An inequality for complex, symmetric matrices with zero diagonal

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Abstract

If the complex symmetric square matrix V has zero diagonal then $2 \parallel |V| \parallel \leq tr(|V|)$.

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The purpose of this note is the proof of the following

Theorem *For an arbitrary $n \times n$ complex matrix $V = (V_{jk})$ which is symmetric –that is $V_{jk} = V_{kj}$ – and has zero diagonal elements, one has*

$$2 \parallel |V| \parallel \leq tr(|V|) \ .$$

Here, $|V|$ denotes the modulus of V (the positive semidefinite square-root of V^*V), $\| \cdot \|$ is the spectral-norm, and tr denotes the trace.

The inequality emerged from our analysis of [1] where it is implicit for $n = 4$. This inequality will play an important role in our ongoing study of quantum-state entanglement. Our proof uses the Takagi diagonalization of symmetric matrices ([2]; p. 204-205), and the following elementary result:

Lemma *Consider $n \geq 1$ non-negative real numbers $c_1 \geq c_2 \geq \cdots \geq c_n \geq 0$. Then*

$$2c_1 \leq \sum_{j=1}^{n} c_j$$

if and only if there are n real numbers θ_j $(j = 1, 2, \cdots, n)$ such that

$$\sum_{j=1}^{n} e^{i\theta_j} c_j = 0 \ .$$

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Proof: For $n = 1$, $0 \leq 2c_1 \leq c_1$ iff $c_1 = 0$. We assume henceforth that $n \geq 2$. If $\sum_{j=1}^n e^{i\theta_j} c_j = 0$ then by the triangle inequality, $c_1 = |e^{i\theta_1} c_1| = | - \sum_{j=2}^n e^{i\theta_j} c_j| \leq \sum_{j=2}^n c_j$, so the condition is sufficent.

The necessity is proved by considering first the cases $n = 2$ and $n = 3$ (which cannot be reduced to $n = 2$) and then using induction on $n \geq 3$. For $n = 2$, the hypothesis and the inequality imply $c_1 = c_2$ so that $\theta_1 = 0$ and $\theta_2 = \pi$ will do.

For $n = 3$, we show that there is α and β such that $c_1 = e^{i\alpha} c_2 + e^{i\beta} c_3$ so that $\theta_1 = 0$, $\theta_2 = \alpha + \pi$ and $\theta_3 = \beta + \pi$ will do. When $c_3 = 0$ we have the case $n = 2$. Otherwise, $c_1 \geq c_2 \geq c_3 > 0$ and the numbers

$$\frac{c_1^2 + c_2^2 - c_3^2}{2c_1c_2}\ ,\quad \frac{c_1^2 + c_3^2 - c_2^2}{2c_1c_3}$$

are both non-negative and not above 1. A straightforward direct calculation shows that

$$\alpha = \pm \arccos\left(\frac{c_1^2 + c_2^2 - c_3^2}{2c_1c_2}\right)\ ,\quad \beta = \mp \arccos\left(\frac{c_1^2 + c_3^2 - c_2^2}{2c_1c_3}\right)\ ,$$

give two possible choices of the phases.

We now proceed with induction. Given $c_1 \geq c_2 \cdots \geq c_n \geq c_{n+1}$, consider $b_1 = c_1 - c_{n+1}$ which is non-negative. If $b_1 \geq c_2$, then by the induction hypothesis, there are $\gamma_1, \cdots, \gamma_n$ such that $e^{i\gamma_1} b_1 + \sum_{j=2}^n e^{i\gamma_j} c_j = 0$; $\theta_j = \gamma_j$ for $j \neq n+1$ and $\theta_{n+1} = \gamma_1 + \pi$ does the job. If $b_1 < c_2$, then consider $a_1 = c_2$ and let a_j for $j = 2, \cdots, n$ be a renumeration of $\{b_1, c_3, \cdots, c_n\}$ such that $a_2 \geq a_3 \geq \cdots a_n$. Then, $a_k = c_1 - c_{n+1} = b_1$ for some $2 \leq k \leq n$. We have $a_1 + c_{n+1} = c_2 + c_{n+1} \leq c_1 + c_n$ or, equivalently, $a_1 \leq b_1 + c_n$, so that $a_1 \leq \sum_{j=2}^n a_j$. The induction hypothesis applied to the a's implies the existence of real numbers γ_j ($j = 1, 2, \cdots, n$) such that $\sum_{j=1}^n e^{i\gamma_j} a_j = 0$. Then $\theta_1 = \gamma_k$, $\theta_{n+1} = \gamma_k + \pi$, and $\theta_j = \gamma_j$ for $j \neq k$, does the job.

We state two inmediate corollaries of the Lemma.

Proposition 1 *If $n \geq 1$ and A is an $n \times n$ positive semidefinite complex matrix with repeated eigenvalues $a_1, a_2, \cdots, a_n$ then $2 \parallel A \parallel \leq tr(A)$ if and only if there are n real numbers θ_j ($j = 1, 2, \cdots, n$) such that $\sum_{j=1}^n e^{i\theta_j} a_j = 0$.*

Proposition 2 *If $n \geq 1$ and $z_1, z_2, \cdots, z_n \in \mathbb{C}$ and $\sum_{j=1}^n z_j = 0$ then $2 \max_j |z_j| \leq \sum_{j=1}^n |z_j|$.*

Another inmediate consequence is

Proposition 3 *If V is an hermitian $n \times n$ complex matrix with $tr(V) = 0$, then $2 \parallel V \parallel \leq tr(|V|)$.*

Proof: Enumerate the eigenvalues of V as $v_1, \cdots, v_n$ according to their multiplicities; then $0 = tr(V) = \sum_{j=1}^n v_j$ implies $\sum_{j=1}^n e^{i\theta_j}|v_j| = 0$ where $\theta_j = 0$ if $v_j > 0$ and $\theta_j = \pi$ for $v_j < 0$. Using the Lemma, $2 \parallel V \parallel = 2 \parallel |V| \parallel = 2\max_j |v_j| \leq \sum_{j=1}^n |v_j| = tr(|V|)$. This can be proved without invoking the Lemma quite simply: $V = V_+ - V_-$ and $tr(V_+) = tr(V_-) \geq \parallel V \parallel$ so that $tr(|V|) = tr(V_+) + tr(V_-) \geq 2 \parallel V \parallel$.

We now proceed with the proof of the theorem. For $n = 1$ the claim is trivially true, so we assume $n \geq 2$. If V is symmetric, that is $V = V^T$, where ^{T} denotes transposition, the Takagi diagonalization (see [2], p. 204-205) insures the existence of a unitary matrix U such that $U^T V U = D$ with D diagonal, that is $D_{jk} = \delta_{jk} d_j$ (the fact that $d_j \geq 0$ does not simplify the argument below). Since $U^T(U^T)^* = (U^*U)^T$, it follows that U^T is unitary and thus $V = (U^T)^* D U^*$. Then, $|V|^2 = V^*V = UD^*U^T(U^T)^*DU^* = U|D|^2U^*$, and thus $|V| = U|D|U^*$. In particular,

(1) $$\parallel |V| \parallel = \parallel |D| \parallel \, , \quad tr(|V|) = tr(|D|) \, .$$

Now, $V_{jj} = 0$ for $j = 1, 2, \cdots, n$ implies

$$0 = \sum_{\ell,m}^n (U^T)_{j\ell} D_{\ell m} U_{mj} = \sum_{m=1}^n d_m U_{mj}^2 \, , \quad j = 1, 2, \cdots, n.$$

By Proposition 2,

$$2 \max_m \{| d_m | | U_{jm} |^2\} \leq \sum_{m=1}^n | d_m | | U_{jm} |^2 \, , \quad j = 1, 2, \cdots, n \, , .$$

Since U and thus U^* is unitary, $\sum_{j=1}^n | U_{jm} |^2 = 1$ for $m = 1, 2, \cdots, n$. But then,

$$2 \max_m | d_m | = 2 \max_m \left[\sum_{j=1}^n | U_{jm} |^2 | d_m | \right] \leq 2 \sum_{j=1}^n \max_m \{| U_{jm} |^2 | d_m |\}$$

$$\leq \sum_{j=1}^n \sum_{m=1}^n | d_m | | U_{jm} |^2 = \sum_{m=1}^n | d_m | \, ;$$

which is exactly $2 \parallel |D| \parallel \leq tr(|D|)$ and the claimed inequality follows from Eq. (1).

The inequality is saturated for all symmetric matrices with zero diagonal and at most two non-zero entries in the upper off-diagonal triangle. We remark that if V is not hermitian but symmetric the condition of zero diagonal on V in the theorem cannot be relaxed to $tr(V) = 0$ (cf. Proposition 3). Consider

$$V = \begin{pmatrix} 1 & , & i \\ i & , & -1 \end{pmatrix} \, ,$$

then 2 and 0 are the eigenvalues of $|V|$ so that $\| \, |V| \, \| = tr(|V|)$.

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References

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