold

for $k < 0$ in our context, we will not develop this case here since the tools needed are very different.

Concerning the question of counting integral solutions inside Euclidean balls, we define

$$(1.7) \quad \tilde{N}_s(Q, -k) := \#\{x \in \mathcal{R}(Q, -k) : \|x\| \leq s\}.$$

By applying formula (1.6) and relation (1.4) (see Remark 5.3), one shows that

$$(1.8) \quad \tilde{N}_s(Q, -k) = 2^{-\rho} C_{Q,k} s^{2\rho} + O(s^\tau),$$

where $C_{Q,k}$ denotes the main coefficient in (1.6). Related results for the leading term of the counting function of integral points in quite general algebraic varieties lying in balls of increasing radius as $s \rightarrow +\infty$, have been obtained, for example, in [DRS93] and [BR95].

In the absence of nonzero exceptional eigenvalues, formulas (1.6) and (1.8) become respectively

$$\begin{aligned} N_t(Q, -k) &= C_{Q,k} t^{2\rho} + O\left(t^{2\rho(1-\frac{1}{n+1})+\varepsilon}\right), \\ \tilde{N}_s(Q, -k) &= 2^{-\rho} C_{Q,k} s^{2\rho} + O\left(s^{2\rho(1-\frac{1}{n+1})+\varepsilon}\right), \end{aligned}$$

with $\varepsilon = 0$ if $\mathbb{F} = \mathbb{R}$ and any $\varepsilon > 0$ if $\mathbb{F} = \mathbb{C}$. We recall that $2\rho = r(n+1) - 2$, where $r = \dim_{\mathbb{R}}(\mathbb{F})$.

Our main tool in this paper is the hyperbolic lattice point theorem of P. Lax and R. Phillips [LP82] in the real case (improved by B. M. Levitan [Le87]), and of R. Bruggeman, R. Miatello and N. Wallach [BMW99] in the general case ($\mathbb{F} = \mathbb{R}, \mathbb{C}$ and \mathbb{H}). For $\mathbb{F} = \mathbb{R}$ we use the best lower bound known for the first eigenvalue of the Laplace-Beltrami operator on $\Gamma_Q^0 \backslash \mathbb{H}_{\mathbb{R}}^n$ (see Theorem 3.3), which was obtained in [EGM90] and in [CLPS91].

After applying the lattice point theorem we obtain (see Proposition 3.6)

$$(1.9) \quad N_t(Q, -k) = C'_{Q,k} \left(\sum_{y \in F} |\Gamma_{Q,y}|^{-1} \right) t^{2\rho} + O(t^\tau),$$

where $C'_{Q,k}$ is a constant depending only on Q and k , F is a representative set of $\mathcal{R}(Q, -k)$ under the action of the group of unimodular matrices Γ_Q (see 2.6) and $\Gamma_{Q,y}$ is the stabilizer of y in Γ_Q .

When $\mathbb{F} = \mathbb{R}$ or \mathbb{C} , by using the theory of C. L. Siegel [Sie44] for indefinite quadratic forms and its generalization for indefinite complex hermitian forms given in [Rag62], formula (1.9) can be made more explicit, yielding our main formula (1.6). This is carried out by expressing the term $\sum_{y \in F} |\Gamma_{Q,y}|^{-1}$ as a product of the local density $\delta(Q, -k)$ times a constant which depends only on Q and k (see Corollary 4.3). The derivation of a formula like (1.6) from (1.9) in the quaternionic case, would first require developing for indefinite \mathbb{H} -hermitian forms the classical theory due to Siegel, as done by Raghavan (see [Rag62]) for indefinite \mathbb{C} -hermitian forms.

The paper is organized as follows. In Section 2 we introduce the geometric context, relating $N_t(Q, -k)$ with the number of elements in an arithmetic subgroup of $\text{Iso}(\mathbb{H}_{\mathbb{F}}^n)$ satisfying a geometric condition. In Section 3, we apply the lattice point theorem to count such lattice points. Section 4 uses Siegel's theory to compute the main term of the formula. We conclude with Section 5, which contains the main theorem, together with some examples and remarks.

2. \mathbb{F} -HYPERBOLIC SPACE

Throughout the paper, given R a ring with identity, we denote by $M(m, n; R)$ the set of $m \times n$ matrices with coefficients in R , just $M(m, R)$ when $n = m$, by $GL(m, \mathbb{F})$ the *general linear group* and by $SL(m, \mathbb{F})$ its derived group, the *special linear group*. For a matrix $C \in M(m, l; \mathbb{F})$ we denote by C^* its conjugate transpose and write $B[C] = C^*BC$, where $B \in M(m, \mathbb{F})$. Let R^m denote the right R -module $M(m, 1; R)$. For $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and \mathbb{H} , let $\mathbb{P}\mathbb{F}^n$ be the n -dimensional projective space over \mathbb{F} , i.e. $\mathbb{P}\mathbb{F}^n = \mathbb{F}^{n+1} \setminus \{0\} / \mathbb{F}^\times$ where \mathbb{F}^\times denotes the nonzero elements of \mathbb{F} .

We now introduce a model for Riemannian symmetric spaces of real rank one and negative curvature (leaving out the Cayley plane). These are the real, complex and quaternionic hyperbolic spaces. For a general reference on this subject see [BH99, II.§10] and [Mo73, §19].

Let Q be the matrix defined as in (1.1). The set

$$(2.1) \quad \mathbb{H}_{\mathbb{F}}^n(Q) = \{[x] \in \mathbb{P}\mathbb{F}^n : Q[x] < 0\}$$

will serve as the set of points for the *Q-Kleinian model* of n -dimensional \mathbb{F} -hyperbolic geometry. Note that the condition $Q[x] < 0$ is well defined on the projective space. The distance function is defined by

$$(2.2) \quad \cosh(d([x], [y])) = \frac{|Q(x, y)|}{|Q[x]|^{1/2} |Q[y]|^{1/2}},$$

where $Q(x, y) = x^*Qy$.

We consider the \mathbb{F} -vector space \mathbb{F}^{n+1} endowed with the form $Q(x, y) = x^*Qy$ of type $(n, 1)$. Let $x^\perp = \{u \in \mathbb{F}^{n+1} : Q(x, u) = 0\}$ be the *Q-orthogonal complement* of $x \in \mathbb{F}^{n+1}$. If $Q[x] < 0$, then the restriction of Q to x^\perp is positive definite. We identify x^\perp with $T_{[x]}\mathbb{H}_{\mathbb{F}}^n(Q)$ using the differential of the natural projection $\mathbb{F}^{n+1} \setminus \{0\} \rightarrow \mathbb{H}_{\mathbb{F}}^n(Q)$. We consider the symmetric positive definite \mathbb{R} -bilinear form on $T_{[x]}\mathbb{H}_{\mathbb{F}}^n(Q)$ given by

$$(2.3) \quad (u, v) = \frac{\operatorname{Re}(Q(u, v))}{|Q[x]|}, \quad x, y \in x^\perp.$$

In this way $\mathbb{H}_{\mathbb{F}}^n(Q)$ is naturally a Riemannian manifold. One can check that the metric associated is (2.2). Moreover, this metric gives constant curvature -1 in the real case, and pinched sectional curvature in the interval $[-4, -1]$ if $\mathbb{F} = \mathbb{C}, \mathbb{H}$.

We denote by

$$U(Q, \mathbb{F}) = \{g \in GL(n+1, \mathbb{F}) : Q[g] = Q\}$$

the *Q-unitary group* and by $SU(Q, \mathbb{F}) = U(Q, \mathbb{F}) \cap SL(n+1, \mathbb{F})$ the *special Q-unitary group*. For $Q = I_{n,1}$, it is well known the classical notation $U(I_{n,1}, \mathbb{F}) = O(n, 1)$, $U(n, 1)$, $Sp(n, 1)$ and $SU(I_{n,1}, \mathbb{F}) = SO(n, 1)$, $SU(n, 1)$, $Sp(n, 1)$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ respectively. The center of the *Q-unitary group* is given by

$$(2.4) \quad Z(U(Q, \mathbb{F})) = \begin{cases} \{\pm I_{n+1}\} & \text{if } \mathbb{F} = \mathbb{R}, \\ \{zI_{n+1} : |z| = 1\} \cong S^1 & \text{if } \mathbb{F} = \mathbb{C}, \\ \{\pm I_{n+1}\} & \text{if } \mathbb{F} = \mathbb{H}. \end{cases}$$

The group $U(Q, \mathbb{F})$ acts transitively on $\mathbb{H}_{\mathbb{F}}^n(Q)$ by

$$(2.5) \quad [x] \mapsto g \cdot [x] = [gx].$$

Indeed, by the distance formula (2.2), its elements act by isometries. Let $\text{Iso}(\mathbb{H}_{\mathbb{F}}^n(Q))$ (resp. $\text{Iso}^+(\mathbb{H}_{\mathbb{F}}^n(Q))$) denote the set of isometries (resp. orientation-preserving isometries) of $\mathbb{H}_{\mathbb{F}}^n(Q)$. It is clear that the elements of $Z(\text{U}(Q, \mathbb{F}))$ act as the identity map on $\mathbb{H}_{\mathbb{F}}^n(Q)$. Moreover, we have that

$$\{1\} \longrightarrow Z(\text{U}(Q, \mathbb{F})) \xrightarrow{\iota} \text{U}(Q, \mathbb{F}) \xrightarrow{\vartheta} \text{Iso}(\mathbb{H}_{\mathbb{F}}^n(Q))$$

is an exact sequence, where ι denotes the inclusion map and ϑ is defined by (2.5). Furthermore, the group $\text{PU}(Q, \mathbb{F}) := \text{U}(Q, \mathbb{F})/Z(\text{U}(Q, \mathbb{F}))$ is, up to finite index, the full isometry group $\text{Iso}(\mathbb{H}_{\mathbb{F}}^n(Q))$.

Remark 2.1. When $\mathbb{F} = \mathbb{R}$ the group $\text{O}(Q) := \text{U}(Q, \mathbb{R})$ has four connected components. The identity connected component is

$$\text{PSO}(Q) := \{g \in \text{SO}(Q) : g_{n+1, n+1} > 0\},$$

which is isomorphic to $\text{Iso}^+(\mathbb{H}_{\mathbb{R}}^n(Q))$.

When $\mathbb{F} = \mathbb{C}$, the group $\text{Iso}(\mathbb{H}_{\mathbb{C}}^n(Q))$ is generated by $\text{PU}(Q, \mathbb{C})$ and the conjugation $[x_1, \dots, x_{n+1}] \rightarrow [\bar{x}_1, \dots, \bar{x}_{n+1}]$. If $\mathbb{F} = \mathbb{H}$, then $\text{Iso}(\mathbb{H}_{\mathbb{H}}^n(Q)) \cong \text{PU}(Q, \mathbb{H})$ for $n > 1$.

We fix a *maximal order* \mathcal{O} in \mathbb{F} ; this means that the subset $\mathcal{O} \subset \mathbb{F}$ satisfies the following conditions:

- (i) \mathcal{O} is a lattice in \mathbb{F} (there is an \mathbb{R} -basis v_1, \dots, v_r of \mathbb{F} such that $\mathcal{O} = \mathbb{Z}v_1 \oplus \dots \oplus \mathbb{Z}v_r$);
- (ii) \mathcal{O} is a subring of \mathbb{F} containing 1;
- (iii) $2\text{Re}(a) \in \mathbb{Z}$ and $a\bar{a} \in \mathbb{Z}$ for $a \in \mathcal{O}$;

and \mathcal{O} is maximal among all orders (subsets of \mathbb{F} satisfying (i)-(iii)). Here $r := \dim_{\mathbb{R}}(\mathbb{F})$.

The elements of \mathcal{O} will be called *integers*. The ring of integers \mathbb{Z} is the only order in \mathbb{R} . In \mathbb{C} , the maximal orders are the rings of integers of the imaginary quadratic extensions $\mathbb{Q}(\sqrt{-D})$ ($D > 0$ squarefree) of \mathbb{Q} . More precisely, they are $\mathbb{Z}[\omega]$ where $\omega = (1 + \sqrt{-D})/2$ if $-D \equiv 1 \pmod{4}$ and $\omega = \sqrt{-D}$ otherwise. When $\mathbb{F} = \mathbb{H}$ there are many orders. As a canonical example in this case, the reader may take

$$\mathcal{O} = \left\{ a + bi + cj + dk \in \mathbb{H} : a, b, c, d \in \mathbb{Z} \text{ or } a, b, c, d \in \mathbb{Z} + \frac{1}{2} \right\},$$

the *Hurwitz integers*.

Let Γ_Q be the set of unimodular matrices in $\text{U}(Q, \mathbb{F})$, that is

$$(2.6) \quad \Gamma_Q = \text{U}(Q, \mathbb{F}) \cap \text{M}(n+1, \mathcal{O}).$$

This is a discrete subgroup of $\text{U}(Q, \mathbb{F})$ with finite center. The action of Γ_Q on $\mathbb{H}_{\mathbb{F}}^n(Q)$ is discontinuous, not free and the quotient $\Gamma_Q \backslash \mathbb{H}_{\mathbb{F}}^n(Q)$ is of finite volume and not compact.

On the other hand, the group Γ_Q acts by left multiplication on the set $\mathcal{R}(Q, k)$ given in (1.3).

Lemma 2.2. *The set of Γ_Q -orbits in $\mathcal{R}(Q, k)$ is finite.*

Proof. This assertion follows by applying [BH-S62, Thm. 6.9]. \square

From now on, we fix $k \in \mathbb{N}$ such that Q represents $-k$, that is, there exists $x \in \mathcal{O}^{n+1}$ satisfying $Q[x] = -k$. Let F be a (finite) set of representatives of the Γ_Q -orbits of $\mathcal{R}(Q, -k)$. Let $\Gamma_{Q, y}$ be the stabilizer of y in Γ_Q , which is finite. We

conclude this section by relating the number $N_t(Q, k)$ defined in (1.5), with the cardinality of subsets of lattice points in Γ_Q .

Proposition 2.3. *For $t > 0$, we have*

$$(2.7) \quad N_t(Q, -k) = \sum_{y \in F} |\Gamma_{Q,y}|^{-1} \# \{g \in \Gamma_Q : d([e_{n+1}], g \cdot [y]) \leq s\},$$

where $s = \operatorname{arccosh}(a^{1/2}k^{-1/2}t) > 0$.

Proof. Put

$$\mathcal{R}_t(Q, -k) = \{x \in \mathcal{R}(Q, -k) : |x_{n+1}| \leq t\}.$$

The cardinality of this set is $N_t(Q, -k)$. Let $x = (x_1, \dots, x_{n+1})^t \in \mathcal{R}(Q, -k)$. By (2.2),

$$(2.8) \quad \cosh(d([e_{n+1}], [x])) = a^{1/2}k^{-1/2}|x_{n+1}|.$$

Fix $t > 0$, thus $\cosh(s) = a^{1/2}k^{-1/2}t$. Let $g \in \Gamma_Q$ and $y \in F$ such that $gy = x$, then $g \cdot [y] = [x]$. Note that (2.8) tells that

$$|x_{n+1}| \leq t \quad \text{if and only if} \quad d([e_{n+1}], g \cdot [y]) \leq s.$$

But the condition on the left ensures that $x \in \mathcal{R}_t(Q, -k)$. We conclude that

$$(2.9) \quad \mathcal{R}_t(Q, -k) = \bigcup_{y \in F} \{gy : g \in \Gamma_Q \text{ and } d([e_{n+1}], g \cdot [y]) \leq s\}.$$

The proposition follows by counting the elements of these sets. \square

3. LATTICE POINT THEOREM

In this section we use lattice point theorems to determine the asymptotic distribution, for $t \rightarrow +\infty$, of the number of elements in the sets

$$\{g \in \Gamma_Q : d([e_{n+1}], g \cdot [y]) \leq s\}.$$

Let

$$(3.1) \quad G = \operatorname{SU}^0(Q, \mathbb{F}),$$

the identity connected component of $\operatorname{SU}(Q, \mathbb{F})$. We have $G = \operatorname{PSO}(Q)$ for $\mathbb{F} = \mathbb{R}$ (see Remark 2.1). When $\mathbb{F} = \mathbb{C}, \mathbb{H}$ the group $\operatorname{SU}(Q, \mathbb{F})$ is connected. Let \mathfrak{g} denote the Lie algebra $\mathfrak{su}(Q, \mathbb{F})$ of G . We have

$$\mathfrak{g} = \{X \in \operatorname{M}(n+1, \mathbb{F}) : X^*Q + QX = 0 \text{ (and } \operatorname{Tr}(X) = 0 \text{ if } \mathbb{F} = \mathbb{C})\}$$

Remark 3.1. If $T = \begin{pmatrix} L & \\ & \sqrt{a} \end{pmatrix}$, where $L^*L = A$, then $I_{n,1}[T] = T^*I_{n,1}T = Q$. It follows that the map $g \mapsto TgT^{-1}$ gives an isomorphism from $\operatorname{U}(Q, \mathbb{F})$ to $\operatorname{U}(n, 1; \mathbb{F}) := \operatorname{U}(I_{n,1}, \mathbb{F})$ and thus also from G to $\operatorname{SU}^0(n, 1; \mathbb{F})$. The corresponding isomorphism at the Lie algebra level $\mathfrak{su}(Q, \mathbb{F}) \rightarrow \mathfrak{su}(n, 1; \mathbb{F})$ is given by $X \mapsto TXT^{-1}$. This isomorphism allows us to consider the Cartan involution θ on \mathfrak{g} by pulling back the standard Cartan involution $X \mapsto -X^*$ on $\mathfrak{su}(n, 1; \mathbb{F})$. It is easy to check that $\theta(X) = -(T^*T)^{-1}X^*(T^*T)$.

The group G is a connected semisimple Lie group of real rank one and finite center. Let θ be the Cartan involution given in Remark 3.1, with corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let H_0 be the matrix in \mathfrak{g} defined as the pull back from $\mathfrak{su}(n, 1; \mathbb{F})$ of the matrix

$$\left(\begin{array}{c|c} & e_n \\ \hline e_n^* & \end{array} \right).$$

Let $\mathfrak{a} = \mathbb{R}H_0$, a maximal abelian subspace of \mathfrak{p} since G has real rank one. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and $G = NAK$ be the corresponding Iwasawa decomposition of \mathfrak{g} and G , respectively. Here K is a maximal compact subgroup of G , with Lie algebra \mathfrak{k} . Let M be the centralizer of A in K with Lie algebra \mathfrak{m} . Set $\zeta = \text{vol}(K/M)$. Let 2ρ denote the sum of the positive roots of G , thus $2\rho = (n+1)r - 2 = n-1, 2n, 4n+2$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ respectively (recall that $r = \dim_{\mathbb{R}}(\mathbb{F})$).

Let B_K denote the Killing form on \mathfrak{g} . We shall work with the inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} defined by

$$(3.2) \quad \langle X, Y \rangle = -\frac{1}{\xi_{\mathbb{F}}} B_K(X, \theta Y), \quad \text{where } \xi_{\mathbb{F}} = \begin{cases} 2(n-1) & \text{for } \mathbb{F}=\mathbb{R}, \\ 4(n+1) & \text{for } \mathbb{F}=\mathbb{C}, \\ 8(n+2) & \text{for } \mathbb{F}=\mathbb{H}. \end{cases}$$

A simple computation shows that $\langle X, Y \rangle = \frac{1}{2} \text{Tr}(X^* Y)$, thus $\langle H_0, H_0 \rangle = 1$. We consider on the homogeneous manifold G/K , the G -invariant Riemannian metric defined by the restriction of $\langle \cdot, \cdot \rangle$ to \mathfrak{p} .

One can check that the action of G on $\mathbb{H}_{\mathbb{F}}^n(Q)$ given in (2.5) is transitive, the element $[e_{n+1}]$ lies in $\mathbb{H}_{\mathbb{F}}^n(Q)$ and its stabilizer subgroup is K . Hence, the map $g \mapsto [ge_{n+1}]$ from G to $\mathbb{H}_{\mathbb{F}}^n(Q)$ gives rise to a G -equivariant bijection between the symmetric space G/K and $\mathbb{H}_{\mathbb{F}}^n(Q)$. Moreover, it follows by standard arguments that this bijection is already an isometry of Riemannian manifolds. This gives to $\mathbb{H}_{\mathbb{F}}^n(Q)$ the structure of a Riemannian symmetric space.

Let Γ be a non-cocompact lattice in G . Set $m_{\Gamma} = |\Gamma \cap Z(G)|$. Let $\text{vol}(\Gamma \backslash \mathbb{H}_{\mathbb{F}}^n(Q))$ denotes the volume on any fundamental domain in $\mathbb{H}_{\mathbb{F}}^n(Q)$ relative to Γ_Q . Let Δ be the Laplace-Beltrami operator on $\Gamma \backslash \mathbb{H}_{\mathbb{F}}^n(Q)$. We identify $-\Delta$ with the Casimir operator C of G , with respect to the inner product defined in (3.2). We fix a complete orthonormal set $\{\varphi_j\}$ of real valued eigenfunctions of C , with eigenvalues λ_j arranged in increasing order and exceptional eigenvalues $0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_N < \rho^2$, which we write as $\lambda_j = \rho^2 - \nu_j^2$, where $0 < \nu_N \leq \nu_{N-1} \leq \dots \leq \nu_1 < \rho$.

Now we can state the *hyperbolic lattice point theorem*. It was proved for the real hyperbolic space by Lax and Phillips [LP82], with an improved error term by Levitan [Le87], and generalized by Bruggeman, Miatello and Wallach [BMW99] for any symmetric space of real rank one.

Theorem 3.2. *In the notation above, for $[x], [y] \in \mathbb{H}_{\mathbb{F}}^n(Q)$, we have that*

$$(3.3) \quad \#\{g \in \Gamma : d([x], g \cdot [y]) \leq s\} = \frac{2^{1-n} m_{\Gamma} \zeta}{2\rho \text{vol}(\Gamma \backslash \mathbb{H}_{\mathbb{F}}^n(Q))} e^{2\rho s} \\ + 2^{1-n} m_{\Gamma} \zeta \sum_{j=1}^N \frac{c(\nu_j)}{\nu_j + \rho} \varphi_j(x) \varphi_j(y) e^{(\rho + \nu_j)s} + O\left(e^{(2\rho \frac{n}{n+1} + \varepsilon)s}\right)$$

as $s \rightarrow +\infty$, for $\varepsilon = 0$ when $\mathbb{F} = \mathbb{R}$ and for any $\varepsilon > 0$ otherwise. Here $c(\nu)$ is the Harish-Chandra c -function.

Note that the summation in (3.3) can be restricted to the indices j such that $\rho + \nu_j > 2\rho \frac{n}{n+1}$ (in the real case we can replace $>$ by \geq). Put

$$(3.4) \quad \tau = \begin{cases} \rho + \nu_1 & \text{if } \rho + \nu_1 \geq 2\rho \frac{n}{n+1} \text{ and } \mathbb{F} = \mathbb{R}, \\ \rho + \nu_1 & \text{if } \rho + \nu_1 > 2\rho \frac{n}{n+1} \text{ and } \mathbb{F} = \mathbb{C}, \mathbb{H}, \\ 2\rho \frac{n}{n+1} + \varepsilon & \text{otherwise,} \end{cases}$$

where ε is zero if $\mathbb{F} = \mathbb{R}$ or any positive value if $\mathbb{F} = \mathbb{C}, \mathbb{H}$. The last case includes the case when there are no exceptional eigenvalues. With this notation we can rewrite (3.3) as

$$(3.5) \quad \#\{g \in \Gamma : d([x], g \cdot [y]) \leq s\} = \frac{2^{1-n} m_\Gamma \zeta}{2\rho \operatorname{vol}(\Gamma \backslash \mathbb{H}_{\mathbb{F}}^n(Q))} e^{2\rho s} + O(e^{\tau s}).$$

We see that in (3.5) the error term depends on the first nonzero eigenvalue of the Laplace-Beltrami operator on $\Gamma \backslash \mathbb{H}_{\mathbb{F}}^n(Q)$. The following theorem was proved in [EGM90, Thm. A] (see also [CLPS91]). The notation $\operatorname{PSO}(Q)$ was introduced in Remark 2.1.

Theorem 3.3. *Let $n \geq 3$ and let Q_0 be a quadratic form with rational coefficients such that Q_0 is of signature $(n, 1)$ and isotropic over \mathbb{Q} . For any congruence subgroup $\Gamma < \operatorname{PSO}(Q_0)$, the first nonzero eigenvalue λ_1 for the Laplace-Beltrami operator on $\Gamma \backslash \mathbb{H}_{\mathbb{R}}^n(Q)$ satisfies $\lambda_1 \geq (2n - 3)/4$.*

Bounds of this kind are not known in the complex and quaternionic case (see [Li91, Cor. 1.4] for a related result in the complex case).

Remark 3.4. Under the assumptions of Theorem 3.3, the value of ν_1 satisfies

$$\nu_1 \leq \sqrt{\left(\frac{n-1}{2}\right)^2 - \frac{2n-3}{4}} = \frac{n-2}{2}.$$

Moreover, in the worst possible case $\nu_1 = \frac{n-2}{2}$, one has $\rho + \nu_1 \geq 2\rho \frac{n}{n+1}$ for all $n \geq 3$, which implies that (3.5) holds for $\tau = \frac{n-1}{2} + \frac{n-2}{2} = n - \frac{3}{2}$ since $\mathbb{F} = \mathbb{R}$.

Before applying Theorem 3.2 to our problem, we give the value of $\zeta = \operatorname{vol}(K/M)$.

Lemma 3.5. *Under the notation above, one has $\zeta = \operatorname{vol}(S^{nr-1})$.*

Proof. It is sufficient to prove the lemma for $Q = I_{n,1}$ since the isomorphism between G and $\operatorname{SU}^0(n, 1; \mathbb{F})$ given in Remark 3.1 preserves the Killing form and the inner product (3.2).

Set $S = \{H \in \mathfrak{p} : \langle H, H \rangle = 1\}$ with the Riemannian metric given by the real inner product $\langle \cdot, \cdot \rangle$ restricted to $\{X \in \mathfrak{p} : \langle X, H \rangle = 0\} \cong T_H S$. The adjoint representation restricted to K leaves the set S invariant, this action is transitive and the stabilizer subgroup of $H_0 \in S$ is M . Then, we have a K -equivariant bijection from the manifold K/M to S , that is already an isometry, considering the Riemannian metric on K/M given by (3.2) restricted to $\mathfrak{k} \cap \mathfrak{m}^\perp$.

It remains to prove that the Riemannian manifold S is isometric to the $(nr - 1)$ -dimensional sphere in \mathbb{R}^{nr} . It can be checked that

$$(3.6) \quad \mathfrak{p} = \left\{ X_v = \begin{pmatrix} & & v \\ & & | \\ & v^* & | \\ & & & \end{pmatrix} : v \in \mathbb{F}^n \right\}.$$

Then $\langle X_v, X_w \rangle = \frac{1}{2} \text{Tr}(X_v^* X_w) = \text{Re}(\sum_{l=1}^n \bar{v}_l w_l)$. Identifying \mathfrak{p} with $\mathbb{F}^n \cong \mathbb{R}^{rn}$ in the obvious way, the inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{p} coincides with the canonical inner product on the (real) vector space \mathbb{F}^n of dimension nr . Hence S is the $(nr - 1)$ -dimensional sphere of radius one in the (standard) Euclidean space \mathbb{R}^{nr} . \square

For a more detailed description of the realizations of the symmetric spaces G/K and K/M , we refer to [KN69, §10, Ch. XI].

Let

$$(3.7) \quad \Gamma_Q^0 = G \cap \Gamma_Q = \text{SU}^0(Q, \mathbb{F}) \cap \text{M}(n+1, \mathcal{O}).$$

This is a finite index subgroup of Γ_Q (see (2.6)). Let us denote by w the number of units in \mathcal{O} when $\mathbb{F} = \mathbb{R}$ or \mathbb{C} . It is clear that

$$\kappa := [\Gamma_Q : \Gamma_Q^0] = \begin{cases} 4 & \text{if } \mathbb{F} = \mathbb{R}, \\ w & \text{if } \mathbb{F} = \mathbb{C}, \\ 1 & \text{if } \mathbb{F} = \mathbb{H}. \end{cases}$$

Let $\{g_1, \dots, g_\kappa\}$ be a set of representatives of the Γ_Q^0 -coclasses of Γ_Q . For example $\left\{ \begin{pmatrix} I_{n-1} & & \\ & \pm 1 & \\ & & \pm 1 \end{pmatrix} \right\}$ if $\mathbb{F} = \mathbb{R}$, $\{(I_n \ \alpha) : \alpha \in \mathcal{O}^\times\}$ if $\mathbb{F} = \mathbb{C}$ and $\{I_{n+1}\}$ if $\mathbb{F} = \mathbb{H}$.

Now we can apply the lattice point theorem to our problem. For each $1 \leq j \leq \kappa$, Theorem 3.2 implies that

$$(3.8) \quad \#\{g \in \Gamma_Q^0 : d([e_{n+1}], g \cdot (g_j \cdot [y])) \leq s\} = \frac{2^{1-n} m_{\Gamma_Q^0} \zeta}{2^\rho \text{vol}(\Gamma_Q^0 \backslash \mathbb{H}_{\mathbb{F}}^n(Q))} e^{2\rho s} + O(e^{s\tau}).$$

A trivial verification shows that $\text{vol}(\Gamma_Q^0 \backslash \mathbb{H}_{\mathbb{F}}^n(Q)) = \xi_{\mathbb{F}} \text{vol}(\Gamma_Q \backslash \mathbb{H}_{\mathbb{F}}^n(Q))$, with $\xi_{\mathbb{R}} = 2$, $\xi_{\mathbb{C}} = m_{\Gamma_Q^0}$ and $\xi_{\mathbb{H}} = 1$. Furthermore, Lemma 3.5 gives $\zeta = \text{vol}(S^{nr-1})$. These considerations imply, by adding formula (3.8) over j , that

$$(3.9) \quad \#\{g \in \Gamma_Q : d([e_{n+1}], g \cdot [y]) \leq s\} = \tilde{w} \frac{2^{1-n} \text{vol}(S^{nr-1})}{2^\rho \text{vol}(\Gamma_Q \backslash \mathbb{H}_{\mathbb{F}}^n(Q))} e^{2\rho s} + O(e^{s\tau})$$

where $\tilde{w} = w$ if $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and $\tilde{w} = 2$ if $\mathbb{F} = \mathbb{H}$. Applying this formula to (2.7), we obtain that

$$(3.10) \quad N_t(Q, -k) = \tilde{w} \frac{2^{1-n} \text{vol}(S^{nr-1})}{2^\rho \text{vol}(\Gamma_Q \backslash \mathbb{H}_{\mathbb{F}}^n(Q))} \left(\sum_{y \in F} |\Gamma_{Q,y}|^{-1} \right) e^{2\rho s} + O(e^{s\tau}).$$

Recall that $\cosh(s) = a^{1/2} k^{-1/2} t$. Notice that we can replace the error term in (3.10) by $O(t^\tau)$ since $e^s \sim 2 \cosh(s) = 2 a^{1/2} k^{-1/2} t$ as $s \rightarrow +\infty$, and furthermore $e^{2\rho s} = 2^{2\rho} a^\rho k^{-\rho} t^{2\rho} + O(t^\tau)$.

Collecting all the information in this section, we have obtained the following formula.

Proposition 3.6. *The number $N_t(Q, -k)$ satisfies the asymptotic estimate*

$$(3.11) \quad N_t(Q, -k) = \tilde{w} \frac{2^{2\rho-(n-1)} a^\rho \text{vol}(S^{nr-1})}{2^\rho k^\rho \text{vol}(\Gamma_Q \backslash \mathbb{H}_{\mathbb{F}}^n(Q))} \left(\sum_{y \in F} |\Gamma_{Q,y}|^{-1} \right) t^{2\rho} + O(t^\tau),$$

as $t \rightarrow +\infty$, where τ is as in (3.4) and $\tilde{w} = w$ if $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and $\tilde{w} = 2$ if $\mathbb{F} = \mathbb{H}$. Moreover, when $\mathbb{F} = \mathbb{R}$ and $n > 2$, (3.11) holds with $\tau = n - 3/2$.

The last assertion follows from Remark 3.4.

4. THE MASS OF THE REPRESENTATION

The object of this section is to obtain a formula for the term $\sum_{y \in F} |\Gamma_{Q,y}|^{-1}$ by using Siegel's theory on quadratic forms and its generalization to complex hermitian forms given by Raghavan. Our main references are [Sie67], [Ram61] and [Rag62]. From now on we make the assumption $\mathbb{F} = \mathbb{R}$ or \mathbb{C} . We denote by $d_{\mathcal{O}}$ the discriminant of the quotient field of \mathcal{O} . The following Remark introduces a canonical volume element in algebraic varieties that will be useful (see [Sie67, §5 Ch IV] for more details).

Remark 4.1. Let x_1, \dots, x_m denote the coordinates of \mathbb{R}^m . Let $y_j = f_j(x_1, \dots, x_m)$ ($1 \leq j \leq n$) be n smooth functions with $n \leq m$. Let $a_1, \dots, a_n \in \mathbb{R}$ such that the surface $\Omega = \{x \in \mathbb{R}^m : y_j(x) = a_j, \text{ for } 1 \leq j \leq n\}$ is non-singular, that is, the matrix with entries $\frac{\partial f_i}{\partial x_j}$ has maximum rank n at every point of Ω . Choose $m - n$ differentiable functions y_{n+1}, \dots, y_m of \mathbb{R}^m so that the Jacobian $J = \det\left(\frac{\partial y_i}{\partial x_j}\right)$ is nonzero at every point of Ω . Then

$$(4.1) \quad d\omega = |J|^{-1} dy_{n+1} \dots dy_m$$

gives a volume element on Ω independently of the choice of y_{n+1}, \dots, y_m . Siegel denoted this volume element by $\frac{\{dx\}}{\{dy\}}$.

We pick $v \in \mathbb{F}^n$ and $R \in M(n, \mathbb{F})$ such that the matrix

$$W = \begin{pmatrix} -k & v \\ v^t & R \end{pmatrix}$$

is \mathbb{F} -hermitian of signature $(n, 1)$. For $y \in \mathbb{F}^{n+1}$ such that $Q[y] = -k$, let $U(Q, \mathbb{F})_y$ denote the set of elements $U \in U(Q, \mathbb{F})$ such that $Uy = y$. Note that if $y \in \mathcal{O}^{n+1}$, the stabilizer of y in Γ_Q is $\Gamma_{Q,y} = U(Q, \mathbb{F})_y \cap \text{GL}(n+1, \mathcal{O})$. Consider the varieties

$$\begin{aligned} \Omega(Q, W) &= \{X \in M(n+1, \mathbb{F}) : Q[X] = W\}, \\ \Omega(Q, W; y) &= \{Y \in M(n+1, n; \mathbb{F}) : Q[(y|Y)] = W\}. \end{aligned}$$

The groups $U(Q, \mathbb{F})$ and $U(Q, \mathbb{F})_y$ act by left multiplication on $\Omega(Q, W)$ and on $\Omega(Q, W; y)$ respectively. On these varieties we fix the volume elements

$$d\omega = \left| \frac{\det(W)}{\det(Q)} \right|^{\frac{2-r}{2}} \frac{\{dX\}}{\{dW\}} \quad \text{and} \quad d\omega^* = \left| \frac{\det(W)}{\det(Q)} \right|^{\frac{2-r}{2}} \frac{\{dX\}}{\{dv\}\{dR\}},$$

respectively, where $\frac{\{dX\}}{\{dW\}}$ and $\frac{\{dX\}}{\{dv\}\{dR\}}$ are the volume elements given in Remark 4.1. In the complex case, we write $X = X^{(1)} + iX^{(2)}$ and $W = W^{(1)} + iW^{(2)}$ with $X^{(1)}$, $X^{(2)}$, $W^{(1)}$ and $W^{(2)}$ real matrices. Thus, the volume element $\frac{\{dX\}}{\{dW\}}$ is defined by considering the algebraic equations $\text{Re}(Q[X^{(1)} + iX^{(2)}]) = W^{(1)}$ and $\text{Im}(Q[X^{(1)} + iX^{(2)}]) = W^{(2)}$. The factor $|\det(W)/\det(Q)|^{\frac{2-r}{2}}$ is included so that $d\omega$ does not depend on W . Similar consideration apply for $\frac{\{dX\}}{\{dv\}\{dR\}}$ and $d\omega^*$.

Siegel (and [Ram61] for the hermitian case) defines *the measure of the representation of $-k \in \mathbb{Z}$ by Q* as

$$(4.2) \quad \mu(Q, -k) = \sum_{y \in F} \mu(y, Q) / \mu(Q).$$

Here $\mu(Q)$ denotes the *measure of the unit group* Γ_Q given by $(r^2/|d_{\mathcal{O}}|)^{(n+1)(n+2)/4}$ times the volume of any fundamental domain for the action of Γ_Q on $\Omega(Q, W)$, and

similarly, $\mu(y, Q)$ is the *measure of the representation* y , given by $(r^2/|d_{\mathcal{O}}|)^{n(n+1)/4}$ times the volume of any fundamental domain for the action of $\Gamma_{Q,y}$ on $\Omega(Q, W; y)$.

We will recover from the right hand side of (4.2) the term $\sum_{y \in F} |\Gamma_{Q,y}|^{-1}$ and then, by applying Siegel's main theorem, we will obtain an explicit formula for this term. By [Sie67, Thm. 7, Ch IV] and [Rag62, (93)] (or [Ram61, (70)]) we have that

$$(4.3) \quad w \mu(Q) = (r^2/|d_{\mathcal{O}}|)^{\frac{(n+1)(n+2)}{4}} \frac{\text{vol}(\Gamma_Q \backslash \mathbb{H}_{\mathbb{F}}^n(Q))}{r^{n+1} |\det Q|^{\frac{r}{2}n+1}} \frac{\pi^{\frac{r}{2}}}{\Gamma(\frac{r}{2})} \prod_{j=1}^n \frac{\pi^{\frac{r}{2}j}}{\Gamma(\frac{r}{2}j)},$$

where $w = \#\mathcal{O}^{\times}$.

Let $\tilde{\mathcal{F}}(y)$ be a fundamental domain of the action of $\Gamma_{Q,y}$ on $\Omega(Q, W; y)$. By definition $\mu(y, Q) = |\det W|^{(2-r)/2} |\det Q|^{-(2-r)/2} (r^2/|d_{\mathcal{O}}|)^{n(n+1)/4} \int_{\tilde{\mathcal{F}}(y)} d\omega^*$, but the measure $d\omega^*$ is invariant by $\Gamma_{Q,y}$, then

$$\mu(y, Q) = \frac{1}{|\Gamma_{Q,y}|} \left| \frac{\det W}{\det Q} \right|^{(2-r)/2} (r^2/|d_{\mathcal{O}}|)^{\frac{n(n+1)}{4}} \int_{\Omega(Q, W; y)} d\omega^*.$$

Using [Sie67, Thm. 6, Ch IV] and [Ram61, Lemma 9], we have that

$$(4.4) \quad \mu(y, Q) = |\Gamma_{Q,y}|^{-1} (r^2/|d_{\mathcal{O}}|)^{\frac{n(n+1)}{4}} \frac{k^{1-\frac{r}{2}(n+1)}}{|\det Q|^{\frac{r}{2}(n-1)+1}} \prod_{j=1}^n \frac{\pi^{\frac{r}{2}j}}{\Gamma(\frac{r}{2}j)}.$$

Finally, (4.3) and (4.4) imply that

$$(4.5) \quad \mu(Q, -k) = w |d_{\mathcal{O}}|^{\frac{n+1}{2}} \frac{k^{1-\frac{r}{2}(n+1)} |\det Q|^{\frac{r}{2}} \Gamma(\frac{r}{2})}{\text{vol}(\Gamma_Q \backslash \mathbb{H}_{\mathbb{F}}^n(Q)) \pi^{\frac{r}{2}}} \sum_{y \in F} |\Gamma_{Q,y}|^{-1}.$$

Now, we will recall Siegel's main theorem for indefinite quadratic and hermitian forms (see [Sie44, Thm. 1] and [Rag62, Thm. 7]). For every rational prime p , the p -adic density of representation of $-k$ by Q is

$$(4.6) \quad \delta_p(Q, -k) = \lim_{j \rightarrow \infty} p^{-j(r(n+1)-1)} \#\{x \in (\mathcal{O}/p^j\mathcal{O})^{n+1} : Q[x] \equiv -k \pmod{p^j}\}.$$

Define $\delta(Q, -k) = \prod_p \delta_p(Q, -k)$, the *local density*, where the product is over all prime numbers.

Theorem 4.2 (*Siegel's mass formula*). *Let Q be an \mathbb{F} -hermitian form as in (1.1) with $n \geq 2$ and let k be a positive integer. Then*

$$(4.7) \quad \mu(Q, -k) = \delta(Q, -k).$$

See [Sie44, Thm. 1] for the real case and [Rag62, Thm. 7] for the complex case.

By combining equation (4.5) and Theorem 4.2, we obtain an expression for the term $\sum_{y \in F} |\Gamma_{Q,y}|^{-1}$, the main goal of this section.

Corollary 4.3. *We have that*

$$\sum_{y \in F} |\Gamma_{Q,y}|^{-1} = w^{-1} |d_{\mathcal{O}}|^{-\frac{n+1}{2}} \frac{k^{\frac{r}{2}(n+1)-1}}{|\det Q|^{\frac{r}{2}}} \text{vol}(\Gamma_Q \backslash \mathbb{H}_{\mathbb{F}}^n(Q)) \frac{\pi^{\frac{r}{2}}}{\Gamma(\frac{r}{2})} \delta(Q, -k)$$

Note that $\pi^{r/2}/\Gamma(r/2) = 1, \pi, \pi^2$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and \mathbb{H} respectively.

5. MAIN THEOREM

We can now state the main result in this paper, which follows by combining Proposition 3.6 and Corollary 4.3. The last assertion is a consequence of Remark 3.4. We first recall some terminology: $r = \dim_{\mathbb{R}}(\mathbb{F})$, $\rho = (n+1)r/2 - 1$ and \mathcal{O} is a maximal order in \mathbb{F} with discriminant $d_{\mathcal{O}}$.

Theorem 5.1. *Let Q be an \mathbb{F} -hermitian matrix as in (1.1), with $n \geq 2$ and $\mathbb{F} = \mathbb{R}$ or \mathbb{C} . We fix $k \in \mathbb{N}$ such that $-k$ is represented by Q . Then, the number $N_t(Q, -k)$ of elements $x \in \mathcal{O}^{n+1}$ such that $Q[x] = -k$ and $|x_{n+1}| \leq t$, satisfies the following asymptotic estimate as $t \rightarrow +\infty$,*

$$(5.1) \quad N_t(Q, -k) = \frac{2^{(r-1)(n+1)} a^{\rho} \text{vol}(S^{nr-1})}{|d_{\mathcal{O}}|^{\frac{n+1}{2}}} \frac{\pi^{\frac{r}{2}}}{2^{\rho} |\det Q|^{\frac{r}{2}} \Gamma(\frac{r}{2})} \delta(Q, -k) t^{2\rho} + O(t^{\tau}),$$

where τ is as in (3.4). Moreover, when $\mathbb{F} = \mathbb{R}$ and $n > 2$, the formula (3.11) holds for $\tau = n - 3/2$.

Remark 5.2. In order to get an explicit value of the main term in (5.1) for a fixed \mathbb{F} -hermitian form Q , one needs to determine $\delta(Q, -k)$. When $\mathbb{F} = \mathbb{R}$, T. Yang [Ya98] computed this local density for any quadratic form Q . For $\mathbb{F} = \mathbb{C}$ see [Hi99].

Remark 5.3. We consider the real norm $\|\cdot\|$ induced by the \mathbb{F} -hermitian form $\begin{pmatrix} A & \\ & a \end{pmatrix}$, i.e. $\|x\|^2 = A[\hat{x}] + a|x_{n+1}|^2$, where $\hat{x} = (x_1, \dots, x_n)^t$. We claim that Theorem 5.1 provides an asymptotic formula with error term for $\tilde{N}_s(Q, -k)$ (see (1.7)) as $s \rightarrow +\infty$, of the number of solutions lying in the Euclidean ball of radius s . Indeed, it is clear that

$$\left. \begin{array}{l} Q[x] = A[\hat{x}] - a|x_{n+1}|^2 = -k \\ \|x\|^2 = A[\hat{x}] + a|x_{n+1}|^2 \leq s^2 \end{array} \right\} \quad \text{imply} \quad |x_{n+1}| \leq \sqrt{\frac{s^2 + k}{2}} =: t_s.$$

Hence, by Theorem 5.1, we have

$$(5.2) \quad \tilde{N}_s(Q, -k) = N_{t_s}(Q, -k) = C_{Q,k} t_s^{2\rho} + O(t_s^{\tau}),$$

where $C_{Q,k}$ denotes the main coefficient of (5.1). By Taylor's expansion at $s = \infty$, $t_s = s/\sqrt{2} + O(s^{-1})$, thus we can replace $O(t_s^{\tau})$ by $O(s^{\tau})$ in (5.2). Moreover, $t_s^{2\rho} = s^{2\rho}/2^{\rho} + O(s^{2\rho-2})$. But $\tau \geq 2\rho n/(n+1) + \varepsilon$ by (3.4), and so it follows immediately that $\tau - (2\rho - 2) > 0$ for $\mathbb{F} = \mathbb{R}$ and \mathbb{C} . Therefore, (5.2) reduces to

$$(5.3) \quad \tilde{N}_s(Q, -k) = 2^{-\rho} C_{Q,k} s^{2\rho} + O(s^{\tau}).$$

Example 5.4. The case $\mathbb{F} = \mathbb{R}$ and $Q = I_{n,1} = \begin{pmatrix} I_n & \\ & -1 \end{pmatrix}$ was considered by J. Ratcliffe and S. Tschantz [RT97]. We have $r = 1$, $|d_{\mathcal{O}}| = 1$, $a = 1$, $|\det(I_{n,1})| = 1$ and $\rho = (n-1)/2$. Theorem 5.1 now yields

$$(5.4) \quad N_t(I_{n,1}, -k) = \frac{\text{vol}(S^{n-1})}{n-1} \delta(I_{n,1}, -k) t^{n-1} + O(t^{n-3/2}).$$

In [RT97, Thm. 12] there is an explicit formula for the local density $\delta(I_{n,1}, -k)$.

Example 5.5. We conclude the article by considering the case of the Lorentzian hermitian form over the Gaussian integers, i.e. $Q = I_{n,1}$ for $\mathbb{F} = \mathbb{C}$ and $\mathcal{O} = \mathbb{Z}[\sqrt{-1}]$. We have $r = 2$, $\rho = n$, $d_{\mathcal{O}} = -4$, $a = 1$ and $|\det(I_{n,1})| = 1$. Furthermore, $\text{vol}(S^{2n-1}) = 2\pi^n/(n-1)!$. Theorem 5.1 now implies that

$$(5.5) \quad N_t(I_{n,1}, -k) = \frac{\pi^{n+1}}{n!} \delta(I_{n,1}, -k) t^{2n} + O(t^{\tau}).$$

The local density $\delta(I_{n,1}, -k)$ is explicitly computable. For example, when $n = 2$ and $k = 1$ we have $\delta(I_{2,1}, -1) = 2^3 3\pi^{-3}$ and consequently $N_t(I_{2,1}, -1) = 12t^4 + O(t^7)$. In a future paper we will compute this term for several examples and testing our formula with experimental computations.

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