ONE-SIDED EXTRAPOLATION AT INFINITY AND SINGULAR INTEGRALS

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Abstract. In this paper we prove, on one hand, extrapolation from infinity for the one-sided classes A_p^+, $A^+(p,q)$, $A^+_{p,q,\alpha,\beta}$, and on the other hand, the BMO-boundedness for one-sided singular integrals. We also provide several applications of our results.

1. Introduction

Extrapolation properties for the Muckenhoupt classes A_p were proved by J.L. Rubio de Francia in 1984, see [14], roughly speaking, if an operator preserves $L^{p_0}(w)$ for all $w \in A_{p_0}$, then it necessarily preserves the $L^p(w)$ space for every $w \in A_p$, and every $1 < p < \infty$. Extensions of this result were obtained by several authors, for instance in [5] and [6], E. Harboure, R.A. Macías and C. Segovia study $A(p,q)$ classes, pairs of weights and extreme cases, $p_0 = \infty$; more recently F. Martín–Reyes, P. Ortega and A. de la Torre, see [9], considered this problem for one-sided classes of weights for $1 \le p_0 < \infty$. (The one-sided classes of weights were first introduced by Sawyer in [13].) Since proving L^{p_0} boundedness for $p_0 = \infty$ is usually much simpler than doing this for a finite p_0, it is of great interest to study that extreme case.

In this paper we study the strong and weak extrapolation properties of one-sided classes of weights, A_p^+, $A^+(p,q)$ and $A^+_{p,q,\alpha,\beta}$ starting from ∞. The results are stated in Paragraph 1, see Theorems I through V. The proofs of these theorems are given in Paragraph 4. Moreover we also study in Paragraph 2 the weighted BMO boundedness of one-sided singular integral operators recently introduced by H. Aimar, L. Forzani and F. Martín–Reyes, (see [1]). These allow us to show several applications which are contained in Paragraph 3, including new proofs of the results appearing in [1], [9], and [10].

Before stating our results we need some definitions. Given $p \in \mathbb{R}\backslash\{0\}$, p' is the conjugate index, $1/p + 1/p' = 1$. A nonnegative function w defined in $\mathbb{R}$ shall be called a weight if it is locally integrable. In the following we consider the right lateral classes. It is easy to see that the left classes can be treated analogously.

A weight w is said to belong to the class A_p^+, $1 < p < \infty$, if and only if there is a constant C such that

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$$\left(\frac{1}{h}\int_{x-h}^{x} w\right)\left(\frac{1}{h}\int_{x}^{x+h} w^{-\frac{1}{p-1}}\right)^{p-1} \leq C\,,$$

for every $h>0$ and $x \in \mathbb{R}$. The class A_1^+ is defined by

$$M^- w(x) = \sup_{h>0}\int_{x-h}^{x} w(t)\,dt \leq Cw(x)\,,$$

for almost every $x \in \mathbb{R}$. For $p=\infty$, we put

$$A_\infty^+ = \cup_{p\geq 1} A_p^+\,.$$

We say that a pair of weights, (u,v), belongs to $A^+(p,q)$, $1<p\leq\infty$, $1\leq q<\infty$, if and only if

$$\left(\frac{1}{h}\int_{x-h}^{x} u^q\right)^{1/q}\left(\frac{1}{h}\int_{x}^{x+h} v^{-p'}\right)^{1/p'} \leq C\,,$$

for all $h>0$ and $x\in\mathbb{R}$. Also $(u,v)\in A^+(p,\infty)$ if and only if

$$\|\chi_{[x-h,x]}u\|_\infty\left(\frac{1}{h}\int_{x}^{x+h} v^{-p'}\right)^{1/p'} \leq C\,,$$

for every $h>0$ and $x\in\mathbb{R}$.
We say that $v\in A^+(p,q)$, $A^+(p,\infty)$, if the pair (v,v) does.

For $0\leq\beta\leq\alpha\leq 1$ and $1<p\leq q<\infty$, $A^+_{p,q,\alpha,\beta}$ is the class of all the pair of weights (u,v), such that

$$\left(\int_a^b u\right)^{1/q}\left(\int_b^c \frac{v^{1-p'}(s)}{(c-s)^{(1-\alpha)p'}}\,ds\right)^{1/p'} \leq C(c-a)^\beta,$$

is satisfied for every $a<b<c$. In the case $p=1$, we let $(u,v)\in A^+_{1,q,\alpha,\beta}$ if and only if

$$\left(\int_a^b u\right)^{1/q} \leq C(c-a)^\beta\ \mathrm{essinf}_{s\in(b,c)} v^{1-p'}(s)(c-s)^{(1-\alpha)}\,,$$

whenever $a<b<c$. In the case $1<p<\infty$ and $q=\infty$, $(u,v)\in A^+_{p,\infty,\alpha,\beta}$ if and only if

$$||\chi_{[a,b]}u||_\infty\left(\int_b^c \frac{v^{1-p'}(s)}{(c-s)^{(1-\alpha)p'}}\,ds\right)^{1/p'} \leq C(c-a)^\beta,$$

is satisfied for every $a<b<c$.

Extrapolation results.

The next theorem is an extension of the one that appears in [8], where the case $1 < p_0 = q_0 < \infty$ is obtained.

Theorem I. *Let T be a sublinear operator defined on $C_0^\infty(\mathbb{R})$. If the inequality*

$$\left(\int |Tf|^{q_0} v^{q_0}\right)^{1/q_0} \leq C(v)\left(\int |f|^{p_0} v^{p_0}\right)^{1/p_0}$$

holds for some pair (p_0, q_0), $1 < p_0 \leq q_0 < \infty$ and for all weights v belonging to the class $A^+(p_0, q_0)$, then for any pair (p, q) $1 < p \leq q < \infty$, satisfying $\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0}$, and for any weight $v \in A^+(p, q)$ the inequality

$$\left(\int |Tf|^{q} v^{q}\right)^{1/q} \leq C(v)\left(\int |f|^{p} v^{p}\right)^{1/p},$$

holds, provided the left hand side is finite.

Given f belonging to $L^1_{loc}(\mathbb{R})$, let

$$f_+^{\#}(x) = \sup_{h>0} \frac{1}{h}\int_x^{x+h}\left(f(y) - \frac{1}{h}\int_{x+h}^{x+2h} f(z)\,dz\right)^+ dy,$$

where $z^+ = \max(z, 0)$. In [9], an one-sided version of the BMO space, BMO^+, is defined as the set of those functions such that $||f_+^{\#}||_\infty < \infty$. We consider the quantity given by

$$|||f|||_{v,+} = \sup_x \sup_{h>0} \|v\chi_{[x-h,x]}\|_\infty \left(\frac{1}{h}\int_x^{x+h}\left(f(y) - \frac{1}{h}\int_{x+h}^{x+2h} f(z)\,dz\right)^+ dy\right).$$

We introduce, $BMO^+(v)$, a weighted version of BMO^+, as being the set of those functions such that $|||f|||_{v,+} < \infty$. Then we are able to give, in Theorems II, III and IV, one-sided versions for the extrapolation from infinity results obtained in [4] and [5].

Theorem II. *Let $1 < p_0 < \infty$ and let T be a sublinear operator defined on $C_0^\infty(\mathbb{R})$ satisfying*

$$\|v\chi_{[x-h,x]}\|_\infty \left(\frac{1}{h}\int_x^{x+h}\left(|Tf|(y) - \frac{1}{h}\int_{x+h}^{x+2h} |Tf|\right)^+ dy\right) \leq C(v)\left(\int |f|^{p_0} v^{p_0}\,dx\right)^{1/p_0}$$

for every $h > 0$, $x \in \mathbb{R}$ and $v \in A^+(p_0, \infty)$, then for every $1 < p < p_0$, $1/p - 1/q = 1/p_0$ and $v \in A^+(p, q)$, the inequality

$$\left(\int |Tf|^{q} v^{q}\right)^{1/q} \leq C(v)\left(\int |f|^{p} v^{p}\right)^{1/p},$$

holds, provided that the left hand side is finite.

Theorem III. *Let T be a sublinear operator defined on $C_0^\infty(\mathbb{R})$ and satisfying*

$$\|v\chi_{[x-h,x]}\|_\infty \left(\frac{1}{h}\int_x^{x+h} \left(|Tf|(y) - \frac{1}{h}\int_{x+h}^{x+2h} |Tf| \right)^+ dy \right) \leq C(v)\|fv\|_\infty$$

for every $h>0$, $x\in\mathbb{R}$ and v such that $v^{-1}\in A_1^-$. Then, if $1<p<\infty$ and $v\in A_p^+$, the inequality

$$\left(\int |Tf|^p v\right)^{1/p} \leq C(v)\left(\int |f|^p v\right)^{1/p},$$

holds, provided the left hand side is finite.

Theorem IV. *Let T be a sublinear operator defined on $C_0^\infty(\mathbb{R})$, with values on the space of measurable functions. Let us assume that T verifies*

$$||aT(f)||_\infty \leq C||fb||_{p_0}\ ,$$

for every pair (a,b) of functions such that $(a^r,b^r)\in A^+(\frac{p_0}{r},\infty)$, $1\leq r<p_0\leq\infty$ with C depending upon the constant of $A^+\left(\frac{p_0}{r},\infty\right)$ of the pair (a^r,b^r). Then if $r<p<p_0$, $\frac{1}{q}=\frac{1}{p}-\frac{1}{p_0}$ and $(u^r,v^r)\in A^+\left(\frac{p}{r},\frac{q}{r}\right)$ there exists C, depending only on $\frac{p}{r}$, $\frac{q}{r}$, and the constant $A^+\left(\frac{p}{r},\frac{q}{r}\right)$ of the pair (u^r,v^r), such that

$$u^q\left(\{x:|Tf(x)|>\lambda\}\right) \leq C\left(\lambda^{-p}\int |f|^p v^p\,dx\right)^{\frac{q}{p}} \quad \forall\lambda>0\ .$$

Recently a general maximal function,

$$M^+_{\alpha,\beta}f(x) = \sup_{c>x}\frac{1}{(c-x)^\beta}\int_x^c \frac{|f(s)|}{(c-s)^{1-\alpha}}\,ds\,,$$

was introduced in [10], where it is shown that the pair $(u,v)\in A^+_{p,q,\alpha,\beta}$, for $0\leq\beta\leq\alpha\leq 1$, $1\leq p\leq q$, or $\frac{1}{p}-\frac{1}{q}=\alpha-\beta$, if and only if

$$u(\{x: M^+_{\alpha,\beta}f(x)>\lambda\}) \leq C\left(\lambda^{-p}\int |f|^p v\right)^{q/p}, \tag{1.1}$$

for all $\lambda>0$. The next theorem proves that also in this case extrapolation from infinity is posible, which allows us to obtain, in Paragraph 3, an easier proof of the boundedness of these maximal functions.

Theorem V. *Let T be a sublinear operator defined in $C_0^\infty(\mathbb{R})$, with values in the space of measurable functions. Let $0\leq r\leq 1$ and $1<p_0<\infty$. Suppose that*

$$||aTf||_\infty \leq C||fb||_{p_0},$$

holds, for every pair of weights (a,b) such that $(a,b^{p_0}) \in A^+_{p_0,\infty,r,r}$, and C depending only on the constant $A^+_{p_0,\infty,r,r}$ of the pair (a,b^{p_0}).
Then for every pair $(u,v) \in A^+_{p,q,\alpha,\beta}$, $0 \le \beta < \alpha \le r \le 1$, $\frac{1}{p} - \frac{1}{q} = \alpha - \beta = \frac{1}{p_0}$, there exists C depending only on p, q, and the constant $A^+_{p,q,\alpha,\beta}$ of the pair (u,v), such that

$$u(\{x : |Tf(x)| > \lambda\}) \le C \left(\lambda^{-p} \int |f|^p v \right)^{q/p},$$

for all $\lambda > 0$.

2. Weighted Boundedness for One Sided Singular Integral

In this section we will prove that one-sided singular integral operators are bounded from $L^\infty(v)$ into $BMO^+(v)$. Then, applying Theorem III we obtain the boundedness of these operators from $L^p(v)$ into $L^p(v)$ if $v \in A^+_p$, with $1 < p < \infty$.

A function $K \in L^1_{loc}(\mathbb{R} - \{0\})$ shall be called a Calderón-Zygmund's kernel if there exist finite constants B_1, B_2, and B_3 satisfying

$$\left| \int_{\epsilon < |x| < N} K(x)\, dx \right| \le B_1 , \tag{2.1}$$

for all ϵ and all N, with $0 < \epsilon < N$,

$$|K(x)| \le \frac{B_2}{|x|}, \tag{2.2}$$

for all $x \neq 0$,

$$|K(x-y) - K(x)| \le B_3 \frac{|y|}{|x|^2} , \tag{2.3}$$

for all x and y with $|x| > 2|y|$. Furthermore we assume that there exists

$$\lim_{\epsilon \to 0} \int_{\epsilon < |x| < 1} K(x)\, dx .$$

Let us consider

$$Tf(x) = \lim_{\epsilon \to 0} T_\epsilon f(x) \qquad \text{and} \qquad T^* f(x) = \sup_{\epsilon > 0} |T_\epsilon f(x)| , \tag{2.4}$$

where

$$T_\epsilon f(x) = \int_{|x-y| > \epsilon} K(x-y) f(y)\, dy .$$

Lateral singular integral operators were recently defined by H. Aimar, L. Forzani, F. Martín–Reyes in [1], where they characterize the weights for which these operators are of strong type $(p,p), 1<p<\infty$. They also give an example of a Calderón-Zygmund kernel with support in the negative real line, namely

$$K(x)=\frac{1}{x}\,\frac{\sin(\log|x|)}{\log(|x|)}\,\chi_{(-\infty,0)}(x)\,.$$

Observe that if K is a Calderón-Zygmund kernel then $K_s(x)=sK(sx)\ s>0$ is as well, with the same constants B_1, B_2, B_3. For completness sake, we state as Theorems A and B, some of the results of [1].

Theorem A. *Let K be a kernel with support in the negative real line that satisfies (2.1), (2.2) and (2.3); then we have:*

(A.1) *given a weight $v\in A_p^+$, $1<p<\infty$, there exists a constant C, depending only on p, B_1, B_2, B_3, and the A_p^+ constant of v, such that*

$$\int_{\mathbb{R}}|T^*f(x)|^p v(x)\,dx\le C\int_{\mathbb{R}}|f(x)|^p v(x)\,dx\ ;$$

(A.2) *given a weight $v\in A_1^+$, there exists a constant C, depending only on B_1, B_2, B_3, and the A_1^+ constant of v, such that*

$$v(\{T^*f(x)>\lambda\})\le\frac{C}{\lambda}\int_{\mathbb{R}}|f(x)|v(x)\,dx\ .$$

Let us observe that T_s^* is the maximal operator associated to K_s, then this Theorem implies that the operators T_s^* , $s>0$, are uniformly bounded in $L^p(v)$, $1<p<\infty$, if $v\in A_p^+$ and are of $L^1(v)$-weak type if $v\in A_1^+$. The following theorem is a kind of reciprocal.

Theorem B. *Let K be a kernel with support in the negative real line that satisfies (2.1), (2.2), (2.3), and $K(x)\not\equiv 0$. Let T_s^* be the maximal operator associated to K_s . If v is a weight and all the operators T_s^* with $s>0$ are of weak type (p,p), $1\le p<\infty$, with respect to v with a constant C not depending on s, then $v\in A_p^+$.*

We can now state and prove our boundedness result from $L^\infty(v)$ into $BMO^+(v)$.

Theorem 2.5. *Let K be a kernel satisfying (2.1), (2.2),(2.3) with support in the negative real line. Let T^* be as in (2.4) and $v^{-1}\in A_1^-$, then*

$$|||T^*f|||_{v,+}\le C||fv||_\infty\,,$$

for all f such that $||fv||_\infty<\infty$.

Proof. First of all we note that

$$T_\epsilon f(x)=\int_{|x-y|>\epsilon}K(x-y)f(y)\,dy=\int_{x+\epsilon}^{\infty}K(x-y)f(y)\,dy\ .$$

Fix $x \in \mathbb{R}$, $h > 0$ and let f, v and $\delta > 0$ be such that $v^{-1} \in A_1^-$, $||fv||_\infty < \infty$ and $(v^{-1})^{1+\delta} \in A_1^-$. Let us put $f_1 = f\chi_{[x,x+8h]}$ and $f = f_1 + f_2$; clearly, $f_1 \in L^{1+\delta}(dx)$. Now, using the strong $L^p(dx)\,, 1 < p < \infty$, boundedness of T^* we have,

$$\begin{aligned}
&\frac{1}{h}\int_x^{x+h}\left(T^*f_1(y) - \frac{1}{h}\int_{x+h}^{x+2h} T^*f_1(z)\,dz\right)^+ dy \\
&\leq C\left(\frac{1}{h}\int_x^{x+2h} |T^*f_1(y)|^{1+\delta}\,dy\right)^{\frac{1}{1+\delta}} \leq C\left(\frac{1}{h}\int_x^{x+8h} |f(y)|^{1+\delta}\,dy\right)^{\frac{1}{1+\delta}} \\
&\leq C||fv||_\infty\left(\frac{1}{h}\int_x^{x+8h} (v^{-1})^{1+\delta}\,dy\right)^{\frac{1}{1+\delta}} .
\end{aligned}$$

Now let us take $t \in [x-h, x]$ such that $||v\chi_{[x-h,x]}||_\infty \leq 2v(t)$; then

$$||v\chi_{[x-h,x]}||_\infty \left(\tfrac{1}{h}\textstyle\int_x^{x+8h}(v^{-1})^{1+\delta}\,dy\right)^{\frac{1}{1+\delta}} \leq Cv(t)\left(M^+((v^{-1})^{1+\delta})\right)^{\frac{1}{1+\delta}}(t) \leq C\ .$$

The last inequality follows from $(v^{-1})^{1+\delta} \in A_1^-$. So we have that

$$\frac{||v\chi_{[x-h,x]}||_\infty}{h}\int_x^{x+h}\left(T^*f_1(y) - \frac{1}{h}\int_{x+h}^{x+2h} T^*f_1(z)\,dz\right)^+ dy \leq C||fv||_\infty\ . \tag{2.6}$$

Now we consider $\frac{1}{h}\displaystyle\int_x^{x+h}\left(T^*f_2(y) - \frac{1}{h}\int_{x+h}^{x+2h} T^*f_2(z)\,dz\right)^+ dy$

$$\leq \tfrac{1}{h}\int_x^{x+h}\frac{1}{h}\int_{x+h}^{x+2h}\sup_{\epsilon>0}\left|\int_{y+\epsilon}^\infty K(y-t)f_2(t)\,dt - \int_{z+\epsilon}^\infty K(z-t)f_2(t)\,dt\right|\,dz\,dy.$$

Let

$$I_\epsilon = \left|\int_{y+\epsilon}^\infty K(y-t)f_2(t)\,dt - \int_{z+\epsilon}^\infty K(z-t)f_2(t)\,dt\right|.$$

If $\epsilon < 6h$

$$I_\epsilon \leq \int_{x+8h}^\infty |(K(y-t) - K(z-t))f(t)|\,dt = I_1\ .$$

When $\epsilon \geq 6h$

$$I_\epsilon \leq \left|\int_{y+\epsilon}^{z+\epsilon} K(y-t)f_2(t)\,dt\right| + \int_{x+8h}^\infty |(K(y-t) - K(z-t))f(t)|\,dt = I_{2,\epsilon} + I_1\ .$$

Since $t - z > 2(y - z)$, applying (2.3), we get

$$\begin{aligned} I_1 &\le \int_{x+8h}^{\infty} \frac{z-y}{(t-z)^2}|f(t)|\,dt \le Ch \sum_{k=3}^{\infty} \frac{1}{(2^k h)^2} \int_{x+2^k h}^{x+2^{k+1}h} |f(t)|\,dt \\ &\le C||fv||_\infty \sum_{k=3}^{\infty} \frac{1}{2^k} \frac{1}{2^{k+1}h} \int_x^{x+2^{k+1}h} v^{-1}\,dt \ . \end{aligned}$$

Let us study $I_{2,\epsilon}$:

$$\begin{aligned} I_{2,\epsilon} &\le \left| \int_{y+\epsilon}^{z+\epsilon} (K(y-t)-K(-\epsilon)) f_2(t)\,dt \right| + \left| \int_{y+\epsilon}^{z+\epsilon} K(-\epsilon) f_2(t)\,dt \right| \\ &\le C \int_{y+\epsilon}^{z+\epsilon} \frac{-\epsilon+t-y}{\epsilon^2} |f_2(t)|\,dt + \int_{y+\epsilon}^{z+\epsilon} \frac{B_2}{\epsilon} |f_2(t)|\,dt \\ &\le \frac{C}{\epsilon} \int_{y+\epsilon}^{z+\epsilon} |f_2(t)|\,dt \le ||fv||_\infty \frac{C}{\epsilon} \int_x^{x+2\epsilon} v^{-1} \ , \end{aligned}$$

where the last inequalities follow from (2.3) and the fact $|-\epsilon| \ge 2\,|\epsilon-(t-y)|$. Therefore

$$\begin{aligned} ||v\chi_{[x-h,x]}||_\infty &\frac{1}{h} \int_x^{x+h} \left(T^* f_2(y) - \frac{1}{h} \int_{x+h}^{x+2h} T^* f_2(z)\,dz \right)^+ dy \\ &\le ||v\chi_{[x-h,x]}||_\infty \frac{1}{h} \int_x^{x+h} \frac{1}{h} \int_{x+h}^{x+2h} \sup_{\epsilon>0} (I_\epsilon)\,dz\,dy \\ &\le C||v\chi_{[x-h,x]}||_\infty ||fv||_\infty \\ &\qquad \times \left(\sum_{k=3}^{\infty} \frac{1}{2^k} \frac{1}{2^{k+1}h} \int_x^{x+2^{k+1}h} v^{-1}\,dt + \sup_{\epsilon>0} \frac{1}{\epsilon} \int_x^{x+2\epsilon} v^{-1}\,dt \right) \\ &\le C||fv||_\infty (\sum_{k=3}^{\infty} \frac{1}{2^k} \frac{||v\chi_{[x-2^{k+1}h,x]}||_\infty}{2^{k+1}h} \int_x^{x+2^{k+1}h} v^{-1}\,dt \\ &\qquad\qquad + \sup_{\epsilon>0} \frac{||v\chi_{[x-2\epsilon,x]}||_\infty}{2\epsilon} \int_x^{x+2\epsilon} v^{-1}\,dt) \\ &\le C||fv||_\infty \ . \end{aligned}$$

Then

$$||v\chi_{[x-h,x]}||_\infty \frac{1}{h} \int_x^{x+h} \left(|T^* f_2(y)| - \frac{1}{h} \int_{x+h}^{x+2h} |T^* f_2(z)|\,dz \right)^+ dy \le C||fv||_\infty \ , \qquad (2.7)$$

so, by (2.6) and (2.7), taking sup we get Theorem 2.5. □

In the following theorem we use the extrapolation properties proved in Theorem III together with the boundedness given by Theorem 2.5, to obtain an easier proof for $(A.1)$.

Theorem 2.8. *Let K be a kernel satisfying (2.1), (2.2),(2.3) with support in the negative real line and T^* be as in (2.4). If $v \in A_p^+$, $1 < p < \infty$, then*

$$\int_{\mathbb{R}} |T^* f(x)|^p v(x)\, dx \le C \int_{\mathbb{R}} |f(x)|^p v(x)\, dx \ .$$

Proof. In order to apply Theorem III, in view of Theorem 2.5, we only have to see that if $v \in A_p^+$ and $f \in C_0^\infty(\mathbb{R})$ then $T^* f \in L^p(v)$. Assume the support of f is contained in $[-N, N]$. Clearly $T^* f(x) = 0$, if $x \ge N$; let $-2N \le x < N$, then

$$\begin{aligned}
T^* f(x) &= \sup_{\epsilon>0} \left| \int_{x+\epsilon}^N K(x-y)(f(y)-f(x))\, dy + f(x) \int_{x+\epsilon}^N K(x-y)\, dy \right| \\
&\le \sup_{\epsilon>0} \int_{x+\epsilon}^N |K(x-y)(f'(z)(y-x))\,|\, dy + \sup_{\epsilon>0} \left| f(x) \int_{x+\epsilon}^N K(x-y)\, dy \right| \\
&\le \|f'\|_\infty \sup_{\epsilon>0} \int_{x+\epsilon}^N \frac{B_2}{|y-x|}(y-x)\, dy + \sup_{\epsilon>0} \left| f(x) \int_{x+\epsilon}^N K(x-y)\, dy \right| \\
&\le B_2 \|f'\|_\infty \sup_{\epsilon>0} \int_{-2N}^N dy + \|f\|_\infty \sup_{\epsilon>0} \left| \int_{x+\epsilon}^N K(x-y)\, dy \right| \\
&\le \left(B_2 3N \|f'\|_\infty + ||f||_\infty \sup_{\epsilon>0} \left| \int_{\epsilon<|t|<N-x} K(t)\, dt \right| \right) \\
&\le (B_2 3N \|f'\|_\infty + B_1 ||f||_\infty) \ .
\end{aligned}$$

Now let us put $x < -2N$,

$$T^* f(x) = \sup_{0<\epsilon} \left| \int_{x+\epsilon}^N K(x-y) f(y)\, dy \right| \le ||f||_\infty \sup_{0<\epsilon} \int_{-N}^N B_2 \frac{1}{|y-x|}\, dy \le B_2 ||f||_\infty \frac{2N}{|x|} \ .$$

Therefore

$$T^* f(x) \le C ||f||_\infty \frac{N}{|x|} \ .$$

Putting all the estimates together we get that $T^* f \in L^p(v)$ provided

$$\int_{x \le -2N} \frac{v(x)}{|x|^p}\, dx < \infty \tag{2.9}$$

is known. The proof of the last inequality can be found in [9]. $\square$

Finally notice that Theorem 2.5 remain true replacing $T^* f$ by $|Tf|$ and Theorem 2.8 is also true replacing T^* by the singular integral operator T (see (2.4)).

3. Applications.

Applications of Theorem II.

Let us consider a lateral version of the fractional integral operator. Given $0 < \alpha < 1$, we put

$$I_\alpha^+ f(x) = \int_x^\infty \frac{f(y)}{(y-x)^{1-\alpha}}\, dy\ .$$

It is shown in [2] (see also [9] and [11]), that I_α^+ is strongly continuous from $L^p(v^p)$ into $L^q(v^q)$ for $1 < p < \alpha^{-1}$ and finite values of q. Our purpose is to give a simpler proof of this fact (see Theorem 3.7). In order to use the extrapolation Theorem II, we need first to show, the boundedness of I_α^+ from $L^{\frac{1}{\alpha}}(v^{\frac{1}{\alpha}})$ into $BMO^+(v)$.

Theorem 3.1. *Let v belong to $A^+(\frac{1}{\alpha},\infty)$, then*

$$||v\chi_{[x-h,x]}||_\infty \frac{1}{h}\int_x^{x+h}\left(|I_\alpha^+ f(y)| - \frac{1}{h}\int_{x+h}^{x+2h}|I_\alpha^+ f(z)|\, dz\right)^+ dy \le C\left(\int |fv|^{\frac{1}{\alpha}}\right)^\alpha , \quad (3.2)$$

for every $f \in L^{\frac{1}{\alpha}}(v^{\frac{1}{\alpha}}),\ x\in\mathbb{R},\ h>0$.

Proof. Let $v \in A^+(\frac{1}{\alpha},\infty)$, $f \in L^{\frac{1}{\alpha}}(v^{\frac{1}{\alpha}})$, $x\in\mathbb{R}$, and $h>0$. We set $f = f_1 + f_2$, where $f_1 = f\chi_{[x,x+4h]}$, then

$$I_\alpha^+ f = I_\alpha^+ f_1 + I_\alpha^+ f_2\ .$$

We will see that (3.2) holds for f_1 and f_2. Let us consider

$$\frac{1}{h}\int_x^{x+h}\left(|I_\alpha^+ f_1(y)| - \frac{1}{h}\int_{x+h}^{x+2h}|I_\alpha^+ f_1(z)|\, dz\right)^+ dy$$

$$\le \frac{1}{h}\int_x^{x+2h}|I_\alpha^+ f_1(y)|\, dy = \frac{1}{h}\int_x^{x+2h}\left|\int_y^{x+4h}\frac{f(t)}{(t-y)^{1-\alpha}}\, dt\right| dy$$

$$\le \frac{1}{h}\int_x^{x+4h}|f(t)|\left(\int_x^t \frac{1}{(t-y)^{1-\alpha}}\, dy\right) dt \le \frac{1}{\alpha}\frac{(4h)^\alpha}{h}\int_x^{x+4h}|f(t)|\, dt.$$

Then

$$||v\chi_{[x-h,x]}||_\infty \frac{1}{h}\int_x^{x+h}\left(|I_\alpha^+ f_1(y)| - \frac{1}{h}\int_{x+h}^{x+2h}|I_\alpha^+ f_1(z)|\, dz\right)^+ dy$$

$$\le \frac{4}{\alpha}||v\chi_{[x-4h,x]}||_\infty (4h)^{\alpha-1}\int_x^{x+4h}|f(t)|\, dt\,.$$

By Hölder for $p=\frac{1}{\alpha}$, and using that $v \in A^+(\frac{1}{\alpha},\infty)$, we have

$$||v\chi_{[x-h,x]}||_\infty \tfrac{1}{h}\int_x^{x+h}\left(|I_\alpha^+ f_1(y)| - \frac{1}{h}\int_{x+h}^{x+2h}|I_\alpha^+ f_1(z)|\,dz\right)^+ dy \le C\left(\int |fv|^{\frac{1}{\alpha}}\right)^\alpha . \quad (3.3)$$

Now we study

$$\begin{aligned}
&\frac{1}{h}\int_x^{x+h}\left(|I_\alpha^+ f_2(y)| - \frac{1}{h}\int_{x+h}^{x+2h}|I_\alpha^+ f_2(z)|\,dz\right)^+ dy \\
&\qquad \le \frac{1}{h^2}\int_x^{x+h}\int_{x+h}^{x+2h}\int_{x+4h}^{\infty}\left|\frac{f(t)}{(t-y)^{1-\alpha}} - \frac{f(t)}{(t-z)^{1-\alpha}}\right| dt\,dz\,dy\,. \qquad (3.4)
\end{aligned}$$

Now using the mean value theorem and the fact that $(t-\xi)^{\alpha-2} \le (t-z)^{\alpha-2}$, for $y \le \xi \le z$, we obtain

$$\begin{aligned}
(3.4) &\le \frac{1}{h^2}\int_x^{x+h}\int_{x+h}^{x+2h}\int_{x+4h}^{\infty}|f(t)|(1-\alpha)2h(t-z)^{\alpha-2}\,dt\,dz\,dy \\
&\le C\int_{x+h}^{x+2h}\int_{x+4h}^{\infty}|f(t)||t-x-2h|^{\alpha-2}\,dt\,dz, \qquad (3.5)
\end{aligned}$$

and by Hölder for exponents $1/\alpha$ and $1/(1-\alpha)$

$$(3.5) \le Ch\left(\int_{x+4h}^{\infty}|f(t)|^{\frac{1}{\alpha}}v^{\frac{1}{\alpha}}\,dt\right)^\alpha\left(\int_{x+4h}^{\infty}|t-x-2h|^{\frac{\alpha-2}{1-\alpha}}v^{-\frac{1}{1-\alpha}}\,dt\right)^{1-\alpha}.$$

On the other hand

$$\begin{aligned}
&||v\chi_{[x-h,x]}||_\infty h\left(\int_{x+4h}^{\infty}|t-x-2h|^{\frac{\alpha-2}{1-\alpha}}v^{-\frac{1}{1-\alpha}}\,dt\right)^{1-\alpha} \\
&\quad \le C||v\chi_{[x-h,x]}||_\infty h\sum_{k=2}^{\infty}\left(\int_{x+2^k h}^{x+2^{k+1}h}(2^{k-1}h)^{\frac{\alpha-2}{1-\alpha}}v^{-\frac{1}{1-\alpha}}\,dt\right)^{1-\alpha} \\
&\quad \le C\sum_{k=2}^{\infty}||v\chi_{[x-2^{k+1}h,x]}||_\infty(2^{k-1}h)^{\alpha-2}h\left(\int_x^{x+2^{k+1}h}v^{-\frac{1}{1-\alpha}}\,dt\right)^{1-\alpha} \\
&\quad \le C\sum\frac{1}{2^k} < \infty\,.
\end{aligned}$$

Putting together the estimates we have

$$||v\chi_{[x-h,x]}||_\infty \tfrac{1}{h}\int_x^{x+h}\left(|I_\alpha^+ f_2(y)| - \frac{1}{h}\int_{x+h}^{x+2h}|I_\alpha^+ f_2(z)|\,dz\right)^+ dy \le C\left(\int |fv|^{\frac{1}{\alpha}}\right)^\alpha \quad (3.6)$$

From (3.3) and (3.6) we get (3.2) for every $x \in \mathbb{R}$ and $h > 0$. $\square$

Now we are ready to apply the extrapolation Theorem II to prove the weighted strong (p,q) type of I_α^+.

Theorem 3.7. *If $1/q = 1/p - \alpha$, $1 < p < 1/\alpha$, and $w \in A^+(p,q)$, then*

$$||I_\alpha^+ fw||_q \leq C||fw||_p.$$

Proof. We shall get the proof as a consequence of Theorem II. In view of Theorem 3.1 we just need to see that $I_\alpha^+ f \in L^q(w^q)$ whenever $f \in C_0^\infty(\mathbb{R})$. Let $1 < p < 1/\alpha$, $1/q = 1/p - \alpha$, $w \in A^+(p,q)$, and f be with support contained in $(-N, N)$. If $x \geq N$ then $I_\alpha^+ f(x) = 0$. Taking $x \in (-2N, N)$, we get

$$|I_\alpha^+ f(x)| \leq CN^\alpha ||f||_\infty \ .$$

Finally if $x \leq -2N$,

$$|I_\alpha^+ f(x)| \leq C2N||f||_\infty \frac{1}{(-N-x)^{1-\alpha}} \ .$$

Since $1 + q/p' = q(1-\alpha)$ and $w^q \in A^+_{q(1-\alpha)}$, using an analogue of (2.9) we get the Theorem. $\square$

Applications of Theorem IV.

1. For $0 < \alpha < 1$ and $1 \leq r < \infty$, we define

$$M_\alpha^{r,+} f(x) = \sup_{h>0} \left(\frac{1}{h^{1-\alpha}} \int_x^{x+h} |f(y)|^r \, dy \right)^{\frac{1}{r}} \ .$$

When $r = 1$, $M_\alpha^{1,+} = M_\alpha^+$.

The boundedness of the bilateral case of $M_\alpha^r f$ was studied in [5]. Here we prove the one-sided result using Theorem IV.

Theorem 3.8. *Let $0 < \alpha < 1$, $1 \leq r < \infty$ and $(u^r, v^r) \in A^+(\frac{p}{r}, \frac{q}{r})$. Then there exists C such that*

$$u^q\left(\{x : M_\alpha^{+,r} f(x) > \lambda\}\right) \leq C \left(\lambda^{-p} \int |f|^p v^p \, dx\right)^{\frac{q}{p}},$$

for all $\lambda > 0$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{r}$ and $r < p < \frac{r}{\alpha}$.

Proof. We shall see that $T = M_\alpha^{r,+}$ satisfies the hypothesis of Theorem IV. Let $p_0 = \frac{r}{\alpha}$, $(a^r, b^r) \in A^+(\frac{p_0}{r}, \infty)$ and $h > 0$. Then, by Hölder's inequality,

$$\frac{1}{h^{1-\alpha}} \int_x^{x+h} |f|^r \leq \frac{1}{h^{1-\alpha}} \left(\int_x^{x+h} |f|^{\frac{r}{\alpha}} b^{\frac{r}{\alpha}} \right)^\alpha \left(\int_x^{x+h} b^{-\frac{r}{1-\alpha}} \right)^{1-\alpha} \ .$$

Note that $a(x)$ is finite almost everywhere. Therefore for every x such that $a^r(x) \leq ||a^r \chi_{[x-h,x]}||_\infty$,

$$
\begin{aligned}
&a(x)\left(\frac{1}{h^{1-\alpha}}\int_x^{x+h}|f|^r\right)^{\frac{1}{r}}\\
&\leq ||a^r\chi_{[x-h,x]}||_\infty^{\frac{1}{r}}\left[\frac{1}{h^{1-\alpha}}\left(\int_x^{x+h}|f|^{\frac{r}{\alpha}}b^{\frac{r}{\alpha}}\right)^{\alpha}\left(\int_x^{x+h}b^{-\frac{r}{1-\alpha}}\right)^{1-\alpha}\right]^{\frac{1}{r}}\\
&\leq C||fb||_{p_0}\ .
\end{aligned}
$$

Then it follows that

$$||aTf||_\infty \leq C||fb||_{p_0}\ . \tag{3.9}$$

Now an application of Theorem IV, proves the theorem. □

Taking $r = 1$ we obtain the known result due to M. Gabidzashvili and V. Kokilashvili, [3]:

Theorem 3.10. *Let $(u,v) \in A^+(p,q)$. Then there exists C such that*

$$u^q\left(\{x : |M_\alpha^+ f(x)| > \lambda\}\right) \leq C\left(\lambda^{-p}\int |f|^p v^p\, dx\right)^{\frac{q}{p}} \quad \forall\lambda > 0\ ,$$

with $\frac{1}{q} = \frac{1}{p} - \alpha$ and $1 < p < \frac{1}{\alpha}$.

2. As another application of Theorem IV, we present next the one-sided version of a result proved in [5].

Theorem 3.11. *Let $Tf(x) = (I_\alpha^+|f|(x))_+^\#$. If $(u,v) \in A^+(p,q)$, with $\frac{1}{p} - \frac{1}{q} = \alpha$, then*

$$u^q(\{x : (I_\alpha^+|f|(x))_+^\# > \lambda\}) \leq C\left(\lambda^{-p}\int |f|^p v^p\, dx\right)^{\frac{q}{p}}$$

for every $\lambda > 0$.

Proof. It is known that (see [9])

$$(I_\alpha^+|f|(x))_+^\# \leq C_\alpha M_\alpha^+ f(x)\,. \tag{3.12}$$

From (3.9) it follows that

$$||aM_\alpha^+ f||_\infty \leq C||fb||_{\frac{1}{\alpha}}\ ,$$

for every pair $(a,b) \in A^+(\frac{1}{\alpha},\infty)$. Using (3.12) we obtain

$$||a(I_\alpha^+|f|)_+^\#||_\infty \leq C_\alpha||fb||_{\frac{1}{\alpha}}\ ,$$

for every pair $(a,b) \in A^+(\frac{1}{\alpha},\infty)$. Then, applying Theorem IV, we have

$$u^q(\{x : (I_\alpha^+|f|(x))_+^\# > \lambda\}) \leq C\left(\lambda^{-p}\int |f|^p v^p\, dx\right)^{\frac{q}{p}}\ ,$$

for all $\lambda > 0$ and $(u,v) \in A^+(p,q)$ with $\frac{1}{p} - \frac{1}{q} = \alpha$. □

Application of Theorem V.

The following Theorem provides a way to give a simpler proof of a result published in [10].

Theorem 3.12. *Let* $(u,v)\in A^+_{p,q,\alpha,\beta}$ *with* $0\le\beta<\alpha\le 1$, *and* $\frac{1}{p}-\frac{1}{q}=\alpha-\beta$, *then*

$$u(\{x: M^+_{\alpha,\beta}f(x)>\lambda\})\le C\left(\lambda^{-p}\int |f|^p v\right)^{q/p}$$

for all $\lambda>0$

Proof. Let $T=M^+_{\alpha,\beta}$, $0\le\beta<\alpha\le 1$, $p_0^{-1}=\alpha-\beta$, and $r=1-\frac{(1-\alpha)}{(1-\alpha+\beta)}$ if $\alpha<1$, and $r=1$ if $\alpha=1$. By Theorem V we only have to show that

$$||aM^+_{\alpha,\beta}f||_\infty\le ||fb||_{\frac{1}{\alpha-\beta}}$$

for every pair of weights (a,b) such that $(a,b^{p_0})\in A^+_{p_0,\infty,r,r}$. In order to see this, notice that if $(a,b^{p_0})\in A^+_{p_0,\infty,r,r}$ then, $a(x)(M^+_{r,r}b^{-p_0'})^{1/p_0'}\le C$. Therefore

$$\begin{aligned} a(x)\frac{1}{(c-x)^\beta}\int_x^c\frac{|f(t)|\,dt}{(c-t)^{1-\alpha}} &\le a(x)||fb||_{p_0}\frac{1}{(c-x)^\beta}\left(\int_x^c\frac{b^{-p_0'}(t)\,dt}{(c-t)^{p_0'(1-\alpha)}}\right)^{1/p_0'}\\ &\le a(x)||fb||_{p_0}(M^+_{r,r}b^{-p_0'}(x))^{1/p_0'}\\ &\le C||fb||_{p_0}. \quad\square \end{aligned}$$

4. Proofs of the Extrapolation Results

In order to shorten notation, we shall denote $||f||_{p,v}=\left(\int|f|^pv\right)^{1/p}$. We will first begin by studing some relations between $|||\,.\,|||_{v,+}$ and $||\,.\,||_{p,v}$.

Lemma 4.1. *Let* $v\ge 0$ *be a locally integrable function, then we have*

$$\|vf^\#_+\|_\infty\ \le\ |||f|||_{v,+}\ \le\ c\|vM^+f\|_\infty\ .$$

Proof. Let x be a Lebesgue point of v with $v(x)>0$, we have for all $h>0$, $0<v(x)\le\|v\chi_{[x-h,x]}\|_\infty$. Thus,

$$\begin{aligned} &v(x)\frac{1}{h}\int_x^{x+h}\left(f(y)-\frac{1}{h}\int_{x+h}^{x+2h}f(z)\,dz\right)^+dy\\ &\qquad\le\|v\chi_{[x-h,x]}\|_\infty\frac{1}{h}\int_x^{x+h}\left(f(y)-\frac{1}{h}\int_{x+h}^{x+2h}f(z)\,dz\right)^+dy\le|||f|||_{v,+}\ . \end{aligned}$$

Then $v(x)f_+^\#(x) \le |||f|||_{v,+}$ for almost every x, so

$$\|vf_+^\#\|_\infty \le |||f|||_{v,+}.$$

We may assume that $\|v\chi_{[x-h,x]}\|_\infty > 0$. Let $0 < M < \|v\chi_{[x-h,x]}\|_\infty$. Then there exist a $t \in [x-h,x]$ such that $M < v(t)$. Then

$$\begin{aligned} &M\frac{1}{h}\int_x^{x+h}\left(f(y) - \frac{1}{h}\int_{x+h}^{x+2h} f(z)\,dz\right)^+ dy \\ &\le v(t)\frac{1}{h}\int_t^{x+2h} |f| \le 3v(t)\cdot M^+f(t) \\ &\le C\|vM^+f\|_\infty\ . \end{aligned}$$

Letting M tend to $\|v\chi_{[x-h,x]}\|_\infty$ we are done. $\square$

Theorem 4.2. *Let $v \in A_\infty^+$ and $f \ge 0$. Assume for some p_0, $1 < p_0 < \infty$, $\int (M^+f)^{p_0} w < \infty$. Then for every $p_0 \le p < \infty$*

$$\int_{-\infty}^{\infty} (M^+f)^p v \le C\int_{-\infty}^{\infty} (f_+^\#)^p v\,.$$

Proof. See [9]. $\square$

Corollary 4.3. *Given $1 < p_0 \le p$, $v \in A_{p_0}^+$ and $f \ge 0$ such that $f \in L^{p_0}(v)$, there exists C independent of p_0 and f such that*

$$\|f\|_{p,v} \le C\|f_+^\#\|_{p,v}\,.$$

Proof. Since $\int (M^+f)^{p_0} w \le \int |f|^{p_0} w < \infty$, (see for instance [7]) and $f(x) \le M^+f(x)$ almost everywhere, by Theorem 4.2

$$\|f\|_{p,v} \le \|M^+f\|_{p,v} \le C\|f_+^\#\|_{p,v}\ . \quad \square$$

We shall also make use of the following lemma proved in [8].

Lemma 4.4.
i. Let $v \in A_p^+$ and $1 \le p_0 < p < \infty$ then for all $h \ge 0$ in $L^{(p/p_0)'}(v)$ there exists $H \ge h$ such that $Hv \in A_{p_0}^+$ and

$$\|H\|_{(p/p_0)',v} \le C\|h\|_{(p/p_0)',v}$$

ii. Let $v \in A_p^-$ and $1 \le p_0 < p < \infty$ then for all $h \ge 0$ in $L^{(p/p_0)'}(v)$ there exists $H \ge h$ such that $Hv \in A_{p_0}^-$ and

$$\|H\|_{(p/p_0)',v} \le C\|h\|_{(p/p_0)',v}\ .$$

Finally, we recall some definitions concerning the Lorentz $L(p,q,\mu)$ spaces. Let f be a measurable function on a measure space $(M,\mathcal{M},\mu)$. The non-increasing rearrangement f^* of f is defined as

$$f^*(t)=\inf\{s:\mu(\{x:|f(x)|>s\})\le t\}\ ,$$

for $t>0$. The function f is said to belong to the Lorentz space $L(p,q,\mu)$ if

$$||f||_{p,q,\mu}=\left(\frac{p}{q}\int_0^\infty [t^{\frac{1}{p}}f^*(t)]^q\frac{dt}{t}\right)^{\frac{1}{q}}<\infty\ ,$$

whenever $1<p<\infty$ and $1<q<\infty$, and

$$||f||_{p,\infty,\mu}=\sup_{t>0}t^{\frac{1}{p}}f^*(t)\ ,$$

when $1<p\le\infty$ and $q=\infty$. For more details see [14].

Proof of Theorem I. Let $v\in A^+(p,q)$. Let us assume first $p>p_0$, thus $q>q_0$ and

$$\left(\int |Tf|^q v^q\right)^{1/q}=\left(\int |Tf|^{q_0}gv^q\right)^{1/q_0}\ ,$$

holds with some $g\ge 0$, $\|g\|_{(q/q_0)',v^q}=1$. We observe that $v\in A^+(p,q)$ if and only if $v^q\in A_r^+$ with $r=1+q/p'$. Let us put $r_0=1+q_0/p_0'$, $h=g$ and $w=v^q$. As $r/r_0=q/q_0$, by Lemma 4.4-i, there exists $H\ge g$ such that $\|H\|_{(q/q_0)',v^q}\le C$ and $Hv^q\in A_{r_0}^+$. This implies $H^{1/q_0}v^{q/q_0}\in A^+(p_0,q_0)$. Therefore, noting that $p_0\,q/q_0=p_0+q\,(1-p_0/p)$,

$$\begin{aligned}\left(\int |Tf|^q v^q\right)^{1/q}&\le\left(\int |Tf|^{q_0}(H^{1/q_0}v^{q/q_0})^{q_0}\right)^{1/q_0}\le C\left(\int |f|^{p_0}(H^{1/q_0}v^{q/q_0})^{p_0}\right)^{1/p_0}\\&=C\left(\int |f|^{p_0}v^{p_0}H^{p_0/q_0}v^{q(\frac{1}{(p/p_0)'})}\right)^{1/p_0}\ .\end{aligned}$$

Using Hölder's inequality

$$\begin{aligned}\left(\int |Tf|^q v^q\right)^{1/q}&\le C\left[\left(\int |f|^p v^p\right)^{\frac{p_0}{p}}\left(\int H^{\frac{p_0}{q_0}(\frac{p}{p_0})'}v^{q\frac{1}{(\frac{p}{p_0})'}(\frac{p}{p_0})'}\right)^{\frac{1}{(p/p_0)'}}\right]^{\frac{1}{p_0}}\\&=C\left(\int |f|^p v^p\right)^{\frac{1}{p}}\left(\int H^{(\frac{r}{r_0})'}v^q\right)^{\frac{1}{p_0}-\frac{1}{p}}\\&\le C\|f\|_{p,v^p}\ .\end{aligned}$$

Now if $p_0>p$, then $q_0>q$ and we have

$$\left(\int |f|^p v^p\right)^{\frac{1}{p}}=\left(\int\left(|fv^{p'}|^{p_0}\right)^{\frac{p}{p_0}}v^{-p'}\right)^{\frac{p_0}{p}\frac{1}{p_0}}\ .$$

Therefore, there exists (see [6], Theorem 210), $g \geq 0$ that satisfies

$$\int g^{\frac{p}{p-p_0}} v^{-p'}\, dx = 1\,, \qquad \text{and} \qquad \left(\int |f|^p v^p\right)^{\frac{1}{p}} = \left(\int |f v^{p'}|^{p_0} g v^{-p'}\right)^{\frac{1}{p_0}} .$$

Let $h = g^{-p_0'/p_0}$, $w = v^{-p'}$, $r = 1 + p'/q$ and $r_0 = 1 + p_0'/q_0$. Since $(r/r_0)'(-p_0'/p_0) = p/(p-p_0)$, we have that $\int h^{(\frac{r}{r_0})'} w\, dx = 1$. On the other hand, $v \in A^+(p,q)$ if and only if $v^{-p'} \in A_r^-$ with $r = 1 + p'/q$. By Lemma 4.4-ii there exists $H \geq h$ such that $\int H^{(\frac{r}{r_0})'} v^{-p'} \leq C$ and $H v^{-p'} \in A_{r_0}^-$. Hence $[Hv^{-p'}]^{-1/p_0'} \in A^+(p_0, q_0)$. Thus

$$\begin{aligned}\left(\int |f|^p v^p\right)^{\frac{1}{p}} &= \left(\int |f v^{p'}|^{p_0} g v^{-p'}\right)^{\frac{1}{p_0}} = \left(\int |f|^{p_0} h^{-\frac{p_0}{p_0'}} v^{-p'(1-p_0)}\right)^{\frac{1}{p_0}} \\ &\geq \left(\int |f|^{p_0} \left[H^{-\frac{1}{p_0'}} v^{\frac{p'}{p_0'}}\right]^{p_0}\right)^{\frac{1}{p_0}} \geq C \left(\int |Tf|^{q_0} \left[H^{-\frac{1}{p_0'}} v^{\frac{p'}{p_0'}}\right]^{q_0}\right)^{\frac{1}{q_0}} .\end{aligned}$$

By Hölder's inequality

$$\begin{aligned}\left(\int |f|^p v^p\right)^{\frac{1}{p}} &\geq C \left(\int |Tf|^q v^q\right)^{\frac{1}{q}} \left(\int H^{(\frac{r}{r_0})'} v^{-p'}\right)^{\frac{q-q_0}{q_0 q}} \\ &\geq C' \left(\int |Tf|^q v^q\right)^{\frac{1}{q}} . \quad \square\end{aligned}$$

Proof of Theorem II. Let $v \in A^+(p,q)$, $1/p - 1/q = 1/p_0$ and $f \in L^p(v^p)$, then

$$\left(\int |f|^p v^p\right)^{\frac{1}{p}} = \left(\int \left(|f v^{p'}|^{p_0}\right)^{p/p_0} v^{-p'}\right)^{\frac{p_0}{p}\frac{1}{p_0}} .$$

So there exists $g \geq 0$ such that $\int g^{(p/p_0)'} v^{-p'} = 1$ and

$$\left(\int |f|^p v^p\right)^{\frac{1}{p}} = \left(\int |f v^{p'}|^{p_0} g v^{-p'}\right)^{1/p_0} .$$

Let $h = g^{-p_0'/p_0}$, thus

$$1 = \int g^{(p/p_0)'} v^{-p'}\, dx = \int h^{q/p_0'} v^{-p'} .$$

Taking $r = 1 + p'/q$, $r_0 = 1$ and $w = v^{-p'}$ then $w \in A_r^-$. Observe that $(r/r_0)' = (1 + p'/q)' = q/p_0'$. Applying Lemma 4.4-ii there exists $H \geq h$ with $Hv^{-p'} \in A_1^-$ and

$$\int H^{q/p_0'} v^{-p'} \leq C . \tag{4.5}$$

Since $v \in A^+(p,\infty)$ if and only if $v^{-p'} \in A_1^-$, we have $H^{-1/p_0'}v^{p'/p_0'} \in A^+(p_0,\infty)$. Hence by Lemma 4.1 and (4.5)

$$\begin{aligned}\left(\int |f|^p v^p\right)^{\frac{1}{p}} &= \left(\int |fv^{p'}|^{p_0} g v^{-p'}\right)^{1/p_0} = \left(\int |f|^{p_0}\left(h^{-1/p_0'}v^{\frac{p'}{p_0'}}\right)^{p_0}\right)^{1/p_0}\\ &\geq \left(\int |f|^{p_0}\left(H^{-1/p_0'}v^{\frac{p'}{p_0'}}\right)^{p_0}\right)^{1/p_0} \geq C\||\, |Tf|\, \||_{H^{-1/p_0'}v^{p'/p_0'},+}\\ &\geq C\|H^{-1/p_0'}v^{p'/p_0'}(Tf)^{\#}_{+}\|_\infty\\ &\geq C\left(\int H^{-q/p_0'}v^{qp'/p_0'}(Tf)^{\#\,q}_{+}H^{q/p_0'}v^{-p'}\right)^{\frac{1}{q}} = C\left(\int (Tf)^{\#\,q}_{+}v^q\right)^{\frac{1}{q}}.\end{aligned}$$

Since $q>1$ and $\int |Tf|^q v^q < \infty$, by Corollary 4.3 we obtain

$$\left(\int |Tf|^q v^q\right)^{\frac{1}{q}} \leq \left(\int |f|^p v^p\right)^{\frac{1}{p}} . \quad \square$$

Before proving Theorem III, we need some previous results.

Lemma 4.6. *If $\Phi \in L^1$ and $\int |\Phi| = 1$, then there exists h such that $\int |h| \leq 2$ and $\|\Phi h^{-1}\|_\infty = 1$.*

Proof. Let

$$h(x) = \begin{cases} \Phi(x) & \text{if } \Phi(x) \neq 0\\ e^{-\pi|x|^2} & \text{if } \Phi(x) = 0\,,\end{cases}$$

clearly $\|\Phi h^{-1}\|_\infty = 1$ and

$$\int |h| = \int_{\{x\,:\,\Phi(x)\neq 0\}} |\Phi| + \int_{\{x\,:\,\Phi(x)=0\}} e^{-\pi|x|^2} \leq 2\,. \quad \square$$

Corollary 4.7. *Given $f \in L^p(v)$, then there exists $g \in L^p(v^{-\frac{1}{p-1}})$, $g \geq 0$, such that*

$$\int g^p v^{-\frac{1}{p-1}} \leq 2 \qquad \text{and} \qquad \left(\int |f|^p v\right)^{\frac{1}{p}} = \|f v^{\frac{1}{p-1}} g^{-1}\|_\infty.$$

Proof. If $\int |f|^p v \neq 0$ take

$$\Phi = \frac{|f|^p v}{\int |f|^p v}\,,$$

then $\int |\Phi| = 1$. Let h be the function given by Lemma 4.6. If we put

$$g^p = \begin{cases} v^{\frac{1}{p-1}} h & v \neq 0\\ 0 & v = 0\,,\end{cases}$$

it follows immediately that $\int g^p v^{-\frac{1}{p-1}} = \int h \le 2$ and

$$1 = \|\Phi h^{-1}\|_\infty = \frac{1}{\int |f|^p v} \| \, |f|^p v^{\frac{p}{p-1}} g^{-p}\|_\infty .$$

The case $\int |f|^p v = 0$ is clear. $\square$

Proof of Theorem III. Let $v \in A_p^+$, $f \in L^p(v)$ and g the function given by Corollary 4.7. Let $w = v^{-\frac{1}{p-1}}$, $r = p'$, $r_0 = 1$ and $h = g \ge 0$. Then we have $w \in A_{p'}^-$, and $\int h^p w = \int h^{(r/r_0)'} w \le 2$. Using Lemma 4.4-ii, there exists $H \ge g$ such that $Hw = Hv^{-\frac{1}{p-1}} \in A_1^-$ and

$$\Big(\int H^p v^{-\frac{1}{p-1}}\Big)^{\frac{1}{p}} \le C .$$

Then

$$\Big(\int |f|^p v \, dx\Big)^{\frac{1}{p}} = \|f v^{\frac{1}{p-1}} g^{-1}\|_\infty \ge \|f v^{\frac{1}{p-1}} H^{-1}\|_\infty .$$

As $Hv^{-\frac{1}{p-1}} \in A_1^-$, by hypothesis and Lemma 4.1, we obtain

$$\| \, |Tf|_+^{\#} v^{\frac{1}{p-1}} H^{-1}\|_\infty \le ||| \, |Tf| \, |||_{v^{\frac{1}{p-1}} H^{-1},+} \le C \|f v^{\frac{1}{p-1}} H^{-1}\|_\infty .$$

Collecting the estimates,

$$\begin{aligned}
\left(\int |f|^p v \, dx\right)^{\frac{1}{p}} &\ge C \| \, |Tf|_+^{\#} v^{\frac{1}{p-1}} H^{-1}\|_\infty \left(\int H^p v^{-\frac{1}{p-1}} \, dx\right)^{\frac{1}{p}} \\
&\ge C \left(\int |Tf|_+^{\# \, p} v^{\frac{p}{p-1}} H^{-p} H^p v^{-\frac{1}{p-1}} \, dx\right)^{\frac{1}{p}} \\
&= C \left(\int |Tf|_+^{\# \, p} v \, dx\right)^{\frac{1}{p}} \ge C \left(\int |Tf|^p v \, dx\right)^{\frac{1}{p}} .
\end{aligned}$$

The last inequality follows from Corollary 4.3. $\square$

Proof of Theorem IV. Let $f \in C_0^\infty(\mathbb{R})$, $0 < m = \int |f|^p v^p$, and $(u^r, v^r) \in A^+\left(\frac{p}{r}, \frac{q}{r}\right)$. We define

$$b(x) = \begin{cases} |f(x)|^{p/p_0 - 1} v(x)^{p/p_0} m^{\frac{1}{q}} & \text{if} \quad |f(x)| > 0 \\ e^{\pi \frac{|x|^2}{q}} v(x) & \text{if} \quad |f(x)| = 0 . \end{cases}$$

Thus

$$||fv||_p = ||fb||_{p_0} \qquad \text{and} \qquad \int b^{-q} v^q \, dx \le 2$$

Let us put

$$a(x) = \left(M^+ b^{-r(\frac{p_0}{r})'}(x)\right)^{-\frac{1}{r(\frac{p_0}{r})'}} .$$

It follows immediately that $(a^r, b^r) \in A^+(\frac{p_0}{r}, \infty)$. Let $E_\lambda = \{x : |Tf(x)| > \lambda\}$. Hence

$$\begin{aligned} u^q(E_\lambda) &= \int_{E_\lambda} u^q = \int \chi_{E_\lambda}(x) a^{-1}(x) a(x) u^q(x)\, dx \\ &\leq ||\chi_{E_\lambda}||_{(1+1/q,1,au^q)} ||a^{-1}||_{(q+1,\infty,au^q)} \ . \end{aligned}$$

In order to estimate the second factor above, we observe that

$$\lambda^{q+1} \int_{\{x: a(x)^{-1} > \lambda\}} au^q \leq \lambda^q \int_{\{x: M^+ b^{-r(\frac{p_0}{r})'}(x) > \lambda^{r(\frac{p_0}{r})'}\}} u^q \ . \tag{4.8}$$

Recalling $(u^r, v^r) \in A^+\left(\frac{p}{r}, \frac{q}{r}\right)$, implies $(u^q, v^q) = (u^{r\frac{q}{r}}, v^{r\frac{q}{r}}) \in A_s^+$, for $s = 1 + (q/r)/(p/r)'$, then it follows that M^+ is weakly bounded from $L^s(v^q)$ into $L^s(u^q)$ (see [13], [8], [7]). Therefore

$$(4.8) \leq C \frac{\lambda^q}{(\lambda^{r(\frac{p_0}{r})'})^s} \int (b^{-r(\frac{p_0}{r})'})^s v^q = C \int b^{-q} v^q \leq 2C \ .$$

We consider the non-increasing rearrangement of a^{-1} respect to the measure au^q,

$$(a^{-1})^*(t) = \inf\{y : \, au^q(\{x : \, |a^{-1}| > y\}) \leq t\} \leq \inf\{y : \, \frac{C}{y^{q+1}} \leq t\} = \left(\frac{C}{t}\right)^{\frac{1}{q+1}} \ ;$$

then, we have

$$||a^{-1}||_{(q+1,\infty,au^q)} = \sup_{t>0} \ t^{\frac{1}{q+1}} (a^{-1})^*(t) \leq C \ .$$

A non-increasing rearragement of χ_{E_λ} with respect to the measure au^q is $\chi_{[0,R)}$ with $R = \displaystyle\int_{E_\lambda} au^q$, then

$$||\chi_{E_\lambda}||_{(1+\frac{1}{q},1,au^q)} = \frac{q}{q+1} \int_0^R t^{\frac{q}{q+1}} \frac{dt}{t} = \frac{q}{q+1} \int_0^R t^{-\frac{1}{q+1}}\, dt = R^{\frac{q}{q+1}}.$$

On the other hand

$$R = \int_{E_\lambda} au^q \leq \lambda^{-1} \int_{E_\lambda} |Tf| au^q \leq \lambda^{-1} ||aTf||_\infty \int_{E_\lambda} u^q \leq C\lambda^{-1} ||fb||_{p_0} \int_{E_\lambda} u^q \ .$$

then

$$||\chi_{E_\lambda}||_{(1+\frac{1}{q},1,au^q)} \leq C\lambda^{-\frac{q}{q+1}} ||fb||_{p_0}^{\frac{q}{q+1}} (\int_{E_\lambda} u^q)^{\frac{q}{q+1}} \ .$$

Since $f \in C_0^\infty(\mathbb{R}))$, it follows that

$$u^q(E_\lambda) \leq ||\chi_{E_\lambda}||_{(1+1/q,1,au^q)} ||a^{-1}||_{(q+1,\infty,au^q)} \leq C\lambda^{-\frac{q}{q+1}} ||fv||_p^{\frac{q}{q+1}} (u^q(E_\lambda))^{\frac{q}{q+1}} \ ,$$

and as $u^q(E_\lambda)$ is finite, we get

$$u^q(\{x : |Tf(x)| > \lambda\}) \leq C(\frac{1}{\lambda^p} \int f^p v^p)^{\frac{q}{p}} \ . \quad \square$$

Proof of Theorem V. The proof is similar to the one given in Theorem IV. Let α, β be such that $0 \leq \beta < \alpha \leq r \leq 1$, $\alpha - \beta = \frac{1}{p_0}$, $\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0}$, and $(u, v) \in A^+_{p,q,\alpha,\beta}$. We only need to produce a weight satisfiying

$$||fv||_p = ||fb||_\beta \qquad \text{and} \qquad \int b^{-q} v^q \, dx \leq 2.$$

Then defining $a(x) = (M^+_{r,r} b^{-p'_0})^{-1/p'_0}(x)$, obviously, we have that the pair $(a, b^{p_0}) \in A^+_{p_0,\infty,r,r}$. Notice that if $(u, v) \in A^+_{p,q,\alpha,\beta}$ then $(u, v^{q/p}) \in A^+_{s,s,\alpha,\alpha}$ with $s = q(1-(\alpha-\beta)) = \frac{q}{p'_0}$. This implies $(u, v^{q/p}) \in A^+_{s,s,r,r}$ for all $1 \geq r \geq \alpha$. In order to get a bound similar to (4.8), we need to use (1.1) for $\alpha = \beta = r$ and $p = q = s$. $\square$

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