An inequality for complex, symmetric matrices with zero diagonal

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Abstract
If the complex symmetric square matrix $V$ has zero diagonal then
$$2 \| |V| \| \leq \text{tr}(|V|).$$

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The purpose of this note is the proof of the following

**Theorem** For an arbitrary $n \times n$ complex matrix $V = (V_{jk})$ which is symmetric (that is $V_{jk} = V_{kj}$) and has zero diagonal elements, one has
$$2 \| |V| \| \leq \text{tr}(|V|).$$

Here, $|V|$ denotes the modulus of $V$ (the positive semidefinite square-root of $V^*V$), $\| \cdot \|$ is the spectral-norm, and $\text{tr}$ denotes the trace.

The inequality emerged from our analysis of [1] where it is implicit for $n = 4$. This inequality will play an important role in our ongoing study of quantum-state entanglement. Our proof uses the Takagi diagonalization of symmetric matrices ([2]; p. 204-205), and the following elementary result:

**Lemma** Consider $n \geq 1$ non-negative real numbers $c_1 \geq c_2 \geq \cdots \geq c_n \geq 0$. Then
$$2c_1 \leq \sum_{j=1}^{n} c_j$$
if and only if there are $n$ real numbers $\theta_j$ ($j = 1, 2, \cdots, n$) such that
$$\sum_{j=1}^{n} e^{\imath \theta_j} c_j = 0.$$
Proof: For \( n = 1, 0 \leq 2c_1 \leq c_1 \) iff \( c_1 = 0 \). We assume henceforth that \( n \geq 2 \). If \( \sum_{j=1}^{n} e^{i\theta_j} c_j = 0 \) then by the triangle inequality, \( c_1 = |e^{i\theta_1} c_1| = | - \sum_{j=2}^{n} e^{i\theta_j} c_j | \leq \sum_{j=2}^{n} c_j \), so the condition is sufficient.

The necessity is proved by considering first the cases \( n = 2 \) and \( n = 3 \) (which cannot be reduced to \( n = 2 \)) and then using induction on \( n \geq 3 \). For \( n = 2 \), the hypothesis and the inequality imply \( c_1 = c_2 \) so that \( \theta_1 = 0 \) and \( \theta_2 = \pi \) will do.

For \( n = 3 \), we show that there is \( \alpha \) and \( \beta \) such that \( c_1 = e^{i\alpha} c_2 + e^{i\beta} c_3 \) so that \( \theta_1 = 0, \theta_2 = \alpha + \pi \) and \( \theta_3 = \beta + \pi \) will do. When \( c_3 = 0 \) we have the case \( n = 2 \). Otherwise, \( c_1 \geq c_2 \geq c_3 > 0 \) and the numbers

\[
\frac{c_1^2 + c_2^2 - c_3^2}{2c_1c_2}, \quad \frac{c_1^2 + c_3^2 - c_2^2}{2c_1c_3}
\]

are both non-negative and not above 1. A straightforward direct calculation shows that

\[
\alpha = \pm \arccos \left( \frac{c_1^2 + c_2^2 - c_3^2}{2c_1c_2} \right), \quad \beta = \mp \arccos \left( \frac{c_1^2 + c_3^2 - c_2^2}{2c_1c_3} \right),
\]

give two possible choices of the phases.

We now proceed with induction. Given \( c_1 \geq c_2 \cdots \geq c_n \geq c_{n+1} \), consider \( b_1 = c_1 - c_{n+1} \) which is non-negative. If \( b_1 \geq c_2 \), then by the induction hypothesis, there are \( \gamma_1, \cdots, \gamma_n \) such that \( e^{i\gamma_j} b_1 + \sum_{j=2}^{n} e^{i\gamma_j} c_j = 0 \); \( \theta_j = \gamma_j \) for \( j \neq n + 1 \) and \( \theta_{n+1} = \gamma_1 + \pi \) does the job. If \( b_1 < c_2 \), then consider \( a_1 = c_2 \) and let \( a_j \) for \( j = 2, \cdots, n \) be a renumeration of \( \{b_1, c_3, \cdots, c_n\} \) such that \( a_2 \geq a_3 \geq \cdots \geq a_n \). Then, \( a_k = c_1 - c_{n+1} = b_1 \) for some \( 2 \leq k \leq n \). We have \( a_1 + c_{n+1} = c_2 + c_{n+1} \leq c_1 + c_n \) or, equivalently, \( a_1 \leq b_1 + c_n \), so that \( a_1 \leq \sum_{j=2}^{n} a_j \).

The induction hypothesis applied to the \( a \)'s implies the existence of real numbers \( \gamma_j \ (j = 1, 2, \cdots, n) \) such that \( \sum_{j=1}^{n} e^{i\gamma_j} a_j = 0 \). Then \( \theta_1 = \gamma_k, \theta_{n+1} = \gamma_k + \pi, \) and \( \theta_j = \gamma_j \) for \( j \neq k \), does the job.

We state two immediate corollaries of the Lemma.

**Proposition 1** If \( n \geq 1 \) and \( A \) is an \( n \times n \) positive semidefinite complex matrix with repeated eigenvalues \( a_1, a_2, \cdots, a_n \) then \( 2 \| A \| \leq \text{tr}(A) \) if and only if there are \( n \) real numbers \( \theta_j \ (j = 1, 2, \cdots, n) \) such that \( \sum_{j=1}^{n} e^{i\theta_j} a_j = 0 \).

**Proposition 2** If \( n \geq 1 \) and \( z_1, z_2, \cdots, z_n \in \mathbb{C} \) and \( \sum_{j=1}^{n} z_j = 0 \) then \( 2 \max_j |z_j| \leq \sum_{j=1}^{n} |z_j| \).

Another immediate consequence is

**Proposition 3** If \( V \) is an hermitian \( n \times n \) complex matrix with \( \text{tr}(V) = 0 \), then \( 2 \| V \| \leq \text{tr}(|V|) \).
Proof: Enumerate the eigenvalues of $V$ as $v_1, \ldots, v_n$ according to their multiplicities; then $0 = tr(V) = \sum_{j=1}^{n} v_j$ implies $\sum_{j=1}^{n} e^{\theta_j} |v_j| = 0$ where $\theta_j = 0$ if $v_j > 0$ and $\theta_j = \pi$ for $v_j < 0$. Using the Lemma, $2 \parallel V \parallel = 2 \parallel |V| \parallel = 2 \max_j |v_j| \leq \sum_{j=1}^{n} |v_j| = tr(|V|)$. This can be proved without invoking the Lemma quite simply: $V = V_+ - V_-$ and $tr(V_+) = tr(V_-) \geq \parallel V \parallel$ so that $tr(|V|) = tr(V_+) + tr(V_-) \geq 2 \parallel V \parallel$.

We now proceed with the proof of the theorem. For $n = 1$ the claim is trivially true, so we assume $n \geq 2$. If $V$ is symmetric, that is $V = V^T$, where $T$ denotes transposition, the Takagi diagonalization (see [2], p. 204-205) insures the existence of a unitary matrix $U$ such that $U^TVU = D$ with $D$ diagonal, that is $D_{jk} = \delta_{jk}d_j$ (the fact that $d_j \geq 0$ does not simplify the argument below). Since $U^T(U^T)^* = (U^*)U^T$, it follows that $U^T$ is unitary and thus $V = (U^*)^*DU^*$. Then, $|V|^2 = V^*V = UD^*U^T(U^T)^*DU^* = U|D|^2U^*$, and thus $|V| = U|D|U^*$. In particular,

$$0 = \sum_{j=1}^{n} (U^T)_{jk}D_{km}U_{mj} = \sum_{m=1}^{n} d_mU_{mj}^2, \quad j = 1, 2, \ldots, n.$$  

By Proposition 2,

$$2 \max_m \{ |d_m| \ | U_{jm} |^2 \} \leq \sum_{m=1}^{n} |d_m| \ | U_{jm} |^2, \quad j = 1, 2, \ldots, n,$$

Since $U$ and thus $U^*$ is unitary, $\sum_{j=1}^{n} | U_{jm} |^2 = 1$ for $m = 1, 2, \ldots, n$. But then,

$$2 \max_m |d_m| = 2 \max_m \left[ \sum_{j=1}^{n} | U_{jm} |^2 | d_m | \right] \leq 2 \max_m \{ | U_{jm} |^2 | d_m | \}
\leq \sum_{j=1}^{n} \sum_{m=1}^{n} |d_m| \ | U_{jm} |^2 = \sum_{m=1}^{n} |d_m|;$$

which is exactly $2 \parallel |D| \parallel \leq tr(|D|)$ and the claimed inequality follows from Eq. (1).

The inequality is saturated for all symmetric matrices with zero diagonal and at most two non-zero entries in the upper off-diagonal triangle. We remark that if $V$ is not hermitian but symmetric the condition of zero diagonal on $V$ in the theorem cannot be relaxed to $tr(V) = 0$ (cf. Proposition 3). Consider

$$V = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix},$$

3
then 2 and 0 are the eigenvalues of $|V|$ so that $||V|| = tr(|V|)$.

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References
