RESTRICTION OF SQUARE INTEGRABLE REPRESENTATIONS: DISCRETE SPECTRUM

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Abstract. In this note, whenever we restrict a square integrable representation of a connected semisimple Lie group to a reductive subgroup, we obtain information about the discrete spectrum.

1. Introduction

Let G be a connected semisimple matrix Lie group. Henceforth, we fix a connected reductive subgroup H of G and a maximal compact subgroup K of G such that $H \cap K$ is a maximal compact subgroup of H. We fix Haar measures in G and H and assume that group G have a nonempty Discrete Series. Let (π, V) be a square integrable representation of G and let (τ, W) its lowest $K-$type [2], [6]. Then (τ, W) has multiplicity one in the restriction of π to K. Let $E := G \times_\tau W \longrightarrow G/K$ be the $G-$homogeneous, hermitian, smooth vector bundle attached to the representation τ. We denote its space of L^2- (resp. smooth) sections by

$$L^2(G,\tau) = \{f : G \longrightarrow W, f(gk) = \tau(k^{-1})f(g),\ g \in G,\ k \in K,\ \int_G |f(g)|^2 dg < \infty\}$$

(resp. $C^\infty(G,\tau)$) The Lie algebra of a Lie group will be denoted by the corresponding German lower case letter, the complexification of a real Lie algebra $\mathfrak{n}$ will be denoted by $\mathfrak{n}_{\mathbb{C}}$. Let Ω be the Casimir element of the universal enveloping algebra of $\mathfrak{g}$ and let $\bar{\Omega}$ denote its closure as a linear operator on $L^2(G,\tau)$. Then,

$$H^2(G,\tau) := \{f \in L^2(G,\tau) : \quad \bar{\Omega}(f) = (\|\lambda\|^2 - \|\rho\|^2)f\ \}$$

is a closed linear subspace of $L^2(G,\tau)$ on which G acts continuously and isometrically. Thus, G acts on $H^2(G,\tau)$ by a unitary representation. In [4] and [1] it is shown that (π, V) is equivalent to the representation of G in $H^2(G,\tau)$. From now on, we think of (π, V) as the representation of G in $H^2(G,\tau)$. Since $\bar{\Omega}$ is an elliptic and real analytic coefficients linear operator we have that $H^2(G,\tau)$ is contained in the space of real analytic sections of the bundle $E \to G/K$. Let $(\tau_\star, W)$ denote the restriction of τ to the subgroup $H \cap K$. Let $F := H \times_{\tau_\star} W \longrightarrow H/(H \cap K)$ denote the associated $H-$homogeneous, hermitian bundle over $H/(H \cap K)$. Owing to our choice, $H/(H \cap K)$ can be thought as the orbit of H through the point eK on G/K and F as a subbundle of E over this orbit. Let

$$r : C^\infty(G,\tau) \longrightarrow C^\infty(H,\tau_\star)$$

denote the restriction map. The first result of this note is:

Theorem 1. *If we further assume that the representation (π, V) is integrable, then*
1) $r(H^2(G,\tau)) \subset L^2(H,\tau_\star)$
2) $r : H^2(G,\tau) \longrightarrow L^2(H,\tau_\star)$ is a continuous map.

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In [14] it is computed the image of r for $G = SO(2n,1)$ and $H = SO(2n-1,1)$; in [9] for G a semisimple Lie group so that (G,K) is a hermitian symmetric pair, H is such that $H/(H\cap K)$ is a real form for the hermitian symmetric space G/K and π a holomorphic discrete series representation with a one dimensional lowest $K-$type; in [5] it is computed the image of r for $G = G_1 \times G_1$, H the diagonal elements and $\pi = \pi_1 \otimes \pi_1^*$, π_1 being a holomorphic Discrete Series representation of G_1.

In order to state the second result we assume, as we may, that H is invariant under the Cartan involution associated to K. Thus, we have the $Ad(H\cap K)-$invariant decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, and $\mathfrak{q} = \mathfrak{q}\cap\mathfrak{k} \oplus \mathfrak{q}\cap\mathfrak{s}$. For each nonnegative integer m let $S^m(\mathfrak{q}\cap\mathfrak{s})$ denote the $m^{th}-$symmetric power of $\mathfrak{q}\cap\mathfrak{s}$. Thus, $S^m(\mathfrak{q}\cap\mathfrak{s})\otimes W$ is a $H\cap K-$module. Let (ρ, Z) be an $H-$irreducible constituent of (π, V). In [13] it is proved that ρ is an square integrable representation for H. We have,

Theorem 2. *There exists $m \geq 0$ and a $H\cap K$ type of Z which is $H\cap K$ type of $S^m(\mathfrak{q}\cap\mathfrak{s})\otimes W$*

For a converse to Theorem 2, in [14] we have shown that if the lowest $H\cap K-$type of a discrete series of H is contained in $(\tau_\star, W)$, then the such a discrete series is a subrepresentation of $res_H(\pi)$. In [8], it is shown that holomorphic Discrete Series representations restricts discretely for suitable reductive subgroups H. Since a $H\cap K$ irreducible representation is contained in at most finitely many inequivalent square integrable representations for H, we have that Theorem 2, in some sense, is the best we can achieve.

Next we state a corollary to the previous theorem. For this, we denote by π_d the closed subspace of V spanned by the irreducible $H-$subrepresentations of the restriction $res_H(\pi)$ of π to H. We recall that the abstract Plancherel theorem let us write $res_H(\pi) = \pi_d \oplus \pi_{cont}$. Thus, if $L := \{i \in \hat{H} : Hom_H(Z_i, V) \neq 0\}$, we may write as a Hilbert sum, $\pi_d = \sum_{i\in L} Z_i \otimes Hom_H(Z_i, V)$.

Corollary 1. *Assume that $H\cap K$ acts trivially on $\mathfrak{q}\cap\mathfrak{s}$. Then L is a finite set.*

We will actually show, for $j \in L$, that the lowest $H\cap K-$type of Z_j is contained in $(\tau_\star, W)$. Examples of pair (G,H) as above are $(SO(2n,1), SO(2n-2k,1))$ for $k \geq 2$ and $(SU(n,1), SU(n-k,1))$ for $k \geq 1$.

Finally, we have the following,

Proposition 1. *Assume that G is either of the groups $SO(2n,1), SU(n,1), U(n,1)$ and H is one of $SO(2k)\times SO(2n-2k,1), SU(k)\times SU(n-k,1), U(k)\times U(n-k,1)$ immersed in the usual way, in the obvious G. Let π be a square integrable representation of G. Then the multiplicity of each discrete factor in the restriction of π to H is finite.*

For certain pairs (G,H) we can assure that multiplicity of the discrete factors is infinite. In fact, assume that the centralizer of H in G contains a semisimple non compact subgroup H_2. Next, we consider the pair $(G, H_2 \times H)$. In [14] it is proved that π restricted to $H_2 \times H$ contains a discrete factor. Since the irreducible representations of $H_2 \times H$ are exterior tensor product of representations of H_2 by representations of H. We obtain that the restriction of π to H contains discrete factors with infinity multiplicity. Examples of such a pairs are $(SO(2p+2q, 2r+2s), SO(2r,2s)), p > 0, q > 0, r > 0, s > 0$.

2. Proof of Theorem 1

For any square integrable representation In [14] it is proven,

(1) If $f \in H^2(G,\tau)$ is a $K-$finite function, then $r(f) \in L^2(H, \tau_\star)$
(2) Let $\mathcal{D} := \{f \in H^2(G,\tau) : r(f) \in L^2(H,\tau_\star)\}$, then $r : \mathcal{D} \longrightarrow L^2(H,\tau_\star)$ is a closed densely defined linear transformation.

Hence, if $r^\star$ denotes the adjoint linear transformation to $r : \mathcal{D} \longrightarrow L^2(H, \tau_\star)$, we have that $r^\star$ is a closed densely defined linear transformation from $L^2(H, \tau_\star)$ to $H^2(G, \tau)$ and we may write the polar decomposition for $r^\star$,

$$r^\star = U(rr^\star)^{\frac{1}{2}}$$

Therefore, the continuity of $(rr^\star)$ gives the continuity of $r^\star$ and hence the continuity of r. In order to verify the continuity of $(rr^\star)$ we recall the reproducing kernel for $H^2(G, \tau)$. Since $\bar{\Omega}$ is an elliptic operator, L^2 convergence in $H^2(G, \tau)$ implies uniform convergence on the induced topology by $C^\infty(G, \tau)$ ([10] Theorem 52.1) Thus, point evaluation are continuous linear functionals in $H^2(G, \tau)$. Therefore, the orthogonal projector of $L^2(G, \tau)$ onto $H^2(G, \tau)$ is an integral operator given by a smooth kernel, $k : G \times G \longrightarrow End_{\mathbb{C}}(W), x, y \in G$. In [15] it is explicitely computed this kernel k. In order to describe the kernel k, we fix a $K-$equivariant immersion $i : W \rightarrow H^2(G, \tau)$ whose adjoint linear map is the linear map $e : H^2(G, \tau) \rightarrow W$ defined via evaluation at the identity of G. Finally, let $P : H^2(G, \tau) \rightarrow i(W)$ denote the orthogonal projector onto the $K-$type W in $H^2(G, \tau)$. Then, we have

$$k(x, y)(v) = e(P(\pi(y^{-1}x)(i(v)))),\ x, y \in G.$$

In particular, we have,

$$(f(z), v)_W = (f, k(?, z)v)_{L^2(G,\tau)},\ f \in H^2(G, \tau),\ z \in G,\ v \in W. \tag{1}$$

Here, $(.., ..)_V$ denotes the inner product in the Hilbert space V. For further use we notice that

$$k(x)(v) := e(P(\pi(x^{-1})(i(v))))\ \in H^2(G, \tau)$$

and it is $K-$finite function. For a semisimple Lie group G let

$$\Xi_G(x) = \int_K e^{-\rho(H(xk))} dk$$

denote the Harish-Chandra $\Xi-$function [6] page 187. We recall that $\Xi_G \in L^{2+\gamma}(G)$ for every $\gamma > 0$. Since (π, V) is an square integrable representation, in [11] it is proved that there exist $\epsilon > 0$, $q \geq 0$, $0 \leq C_v < \infty$ so that

$$\|k(x)(v)\| \leq C_v\, \Xi_G^{1+\epsilon}(x)(1 + \|x\|)^q,\ x \in G, v \in W.$$

Whenever (π, V) is integrable, in [6] page 256, we find the estimate

$$\|k(x)(v)\| \leq C_v\ \Xi_G^{2+\epsilon}(x)(1 + \|x\|)^{q'},\ x \in G, v \in W.$$

Since H is a reductive Lie subgroup of G, for an integrable representation, from the estimate in [14], it follows that $k \in L^2(H, \tau_\star) \cap L^1(H, \tau_\star)$. Therefore, the linear operator

$$G(z) := \int_H k(z, h) g(h) dh$$

is bounded in $L^2(H, \tau_\star)$. We now verify that

$$(rr^\star)(z) = r^\star(z) = \int_H k(z, h) g(h) dh,\ z \in H,\ g \in Domain(r^\star).$$

In fact, the equality (1) applied to $r^\star(g)$ yields

$$(r^\star(g)(z), v)_W = (r^\star(g), k(?, z)v)_{L^2(G,\tau)}$$

Since $k(?,z)v$ is $K-$finite, it belongs to the domain of r, hence we have

$$
\begin{aligned}
(r^\star(g), k(?,z)v)_{L^2(G,\tau)} &= (g, r(k(?,z)v)_{L^2(H,\tau_\star)} \\
&= \int_H (g(h), k(h,z)v)_W dh \\
&= \int_H (k(h,z)^\star g(h), v)_W dh \\
&= \int_H (k(z,h)g(h), v)_W dh \qquad (1) \\
&= \int_H (k(h^{-1}z)g(h), v)_W dh \\
&= (\int_H k(h^{-1}z)g(h)dh, v)_W .
\end{aligned}
$$

and we have conclude the proof of theorem 1.

Remark 1. *Using the estimate $\|f \star g\|_\infty \leq \|f\|^2\|g\|^2$, $f \in L^2, g \in L^2$ and that U is a partial isometry, it follows, for any square integrable representation, that r is a continuous map from $H^2(G,\tau)$ into $C^\infty(H,\tau_\star)$, here, in the last space we have set the smooth uniform convergence on compacts topology.*

3. Proof of theorem 2 and Proposition 1

Let $\mathcal{S}(\mathfrak{g})$ (resp. $\mathcal{U}(\mathfrak{g})$)be symmetric algebra of $\mathfrak{g}$ (resp. the universal enveloping algebra of $\mathfrak{g}$. Let $\lambda : \mathcal{S}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ be the symmetrization. For any $D \in \mathcal{U}(\mathfrak{g})$, R_D will denote infinitesimal right translation by D. Let

$$r_m : C^\infty(G,\tau) \to C^\infty(H, Hom_{\mathbb{C}}(S^m(\mathfrak{s}\cap\mathfrak{q}), W))$$

the linear map defined by the rule

$$r_m(f)(h)(X_1, \cdots, X_m) = (R_{\lambda(D)}f)(h).$$

The action, via the Adjoint representation, of $H \cap K$ in $\mathfrak{s}\cap\mathfrak{q}$ gives rise to a representation of $H\cap K$ in $Hom_{\mathbb{C}}(S^m(\mathfrak{s}\cap\mathfrak{q}), W)$. We denote this representation by $\bullet$. It readily follows that $r_m(f)(hk) = k^{-1} \bullet r_m(g)(h), h \in H, k \in H \cap K$. Thus, r_m maps $C^\infty(G,\tau)$ into $C^\infty(H,\bullet)$.

Remark 2. *For any pair G,H and π a square integrable representation we may prove i) r_m is a closed densely defined linear transformation from $H^2(G,\tau)$ into $L^2(H,\bullet)$ whose domain contains the $K-$finite vectors. Indeed, this follows from the estimate in Corollary 7.4 in* [11] *and the proof of Theorem 1 in* [14]. *ii) If further π is an integrable representation, then r_m extends to a continuous linear map from $H^2(G,\tau)$ into $L^2(H,\bullet)$. In fact, as in the proof of Theorem 1, we have that $r_m r_m^\star$ is an integral operator given by an integrable kernel. In this case the kernel is $k_m(x,y)(D\otimes v) = k(x,y)(\dot\pi(\lambda(D))(v))$. In particular, this remark shows Theorem 2 for integrable representations.*

We now show Theorem 2, let (ρ, Z) be an $H-$irreducible subrepresentation of $H^2(G,\tau)$. For each left $K\cap K-$finite element f of Z, we claim that $r_m(f)$ is tempered function on H. In fact, $r_m(f)$ is a smooth, $H\cap K$ and $\mathcal{B}_\mathfrak{h}-$finite because r_m is a continuous map on C^∞ topology of uniform convergence on compact sets. On the other hand, Z is a square integrable representation of H ([13]). Therefore for each $H\cap K-$finite continuous linear functional on Z we have that the function $h \to \lambda(L_h(f))$ is tempered in H (for a proof cf. [6]). Since point evaluation at e is $K-$finite and continuous linear functional, we obtain that $r_m(f)$ is tempered. Since tempered functions are square integrable, we have that $r_m(Z_{H\cap K-finite})$ is a linear subspace of $L^2(H,\bullet)$. Moreover, the elements in $H^2(G,\tau)$ are real analytic functions, hence $r_m(f)$ is nonzero for some m (for a proof cf [7], lemma 2.2) Frobenius reciprocity conclude the proof of Theorem 2.

In order to show Corollary 1 we consider an irreducible $H-$subrepresentation (ρ, Z) of π. Then theorem 2 tell us that a $H\cap K-$type of Z lies in $S^m(\mathfrak{s}\cap\mathfrak{q})\otimes W$ for some m. Since, $H\cap K$ acts trivially on $\mathfrak{s}\cap\mathfrak{q}$ we have that the irreducible constituents of $S^m(\mathfrak{s}\cap\mathfrak{q})\otimes W$ are the same as those of W. We now recall that there are finitely many equivalence class of irreducible square integrable representations having in common a given $K-$type. Thus L is a finite set.

We begin to show Proposition 1, for this end, we consider the usual maximal compact subgroup of G and we fix (ρ, Z) an $H-$irreducible subrepresentation of $H^2(G,\tau)$, and let σ denote the $H\cap K$ lowest $K-$type for ρ. Let χ_0 (resp. χ) denote the infinitesimal character of ρ (resp. $H^2(G,\tau)$). Let d_σ (resp. χ_σ) denote the dimension of σ (resp. the character of σ). For any Lie algebra $\mathfrak{g}$ let $\mathcal{B}_\mathfrak{g}$ denote the center of $\mathcal{U}(\mathfrak{g})$. As a matter of convenience, we fix $H^2(G,\tau) = \{f : f(kg) = \tau(k)f(g)\}$. Hence, the action by G becomes right translation. Since $H^2(G,\tau)$ is a subspace of the space of smooth sections (the kernel of an elliptic operator) we have that $R_Df = \chi(D)f$, for every $f \in H^2(G,\tau), D \in \mathcal{B}_\mathfrak{g}$ and that $R_D(f) = \chi_0(D)f$ for every $f \in Z$, $D \in \mathcal{B}_\mathfrak{h}$.

Lemma 1. *The vector space*

$$\begin{aligned}\mathcal{A} = \{f : G \to W : f\textit{real analytic},\\ f(kg) = \tau(k)f(g),\ k\in K, g\in G,\ R_Z(f) = \chi(Z)f,\ \forall\, Z \in \mathcal{B}_\mathfrak{g},\\ R_Zf = \chi_0(D)f,\ \forall\, Z \in \mathcal{B}_\mathfrak{h},\ f\star d_\sigma\chi_\sigma = f\}\end{aligned}$$

is finite dimensional.

For the proof of the lemma we follow Harish-Chandra and van den Ban [12]. We write K_0 for one of the three groups $SO(2k), SU(k), U(k)$, and H_1 for either of $SO(2n-2k,1), SU(n-k,1), U(n-k,1)$. We denote by K_1 the usual maximal compact subgroup of H_1. Let $H := K_0\times H_1$. From now on, we think of H immersed in G in the usual manner. Thus, (G,H) is a rank one symmetric pair and $K_0\times K_1$ is a maximal compact subgroup for H. Let $\mathfrak{g} = \mathfrak{k}+\mathfrak{s} = \mathfrak{h}+\mathfrak{q}$ be the associated "Cartan" decompositions. Let $\mathfrak{a}_{pq}$ be a maximal abelian subspace in $\mathfrak{s}\cap\mathfrak{q}$. Let L denote the centralizer of $\mathfrak{a}_{pq}$ in G. In our case we have that $(K_0, L\cap K_0)$ is a compact, rank one symmetric pair. Also that $L\cap H_1 = H_1\cap K = K_1$. Moreover K_1 acts trivially on $\mathfrak{s}\cap\mathfrak{q}$. Let $\mu : \mathcal{B}_\mathfrak{g} \to \mathcal{B}_\mathfrak{l}$ be the map defined in [12] Lemma 3.7. as in [12] we fix $v_1,\cdots,v_r \in \mathcal{B}_\mathfrak{l}$ so that $\mathcal{B}_\mathfrak{l} = \sum_{j=1}^{j=r}\mu(\mathcal{B}_\mathfrak{g})v_j$. In Lemma 3.8 in [12] it is proven for each $D \in \mathcal{U}(\mathfrak{g})$ that there exists $D_0 \in \mathcal{U}(\mathfrak{k}\cap\mathfrak{l})(\sum_j \mathcal{B}_\mathfrak{g}v_j)\mathcal{U}(\mathfrak{h})$ and finitely many functions $f_i : A_{pq} \to \mathbb{C}$, $\xi_i \in \mathcal{U}(\mathfrak{k})$, $\eta_i \in (\sum_j \mathcal{B}_\mathfrak{g}v_j)\mathcal{U}(\mathfrak{h})$ such that $D = D_0 + \sum_i f_i(a)\xi_i^{a^{-1}}\eta_i$ for all $a \in A^+_{pq}$. From now on we fix $a \in A^+_{pq}$. Thus, for a real analytic function on G we have that $R_D(f) := f(a;D) = 0,\ \forall\, D \in \mathcal{U}(\mathfrak{g})$ if and only if $f\equiv 0$. We write $D_0 = \sum_k \Theta_k Z_k v_k H_k$, with $\Theta_k \in \mathcal{U}(\mathfrak{k}\cap\mathfrak{l}), Z_k \in \mathcal{B}_\mathfrak{g}, H_k \in \mathcal{U}(\mathfrak{h})$ and $\eta_i = \sum_r Z_{i,r}v_rH_{i,r}$ with $Z_{ir} \in \mathcal{B}_\mathfrak{g}$, $H_{ir} \in \mathcal{U}(\mathfrak{h})$. Thus, we obtain $f(a;D) = \sum_k \dot\tau(\Theta_k)\chi(Z_k)f(a;v_kH_k) + \sum_i f_i(a)\dot\tau(\xi_i)\sum_r \chi(Z_{i,r})f(a;v_rH_{i,r})$. Therefore, each $f \in \mathcal{A}$ is determinated by the functions

$$\{H \ni h \to f(a;v_j;h) =: G_{f,j}(h),\ j = 1,\cdots,r\}.$$

Next, for a fixed j, we verify that the vector space $\{G_{f,j} : f \in \mathcal{A}\}$ is finite dimensional. Henceforth, we drop the letter j from $G_{f,j}$. Let $k_0 \in K$ be fixed and define for $h_1 \in H_1$, $g_{f,k_0}(h) := G_f(k_0h_1)$. Obviously, g_{f,k_0} is a $W-$valued real analytic function on H_1. We write $\sigma = \sigma_0\hat\otimes\sigma_1$ with $\sigma_0 \in \hat{K_0}, \sigma_1 \in \hat{K_1}$. Let $\mathcal{J}$ denote the kernel of $\dot{\sigma_1}$ in $\mathcal{U}(\mathfrak{k}_1)$. Thus $\mathcal{J}$ is a finite codimension two sided ideal in $\mathcal{U}(\mathfrak{k}_1)$. We claim that

$$g_{f,k_0}(k_1h_1) = \tau(k_1)g_{f,k_0}(h_1),\ k_1 \in K_1, h_1 \in H_1;$$
$$R_D g_{f,k_0} = 0,\ \ D \in \mathcal{J};$$
$$R_D g_{f,k_0} = \chi_0(D)g_{f,k_0},\ \ \ D \in \mathcal{B}_{\mathfrak{h}_1}.$$

In fact, if $k_1 \in K_1$, then $Ad(k_1)$ acts trivially on $\mathfrak{s} \cap \mathfrak{q}$, thus, $k_1 \in L$ and $g_{f,k_0}(k_1h_1) = f(a, v_j, k_0k_1h_1) = f(k_1a; v_j; h_1) = \tau(k_1)g_{f,k_0}(h_1)$. Since $f \star d_\sigma\ \chi_\sigma = f$, we have that $f \star d_{\sigma_1}\chi_{\sigma_1} = f$. Hence $R_D(g_{f,k_0}) = 0\, \forall\, D \in \mathcal{J}$. Finally, for $D \in \mathcal{B}_{\mathfrak{h}_1}, R_D(g_{f,k_0})(h_1) = f(a; v_j, k_0h_1; D) = \chi_0(D)g_{f,k_0}(h_1)$. Owing to a Theorem of Harish-Chandra [3], for each $k_0 \in K_0$, we obtain that $\mathcal{S}_{k_0} := \{g_f, k_0; f \in \mathcal{A}\}$ is contained in the finite dimensional vector space of spherical functions on H_1 of the same type $(\tau \otimes \sigma_1, \chi_0)$. Let $\mathcal{S}$ be the finite dimensional vector space spanned by the union of the $\mathcal{S}_{k_0}$, $k_0 \in K_0$. We now define a double representation ϕ of $K_0 \cap L$ in $\mathcal{S}$ by the rule $\phi(c, d)(g_{f,k_0})(h_1) :== \tau(c)(g_{f,k_0d}(h_1))$. From the above formula, we have that the function from K_0 into $\mathcal{S}$ defined by $k_0 \to g_{f,k_0}$ is spherical of type (ϕ, χ_0). Indeed, by definition the function is of type ϕ, and if $D \in \mathcal{B}_{\mathfrak{k}_0}$, then D is in $\mathcal{B}_{\mathfrak{h}}$ and R_D acts on the function by $\chi_0(D)$. Therefore, [3] implies that $\{G_{f,j}, f \in \mathcal{A}\}$ is contained in a finite dimensional vector space of $\mathcal{S}-$valued functions on K_0. Thus, we have conclude the proof of the lemma.

Remark 3. *We would like to point out that the proof of the lemma shows that it holds under the hypothesis:* (G, H) *is a symmetric pair,* $H = K_0 \times H_1$, K_0 *being a compact group and* $K \cap H_1$ *acts trivially on* $\mathfrak{s} \cap \mathfrak{q}$.

Proposition 1 follows readily from lemma, because if we fix a nonzero vector v in the lowest $H \cap K-$type of Z, each $T \in Hom_H(Z, H^2(G, \tau))$ is determinated by the function $T(v)$ and this function lies in $\mathcal{A}$.

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