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Asymptotic Behavior of RA-estimates in Autoregressive 2D Processes

by

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Abstract

In this work we study the asymptotic behavior of a robust class of estimator of coefficients of AR-2D process. We established the consistency and asymptotic normality of the RA estimator under precise conditions. This class of models has diverse applications in image modeling and statistical image processing.

Keywords and phrases: Image Processing, Bidimensional Processes, AR-2D Models, Polynomial Coefficients, Robust Estimators, Residual Autocovariance, Asymptotic Normality and Consistency.

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1 INTRODUCTION

Robust inference techniques appear in a diversity of contexts and applications, though the terms "robust" and "robustness" are quite freely used in the image processing and computer vision literature, not necessarily with the usual statistical meaning.

The median and similar order-based filters are basic tools in image processing (Aysal and Barner (2006); Huang and Lee (2006); Palenichka et al. (2000), (1998)), and in some cases particular attention has been devoted to obtain the distribution of those estimators (Steland (2005)). Frery et al. (1997) derived a family of robust estimators for a class of low signal-to-noise ratio images, while Vallejos and Mardesic (2004) proposed the robust estimation of structural parameters for their restoration.

Other resistant approaches have proved being successfull in image restoration (see, for instance, Ben Hamza and Krim (2001); Chu et al. (1998); Koivunen (1995); Marroquin et al. (1998); Rabie (2005); Tarel et al. (2002); Voloshynovskiy et al. (2000); Zervakis and Kwon (1992)). A common challenge in these applications is that the number of observations is reduced to a few, typically less than a hundred points.

When it comes to image analysis, many robust techniques have been proposed. In this case, the sample size is usually larger than the one available in filters and, frequently, structure and topology do not impose heavy requirements or constraints. In some cases, strong hypothesis are made on the laws governing the observed process (Allende and Pizarro (2003); Brunelli and Messelodi (1995); Bustos et al. (2002); Butler (1998); Dryden et al. (2002); Van de Weijer and Van den Boomgaard (2005)); other approaches can be seen in the works by Bouzouba and Radouane (2000); Brandle et al. (2003); Nirel et al. (1998); Sim et al. (2004); Tohka et al. (2004); Xu (2005) and Zervakis et al. (1995).

High-level image analysis, or vision, also benefits from the use of robust estimation techniques, as can be seen in Black and Rangarajan (1996), Black et al. (1997), Chen et al. (2003), Comport et al. (2006), Glendinning (1999), Gottardo et al. (2006), Hasler et al. (2003), Kim and Han (2006), Li et al. (1998), Meer et al. (1991), Mirza and Boyer (1993), Prastawa et al. (2004), Roth (2006), Singh et al. (2004), Stewart (1999), Torr and Zisserman (2000) and Wang and Suter (2004a,b).

In a wide variety of situations such as image analysis, remote sensing and agricultural field trials, observations are obtained on two-dimensional lattices or grids. A class of two-dimensional autoregressive processes has been suggested (Whittle (1954)) as a source of models for the spatial correlation in such data (Tjostheim (1978)). These models are natural extensions of the autoregressive processes used in time series analysis (Basu and Reinsel (1993)).

Most robust techniques developed for parametric models in time series have been implemented for spatial parametric models when the process has been contaminated with innovation or additive outliers (Kashyap and Eom (1988)). Since a single outlier can produce bias and a large variance in the estimators, most of those proposals aim at providing estimators that are more resistant to the presence of contamination.

There are at least three classes of robust estimators that have been studied in this context, namely, the M, GM and RA estimators. Kashyap and Eom (1988) introduced M estimators for two-dimensional autoregressive models, and a recursive image restoration algorithm was implemented using a robust M estimators. Allende et al. (1998) studied the implementation of Generalized M (GM) estimators for the same class of models. The image restoration algorithm previously developed by Kashyap and Eom (1988) was generalized by Allende et al. (2001).

Robust Residual Autocovariance (RA) estimators were introduced by Bustos and Yohai (1986) in the context of time series. They are based on "cleaning" the residuals by the application of a robustifying ψ function in the recursive estimation procedure. Ojeda (1999) studied the extension of the RA estimators for spatial unilateral autoregressive models. The computational implications of that extension have been studied by Ojeda et al. (2002) and Vallejos et al. (2006). Monte Carlo simulation studies show that the performance of the RA estimators is better than the M estimators and slightly better than the GM estimators when the model has been contaminated with additive outliers.

Although the performance of the M and GM estimators is reasonable under innovation outliers, the asymptotic properties are still open problems.

In this paper we study the asymptotic behavior of the RA estimators for unilateral autoregressive spatial processes, generalizing the results for one-dimensional time series asymptotic behavior established by Bustos et al. (1984). We give precise conditions for the consistency and asymptotic normality of the RA estimators.

The paper is organized as follows. In Section 2, the model and some preliminary notation are established. In Section 3 the RA estimators of a two-dimensional autoregressive process is introduced. Section 4 establishes the Strong Consistency of the RA estimators, while Section 5 proves its Asymptotic Normality. The proofs of the results are organized in two subsections of the Appendix (Section 6). The paper concludes in Section 7 with some final remarks and directions for future work.

2 AR-2D MODELS

Throughout this paper we assume that the random variables are defined on the same probability space $(\Omega, \mathbb{F}, P)$.

If $\underline{m} = (m_1, m_2)'$ and $\underline{k} = (k_1, k_2)' \in \mathbb{Z}^2$, we write $\underline{m} \leq \underline{k}$ if $m_i \leq k_i$ for $i = 1, 2$. Let

$$I_0 = \{\underline{m} \in \mathbb{Z}^2 : \ \underline{0} \leq \underline{m} \text{ and } \underline{m} \neq \underline{0} = (0,0)'\},$$

and let

$$T = \{\underline{t}_1, \ldots, \underline{t}_L\} = \{(t_{1,1}, t_{1,2})', \ldots, (t_{L,1}, t_{L,2})'\} \subset I_0,$$

be a finite and not empty set.

Consider $\boldsymbol{\phi}_0 = (\phi_{01}, \ldots, \phi_{0L})' \in \mathbb{R}^L$ such that

$$\sup_{1 \leq p \leq L} \{|\phi_{0p}|\} < \frac{1}{L}, \tag{1}$$

and let $P_{\boldsymbol{\phi}_0}$ be a polynomial defined over $\mathbb{C}^2$ as follows

$$P_{\boldsymbol{\phi}_0}(z, w) = 1 - \sum_{p=1}^{L} \phi_{0p} z^{t_{p,1}} w^{t_{p,2}}. \tag{2}$$

For each $\underline{m} \in \mathbb{Z}^2$ let $X_{\underline{m}} : \Omega \to \mathbb{R}$ be a random variable such that $\widetilde{\mathbf{X}} = \left(X_{\underline{m}}\right)_{\underline{m} \in \mathbb{Z}^2}$ is a $AR(P_{\boldsymbol{\phi}_0}, \widetilde{\boldsymbol{\varepsilon}})$ process; that is, for each $\omega \in \Omega$ and $\underline{m} \in \mathbb{Z}^2$, $X_{\underline{m}}$ satisfies

$$X_{\underline{m}}(\omega) = \mu + \sum_{p=1}^{L} \phi_{0p} X_{\underline{m} - \underline{t}_p}(\omega) + \varepsilon_{\underline{m}}(\omega), \tag{3}$$

where $\widetilde{\boldsymbol{\varepsilon}} = \left(\varepsilon_{\underline{m}}\right)_{\underline{m} \in \mathbb{Z}^2}$ is a white noise, i.e., the components of $\widetilde{\boldsymbol{\varepsilon}}$ are independent and identically distributed random variables with common distribution function F_ε (not necessarily Gaussian), zero mean and finite variance $\sigma_\varepsilon^2 > 0$. Whenever possible we will drop the argument ω.

The following condition is required:

Assumption 1. *The distribution function of the errors, F_ε, is absolutely continuous with density f_ε.*

Our interest is the estimation of the coefficients $\boldsymbol{\phi}_0$ from the observed values of $\widetilde{\mathbf{X}}$, so we assume in this paper that μ and σ_ε are known and without loss of generality we take $\mu = 0$. If the position μ and the scale σ_ε are unknown they can be estimated using (possibly robust) estimators.

Example 1. *In practice, typical values for L are 1, 2 or at most 3. The following case has been frequently found in recent applications (see Ojeda et al. (2002)): consider $T = \{(1,0)', (1,1)', (0,1)'\}$ and set $\underline{t}_1 = (1,0)'$, $\underline{t}_2 = (1,1)'$ and $\underline{t}_3 = (0,1)'$. So, the polynomial in (2) is given by*

$$P_{\phi_0}(z, w) = 1 - \phi_{01} z - \phi_{02} z w - \phi_{03} w.$$

Hence, the model is described by

$$X_{(m_1,m_2)'} = \phi_{01} X_{(m_1-1,m_2)'} + \phi_{02} X_{(m_1-1,m_2-1)'} + \phi_{03} X_{(m_1,m_2-1)'} + \varepsilon_{(m_1,m_2)'}.$$

We will now show an important characterization of $\widetilde{\mathbf{X}}$. Assuming that $X_{\underline{m}} \in L^2(\Omega, \mathbb{F}, P)$ with

$$\begin{aligned} \left\|X_{\underline{m}}\right\|_{L^2}^2 &= \text{Variance of } X_{\underline{m}} = \sigma_X^2, \\ E(X_{\underline{m}}) &= 0, \end{aligned}$$

(see Guyon (1993), Chapter 1) , we have that, for each $\underline{m}$, $X_{\underline{m}}$ can be expressed as a series of terms of $\left(\varepsilon_{\underline{n}}\right)_{\underline{n} \in \mathbb{Z}^2}$ in the $L^2(\Omega, \mathbb{F}, P)$ space.

Indeed, for each $\omega \in \Omega$ and $\underline{m} \in \mathbb{Z}^2$, after one iteration in (3), $X_{\underline{m}}(\omega)$ can be written as

$$\begin{aligned}
X_{\underline{m}}(\omega) &= \sum_{k_1=1}^{L} \phi_{0k_1} X_{\underline{m}-\underline{t}_{k_1}}(\omega) + \varepsilon_{\underline{m}}(\omega) \\
&= \sum_{k_1=1}^{L} \phi_{0k_1} \left[\sum_{k_2=1}^{L} \phi_{0k_2} X_{\underline{m}-\underline{t}_{k_1}-\underline{t}_{k_2}}(\omega) + \varepsilon_{\underline{m}-\underline{t}_{k_1}} \right] + \varepsilon_{\underline{m}}(\omega) \\
&= \sum_{k_1,k_2=1}^{L} \phi_{0k_1}\phi_{0k_2} X_{\underline{m}-\underline{t}_{k_1}-\underline{t}_{k_2}}(\omega) + \sum_{k_1=1}^{L} \phi_{0k_1} \varepsilon_{\underline{m}-\underline{t}_{k_1}} + \varepsilon_{\underline{m}}(\omega);
\end{aligned}$$

now, if we iterate twice we have

$$\begin{aligned}
X_{\underline{m}}(\omega) &= \sum_{k_1,k_2,k_3=1}^{L} \phi_{0k_1}\phi_{0k_2}\phi_{0k_3} X_{\underline{m}-\underline{t}_{k_1}-\underline{t}_{k_2}-\underline{t}_{k_3}}(\omega) + \\
&\quad \sum_{k_1,k_2=1}^{L} \phi_{0k_1}\phi_{0k_2} \varepsilon_{\underline{m}-\underline{t}_{k_1}-\underline{t}_{k_2}} + \sum_{k_1=1}^{L} \phi_{0k_1} \varepsilon_{\underline{m}-\underline{t}_{k_1}} + \varepsilon_{\underline{m}}(\omega).
\end{aligned}$$

So, after $h-1 \geq 2$ iterations, we arrive to the following expression:

$$X_{\underline{m}}(\omega) = \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^h} I_{h,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0) X_{\underline{m}-s(h,\widetilde{\mathbf{k}})}(\omega) + \sum_{j=1}^{h-1} \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0) \varepsilon_{\underline{m}-s(j,\widetilde{\mathbf{k}})}(\omega) + \varepsilon_{\underline{m}}(\omega),$$

where $\mathbb{L} = \{1, 2, \ldots, L\}$, for each $j \in \mathbb{N}$ and $\widetilde{\mathbf{k}} = (k_1, \ldots, k_j)' \in \mathbb{L}^j$

$$s(j, \widetilde{\mathbf{k}}) = \sum_{i=1}^{j} \underline{t}_{k_i},$$

and $I_{h,\widetilde{\mathbf{k}}} \colon \Theta \to \mathbb{R}$ is the function defined on

$$\Theta = \left\{ \boldsymbol{\phi} = (\phi_1, \ldots, \phi_L)' \in \mathbb{R}^L \colon \sup_{1 \leq p \leq L} \left| \phi_p \right| < \frac{1}{L} \right\}, \tag{4}$$

such that

$$I_{h,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}) = \phi_{k_1} \ldots \phi_{k_h}, \forall \boldsymbol{\phi} \in \Theta.$$

Considering that

$$\left\| \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^h} I_{h,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0) X_{\underline{m}-s(h,\widetilde{\mathbf{k}})} \right\|_{L^2} \leq \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^h} \left| I_{h,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0) \right| \sigma_X = \sigma_X \left(\sum_{p=1}^{L} |\phi_{0p}| \right)^h,$$

and that, by (1), $\lim_{h\to\infty} \left(\sum_{p=1}^{L} \left| \phi_{0p} \right| \right)^h = 0$, the following proposition is valid.

Proposition 1. *For all $\underline{m} \in \mathbb{Z}^2$ holds that*

$$X_{\underline{m}} = \varepsilon_{\underline{m}} + \sum_{j=1}^{\infty} \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0) \varepsilon_{\underline{m}-s(j,\widetilde{\mathbf{k}})}, \tag{5}$$

where the series converges in the $L^2(\Omega, \mathbb{F}, P)$ space.

3 ESTIMATORS OF POLYNOMIAL COEFFICIENTS $\boldsymbol{\phi}_0$

In order to define the estimators of $\boldsymbol{\phi}_0$ we need to introduce some general notation. Let $\mathbf{H} \colon \Theta \to \mathbb{R}^L$ be a differentiable function and, for each $1 \leq p \leq L$, let $H_p \colon \Theta \to \mathbb{R}$ be its p-th component; that is, $H_p(\boldsymbol{\phi}) = (\mathbf{H}(\boldsymbol{\phi}))_p$. Denote the derivative of $\mathbf{H}$ by $\mathbb{D}\mathbf{H}$ and, for each $1 \leq p, q \leq L$, let $D_q(H_p)(\boldsymbol{\phi})$ be the partial derivative with respect to the q-th component of the function H_p evaluated at $\boldsymbol{\phi}$. Similarly, if $h\colon \Theta \to \mathbb{R}$ is a differentiable function, $D_q(h)(\boldsymbol{\phi})$ and $\nabla h(\boldsymbol{\phi})$ denote the partial derivative with respect to the q-th component and the gradient of h evaluated at $\boldsymbol{\phi}$, respectively.

Let $\widetilde{\mathbf{X}}$ be the set of all possible realizations of the random variables $X_{\underline{m}}$, $\underline{m} \in I_0$. Since for all $\underline{m} \in \mathbb{Z}^2$, $E(X_{\underline{m}}) = 0$, and for convenience (it will be clear in the notation introduced below) we assume in this section that

$$X_{\underline{m}}(\omega) = 0 \text{ if } \underline{m} \notin I_0, \text{ for all } \omega \in \Omega. \tag{6}$$

For each $\underline{m} \in \mathbb{Z}^2$, let $r_{\underline{m}}\colon \Omega \times \Theta \to \mathbb{R}$ be a function defined as follows:

$$r_{\underline{m}}(\omega, \boldsymbol{\phi}) = X_{\underline{m}}(\omega) - \sum_{p=1}^{L} \phi_p X_{\underline{m}-\underline{t}_p}(\omega).$$

Then, by (3) and (6), for every $\boldsymbol{\phi} \in \Theta$, the residual with respect to $\boldsymbol{\phi}$, $r_{\underline{m}}(\boldsymbol{\phi})$, is

$$\begin{array}{rcl} r_{\underline{m}}(\boldsymbol{\phi}_0) & \equiv & \varepsilon_{\underline{m}}, \text{ if } \left(\underline{m} - \underline{t}_p\right) \in I_0, \ \forall\ 1 \leq p \leq L, \\ r_{\underline{m}}(\boldsymbol{\phi}) & \equiv & 0, \quad \text{if } \underline{m} \notin I_0. \end{array} \tag{7}$$

For each positive integer M let us define the square window of M order as

$$W_M = \{\underline{m} \in I_0 : \underline{m} \leq (M, M)\},$$

and

$$\widetilde{\mathbf{X}}_M = (X_{\underline{m}})_{\underline{m} \in W_M},$$

denotes the observed process in W_M.

Remark 1. *We assume that M is large enough to have a well defined estimation problem; for example, consider $M \geq M_0 = \inf\{M'\colon T \subset W_{M'}\}$.*

Now we introduce the classical definition of least squares estimator as follows:

Definition 1. *The least squares estimator of $\boldsymbol{\phi}_0$ based on $\widetilde{\mathbf{X}}_M$, with domain $\widehat{\Omega}_M \subset \Omega$ is defined as the function $\widehat{\boldsymbol{\phi}}_M\colon \widehat{\Omega}_M \to \Theta$ such that*

$$\sum_{\underline{m} \in W_M} \left(r_{\underline{m}}(\omega, \widehat{\boldsymbol{\phi}}_M(\omega))\right)^2 \leq \sum_{\underline{m} \in W_M} \left(r_{\underline{m}}(\omega, \boldsymbol{\phi})\right)^2,$$

for all $\omega \in \widehat{\Omega}_M$ and $\boldsymbol{\phi} \in \Theta$.

Considering that, for each $\omega \in \widehat{\Omega}_M$ and $\underline{m} \in \mathbb{Z}^2$ the function $\boldsymbol{\phi} \mapsto r_{\underline{m}}(\omega, \boldsymbol{\phi})$ is continuously differentiable, then $\widehat{\boldsymbol{\phi}}_M$ satisfies

$$\sum_{\underline{m} \in W_M} \left(r_{\underline{m}}(\omega, \widehat{\boldsymbol{\phi}}_M(\omega))\right) \cdot \nabla \left(r_{\underline{m}}(\omega, \widehat{\boldsymbol{\phi}}_M(\omega))\right) = \mathbf{0},$$

for all $\omega \in \widehat{\Omega}_M$. Equivalently, by (7),

$$\sum_{\underline{m} \in W_M} \left(r_{\underline{m}}(\omega, \widehat{\boldsymbol{\phi}}_M(\omega))\right) \cdot X_{\underline{m}-\underline{t}_p}(\omega) = 0, \tag{8}$$

for all $1 \leq p \leq L$ and $\omega \in \widehat{\Omega}_M$. But, by Proposition 1, (6) and (7),

$$X_{\underline{m}-\underline{t}_p}(\omega) = r_{\underline{m}-\underline{t}_p}(\omega, \boldsymbol{\phi}_0) + \sum_{j=1}^{M-1} \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0) r_{\underline{m}-\underline{t}_p - s(j,\widetilde{\mathbf{k}})}(\omega, \boldsymbol{\phi}_0),$$

for all $\omega \in \widehat{\Omega}_M$. Replacing this expression in (8), the following equality holds:

$$\sum_{\underline{m} \in W_M} \left\{ r_{\underline{m}} \left(\omega, \widehat{\boldsymbol{\phi}}_M(\omega)\right) r_{\underline{m}-\underline{t}_p}(\omega, \boldsymbol{\phi}_0) + \sum_{j=1}^{M-1} \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0) r_{\underline{m}}(\omega, \widehat{\boldsymbol{\phi}}_M(\omega)) r_{\underline{m}-\underline{t}_p - s(j,\widetilde{\mathbf{k}})}(\omega, \boldsymbol{\phi}_0) \right\} = 0, \tag{9}$$

for all $1 \leq p \leq L$ and $\omega \in \widehat{\Omega}_M$. Now, replacing the parameter by its estimator in (9) we have the following definition:

Definition 2. *The least squares estimator of $\boldsymbol{\phi}_0$ based on the covariance of the residuals corresponding to the observations $\widetilde{\mathbf{X}}_M$, with domain $\Omega_{MQ,M} \subset \Omega$ is defined as the function $\widehat{\boldsymbol{\phi}}_{Q,M} \colon \Omega_{MQ,M} \to \Theta$ such that*

$$\sum_{\underline{m} \in W_M} \left(\boldsymbol{\chi}_{\underline{m},M} \left(\omega, \widehat{\boldsymbol{\phi}}_{Q,M}(\omega) \right) \right)_p = 0,$$

for all $1 \leq p \leq L$ and $\omega \in \Omega_{MQ,M}$, where $\boldsymbol{\chi}_{\underline{n},N} \colon \Omega \times \Theta \to \mathbb{R}^L$ is given by

$$\begin{aligned} \left(\boldsymbol{\chi}_{\underline{n},N}(\omega, \boldsymbol{\phi}) \right)_p &= r_{\underline{n}}(\omega, \boldsymbol{\phi}) . r_{\underline{n}-\underline{t}_p}(\omega, \boldsymbol{\phi}) + \\ & \sum_{j=1}^{N-1} \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}) r_{\underline{n}}(\omega, \boldsymbol{\phi}) \cdot r_{\underline{n}-\underline{t}_p - s(j,\widetilde{\mathbf{k}})}(\omega, \boldsymbol{\phi}), \end{aligned}$$

$1 \leq N$, $\omega \in \Omega$, $\underline{n} \in \mathbb{Z}^2$, $\boldsymbol{\phi} \in \Theta$ and $1 \leq p \leq L$.

Remark 2. *If we assume that $\widetilde{\mathbf{X}} = \left(X_{\underline{m}} \right)_{\underline{m} \in \mathbb{Z}^2}$ is a $AR(P_{\boldsymbol{\phi}_0}, \widetilde{\boldsymbol{\varepsilon}})$ Gaussian process, then the asymptotic properties of this estimator can be derived from more general results as in Guyon (1993), Chapter 3.*

It is a well known fact that least squares estimator based on the covariances are not robust. Hence, the idea is to make them robust using adequate continuous and bounded score functions.

Definition 3. *The RA estimator of $\boldsymbol{\phi}_0$ based on the observations $\widetilde{\mathbf{X}}_M$ with domain $\Omega_{RA,M} \subset \Omega$ is defined as the function $\widehat{\boldsymbol{\phi}}_M^{RA} : \Omega_{RA,M} \to \Theta$, such that*

$$0 = \sum_{\underline{m} \in W_M} \left(\boldsymbol{\chi}^{\eta}_{\underline{m},M}(\omega, \widehat{\boldsymbol{\phi}}_M^{RA}(\omega)) \right)_p ,$$

for all $1 \leq p \leq L$ and $\omega \in \Omega_{RA,M}$, where $\boldsymbol{\chi}^{\eta}_{\underline{n},N} \colon \Omega \times \Theta \to \mathbb{R}^L$ is given by

$$\begin{aligned} \left(\boldsymbol{\chi}^{\eta}_{\underline{n},N}(\omega, \boldsymbol{\phi}) \right)_p &= \eta \left(r^*_{\underline{n}}(\omega, \boldsymbol{\phi}), r^*_{\underline{n}-\underline{t}_p}(\omega, \boldsymbol{\phi}) \right) + \\ & \sum_{j=1}^{N-1} \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}) \eta \left(r^*_{\underline{n}}(\omega, \boldsymbol{\phi}), r^*_{\underline{n}-\underline{t}_p - s(j,\widetilde{\mathbf{k}})}(\omega, \boldsymbol{\phi}) \right), \end{aligned}$$

and

$$r^*_{\underline{n}}(\omega, \boldsymbol{\phi}) = r_{\underline{n}}(\omega, \boldsymbol{\phi}) / \sigma_{\varepsilon},$$

$1 \leq N$, $\omega \in \Omega$,$\underline{n} \in \mathbb{Z}^2, \boldsymbol{\phi} \in \Theta$, $1 \leq p \leq L$ and $\eta \colon \mathbb{R}^2 \to \mathbb{R}$ is a score function.

Score functions can be chosen from several families or types (see Bustos and Yohai (1986)). The Mallows type score functions are defined as $\eta_M(u,v) = \psi_1(u)\psi_2(v)$, while the Hampel type is $\eta_H(u,v) = \psi(uv)$, where the ψ functions are continuous and odd, and they may be chosen, for example, from the Huber family:

$$\psi_{H,k}(u) = sgn(u) \cdot \min(|u|, k),$$

where $sgn(u)$ is the sign function and k is a constant; or in a redescending familiy, for example the bisquare family defined by

$$\psi_{B,k}(u) = k \cdot \psi_B \left(\frac{u}{k} \right),$$

where k is a constant and

$$\psi_B(u) = \begin{cases} u \left(1 - u^2 \right)^2, & 0 \leq |u| \leq 1 \\ 0, & |u| > 1. \end{cases}$$

Note that if $\eta(u,v) = uv$ then the RA estimator defined above coincides with the previously defined least squares estimator of $\boldsymbol{\phi}_0$ based on the covariance of the residuals.

Continuation of Example 1. Let $\Theta = \{(\phi_1, \phi_2, \phi_3)' \in \mathbb{R}^3 : \sup_{1\le p\le 3}\{|\phi_p|\} < 1/3\}$. The RA estimator of $\boldsymbol{\phi}_0$ based on the observations $\widetilde{\mathbf{X}}_M$ is the random vector $\widehat{\boldsymbol{\phi}}_M^{RA} = (\widehat{\phi}_{M,1}^{RA}, \widehat{\phi}_{M,2}^{RA}, \widehat{\phi}_{M,3}^{RA})' \in \Theta$, such that

$$\begin{aligned} 0 &= \sum_{\underline{m}\in W_M} \eta\left(r^*_{\underline{m}}(\widehat{\boldsymbol{\phi}}_M^{RA}), r^*_{\underline{m}-\underline{t}_p}(\widehat{\boldsymbol{\phi}}_M^{RA})\right) + \\ & \sum_{\underline{m}\in W_M}\sum_{j=1}^{2M}\sum_{u=0}^{j}\sum_{v=0}^{u}\left(\widehat{\phi}_{M,1}^{RA}\right)^{v}\left(\widehat{\phi}_{M,2}^{RA}\right)^{j-u}\left(\widehat{\phi}_{M,3}^{RA}\right)^{u-v}\eta\left(r^*_{\underline{m}}\left(\widehat{\boldsymbol{\phi}}_M^{RA}\right), r^*_{\underline{m}-\underline{t}_p-(v+j-u,u-v)'}\left(\widehat{\boldsymbol{\phi}}_M^{RA}\right)\right), \end{aligned}$$

for all $p = 1, 2, 3$.

4 CONSISTENCY OF RA ESTIMATORS

In order to study the asymptotic behavior of RA estimator we will introduce some changes in the previous definitions. First, we eliminate the condition (6) and we assume that the process $\widetilde{\mathbf{X}}$ is observed in $\mathbb{Z}^2$. Now we redefine the residuals as follows

Definition 4. *For all $\underline{m} \in \mathbb{Z}^2$, the residual with respect to $\boldsymbol{\phi}$ is the function $R_{\underline{m}}: \Omega \times \Theta \to \mathbb{R}$ defined as*

$$R_{\underline{m}}(\omega, \boldsymbol{\phi}) = \frac{1}{\sigma_\varepsilon}\left(X_{\underline{m}}(\omega) - \sum_{p=1}^{L}\phi_p X_{\underline{m}-\underline{t}_p}(\omega)\right),$$

with $\omega \in \Omega$ and $\boldsymbol{\phi} \in \Theta$.

Note that $R_{\underline{m}}(\boldsymbol{\phi})$ is $r^*_{\underline{m}}(\boldsymbol{\phi})$ as given in Definition 3 when $\left(\underline{m} - \underline{t}_p\right) \in I_0$ for all $1 \le p \le L$. Also

$$R_{\underline{m}}(\boldsymbol{\phi}_0) = \varepsilon_{\underline{m}}/\sigma_\varepsilon, \text{ for all } \underline{m}. \tag{10}$$

Definition 5. *The RA estimator of $\boldsymbol{\phi}_0$ based on the observations $(X_{\underline{m}})_{\underline{m}\le(M,M)'}$ with domain $\Omega'_{RA,M} \subset \Omega$ is defined as the function $\widehat{\boldsymbol{\phi}}_M^{RA} : \Omega'_{RA,M} \to \Theta$, such that*

$$\frac{1}{\#(W_M)}\sum_{\underline{m}\in W_M}\boldsymbol{\Gamma}^{\eta}_{\underline{m},M}(\omega, \widehat{\boldsymbol{\phi}}_M^{RA}(\omega)) = \mathbf{0}, \tag{11}$$

for all $\omega \in \Omega'_{RA,M}$, and $\boldsymbol{\Gamma}^{\eta}_{\underline{n},N}: \Omega \times \Theta \to \mathbb{R}^L$ is given by

$$\left(\boldsymbol{\Gamma}^{\eta}_{\underline{n},N}(\omega, \boldsymbol{\phi})\right)_p = \eta\left(R_{\underline{n}}(\omega, \boldsymbol{\phi}), R_{\underline{n}-\underline{t}_p}(\omega, \boldsymbol{\phi})\right) + \sum_{j=1}^{N-1}\sum_{\widetilde{\mathbf{k}}\in\mathbb{L}^j} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi})\ \eta\left(R_{\underline{n}}(\omega, \boldsymbol{\phi}), R_{\underline{n}-\underline{t}_p-s(j,\widetilde{\mathbf{k}})}(\omega, \boldsymbol{\phi})\right),$$

where η is a score function, $1 \le N \le \infty$, $\underline{n} \in \mathbb{Z}^2$, $\omega \in \Omega$,$\boldsymbol{\phi} \in \Theta$ and $1 \le p \le L$.

Remark 3.

(a) *For $N = \infty$ the convergence of the series involved in the definition of $\boldsymbol{\Gamma}^{\eta}_{\underline{n},\infty}$ is granted because of the definition of Θ and the assumptions about the η function that we describe below.*

(b) *As we mention before, when σ_ε is unknown, the computation of $R_{\underline{m}}(\boldsymbol{\phi}_0)$ can be done by plugging in (10) a robust estimator of the scale of F_ε.*

Let us consider the following assumptions about η:

Assumption 2.

(i) *$\eta\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuously differentiable function satisfying*

$$\begin{aligned}\eta(0,v) &= 0,\\ \eta(u,0) &= 0,\\ |\eta(u,v)| &\le K_\eta,\end{aligned}$$

for some constant $K_\eta < \infty$ and for all $(u,v) \in \mathbb{R}^2$.

(ii) *Let $\eta_1(u,v) = D_1\eta(u,v)$ and $\eta_2(u,v) = D_2\eta(u,v)$ the partial derivatives of η with respect to the first and second components respectively. One of the two following conditions is satisfied:*

(a) *There are constants K_1, K_2 such that*

$$\begin{aligned}|\eta_1(u,v)| &\le K_1,\\ |\eta_2(u,v)| &\le K_2,\end{aligned}$$

for all (u,v) in $\mathbb{R}^2$.

(b) *There exists a constant $0 < K_3 < \infty$, such that*

$$\begin{aligned}|\eta_1(u,v)| &\le K_3|v|,\\ |\eta_2(u,v)| &\le K_3|u|.\end{aligned}$$

(iii)

$$E\left(\eta\left(\frac{\varepsilon_{\underline{m}}}{\sigma_\varepsilon},\frac{\varepsilon_{\underline{m}'}}{\sigma_\varepsilon}\right)\right) = 0 \text{ if } \underline{m} \neq \underline{m}',$$

where $\varepsilon_{\underline{m}}$ and $\varepsilon_{\underline{m}'}$ are independent random variables with distribution F_ε.

(iv)

$$E\left(\eta_1\left(\frac{\varepsilon_{\underline{m}}}{\sigma_\varepsilon},\frac{\varepsilon_{\underline{m}'}}{\sigma_\varepsilon}\right)\varepsilon_{\underline{m}'}\right) \neq 0 \text{ if } \underline{m} \neq \underline{m}',$$

where $\varepsilon_{\underline{m}}$ and $\varepsilon_{\underline{m}'}$ are independent random variables with distribution F_ε.

Now, we state the main result of this section:

Theorem 1 (Existence and Consistency of the RA Estimator)**.** *Given $M_0 < \infty$, there exists $\Omega'' \subset \Omega$ with $P(\Omega'') = 1$ such that*

$$\begin{aligned}&\Omega'' \subset \Omega''' = \{\omega \in \Omega\colon \text{ there exists } N > M_0 \text{ such that}\\ &M \ge N \Rightarrow \exists \widehat{\boldsymbol{\phi}}_M^{RA}(\omega) \in \Theta \text{ with } \frac{1}{\#(W_M)}\sum_{\underline{m}\in W_M} \boldsymbol{\Gamma}^\eta_{\underline{m},M}(\omega,\widehat{\boldsymbol{\phi}}_M^{RA}(\omega)) = \mathbf{0}, \text{ and } \lim_{M\to\infty}\widehat{\boldsymbol{\phi}}_M^{RA}(\omega) = \boldsymbol{\phi}_0\}.\end{aligned}$$

The proof of this Theorem is given in the Subsection 6.1 of the Appendix.

5 ASYMPTOTIC NORMALITY OF RA ESTIMATORS

We will prove in the appendix that the covariance matrix of $\mathbf{X}$,say $\Sigma(\mathbf{X}_T)$, is a positive definite matrix. More precisely, $\Sigma(\mathbf{X}_T)$ is given by $\Sigma(\mathbf{X}_T) = \sigma_\varepsilon^2 \mathbf{I}_\infty(\boldsymbol{\phi}_0)$ (see Lemma 1 in Subsection 6.1) where $\mathbf{I}_\infty(\boldsymbol{\phi}_0)$ is defined as follows.

Let

$$T - T = \{\underline{t} - \underline{s} : \underline{t} \text{ and } \underline{s} \text{ in } T\}, \tag{12}$$

and, for each $\underline{v} \in T - T$, let

$$\begin{aligned}(T-T)_1(\underline{v}) &= \left\{(j,\widetilde{\mathbf{k}})\colon j \ge 1, \widetilde{\mathbf{k}} \in \mathbb{L}^j \text{ and } s(j,\widetilde{\mathbf{k}}) = \underline{v}\right\},\\ (T-T)_2(\underline{v}) &= \left\{(j,\widetilde{\mathbf{k}},l,\widetilde{\mathbf{h}})\colon j,l \ge 1, \widetilde{\mathbf{k}} \in \mathbb{L}^j, \widetilde{\mathbf{h}} \in \mathbb{L}^l \text{ and } s(j,\widetilde{\mathbf{k}}) - s(l,\widetilde{\mathbf{h}}) = \underline{v}\right\}.\end{aligned}$$

Note that

$$\underline{v} \notin I_0 \Rightarrow (T-T)_1(\underline{v}) = \emptyset. \tag{13}$$

So, $\mathbf{I}_\infty(\boldsymbol{\phi}_0)$ is the $L \times L$ matrix given by

$$\mathbf{I}_\infty(\boldsymbol{\phi}_0)_{p,p} = 1 + \sum_{(j,\widetilde{\mathbf{k}},l,\widetilde{\mathbf{h}}) \in (T-T)_2(\underline{0})} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0) I_{l,\widetilde{\mathbf{h}}}(\boldsymbol{\phi}_0), \tag{14}$$

for all $1 \leq p \leq L$, and

$$\begin{aligned}\mathbf{I}_\infty(\boldsymbol{\phi}_0)_{p,q} \quad = \quad & \sum_{(j,\widetilde{\mathbf{k}}) \in (T-T)_1(\underline{t}_p - \underline{t}_q)} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0) + \sum_{(j,\widetilde{\mathbf{k}}) \in (T-T)_1(\underline{t}_q - \underline{t}_p)} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0) + \\ & \sum_{(j,\widetilde{\mathbf{k}},l,\widetilde{\mathbf{h}}) \in (T-T)_2(\underline{t}_p - \underline{t}_q)} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0) I_{l,\widetilde{\mathbf{h}}}(\boldsymbol{\phi}_0),\end{aligned} \tag{15}$$

for all $p \neq q$, $1 \leq p, q \leq L$.

Now, in addition to the conditions given in Assumption 2 we need to introduce the following restriction on the score function.

Assumption 3.

$$E\left(\eta\left(\frac{\varepsilon}{\sigma_\varepsilon}, \frac{\varepsilon'}{\sigma_\varepsilon}\right) \cdot \eta\left(\frac{\varepsilon}{\sigma_\varepsilon}, \frac{\varepsilon''}{\sigma_\varepsilon}\right)\right) = E\left(\eta\left(\frac{\varepsilon}{\sigma_\varepsilon}, \frac{\varepsilon'}{\sigma_\varepsilon}\right) . \eta\left(\frac{\varepsilon''}{\sigma_\varepsilon}, \frac{\varepsilon}{\sigma_\varepsilon}\right)\right) = 0,$$

where ε, ε' and ε'' are independent random variables with distribution function F_ε.

We now state the asymptotic distribution of the RA estimator.

Theorem 2 (Asymptotic Normality of the RA Estimator)**.** *Let $\left(\widehat{\boldsymbol{\phi}}_M^{RA}\right)_{M \geq M_0}$ be a sequence of random variables in Θ such that*

$$\sqrt{\#(W_M)}\left[\frac{1}{\#(W_M)} \sum_{\underline{m} \in W_M} \boldsymbol{\Gamma}^\eta_{\underline{m},M}\left(\widehat{\boldsymbol{\phi}}_M^{RA}\right)\right] \xrightarrow{P} \mathbf{0}, \text{ as } M \to \infty, \tag{16}$$

and

$$\widehat{\boldsymbol{\phi}}_M^{RA} \xrightarrow{P} \boldsymbol{\phi}_{0,} \text{ as } M \to \infty.$$

Then

$$\sqrt{\#(W_M)}\left(\widehat{\boldsymbol{\phi}}_M^{RA} - \boldsymbol{\phi}_0\right) \xrightarrow{D} N\left(\mathbf{0}, \sigma_A^2\right), \quad \text{as} \quad M \to \infty,$$

where the asymptotic variance is given by

$$\sigma_A^2 = \sigma_\varepsilon^2 \frac{E\left(\left(\eta\left(\frac{\varepsilon}{\sigma_\varepsilon}, \frac{\varepsilon'}{\sigma_\varepsilon}\right)\right)^2\right)}{\left(E\left(\eta_1\left(\frac{\varepsilon}{\sigma_\varepsilon}, \frac{\varepsilon'}{\sigma_\varepsilon}\right) . \varepsilon'\right)\right)^2} \left(\mathbf{I}_\infty(\boldsymbol{\phi}_0)\right)^{-1},$$

with ε, ε' are independent random variables with distribution function F_ε.

The proof of Theorem 2 is given in the Subsection 6.2 of the Appendix.

Remark 4. *Theorem 2 gives the asymptotic distribution of the least squares estimator of ϕ_0 based on the covariance of the residuals (LS estimator) when $\eta(u,v) = uv$. Hence, the efficiency of the RA estimator with respect to the LS estimator is given by*

$$\frac{1}{\sigma_\varepsilon^2 \frac{E\left(\left(\eta\left(\frac{\varepsilon}{\sigma_\varepsilon}, \frac{\varepsilon'}{\sigma_\varepsilon}\right)\right)^2\right)}{\left(E\left(\eta_1\left(\frac{\varepsilon}{\sigma_\varepsilon}, \frac{\varepsilon'}{\sigma_\varepsilon}\right) . \varepsilon'\right)\right)^2}}.$$

So, as discussed by in Bustos and Yohai (1986), the constants involved in the definition of the η function can be tuned.

6 APPENDIX: PROOFS

In order to simplify the presentation we introduce some additional notation and considerations. Let

$$\begin{aligned}
\mathbf{G}_{M,N}(\omega,\boldsymbol{\phi}) &= \frac{1}{\#(W_M)}\sum_{\underline{m}\in W_M}\boldsymbol{\Gamma}^{\eta}_{\underline{m},N}(\omega,\boldsymbol{\phi}),\\
\mathbf{g}_N(\boldsymbol{\phi}) &= E\left(\Gamma^{\eta}_{\underline{0},N}(\boldsymbol{\phi})\right),\\
\mathbf{F}_M(\omega,\boldsymbol{\phi}) &= \mathbf{G}_{M,M}(\omega,\boldsymbol{\phi}),\\
\mathbf{f}(\boldsymbol{\phi}) &= \mathbf{g}_\infty(\boldsymbol{\phi}),
\end{aligned} \tag{17}$$

where $1\leq N\leq\infty$, $M_0\leq M<\infty,\omega\in\Omega$ and $\boldsymbol{\phi}\in\Theta$. Note that from (10), and (i) and (iii) of Assumption 2, it follows that

$$\mathbf{f}(\boldsymbol{\phi}_0)=\mathbf{0}. \tag{18}$$

Since η is continuously differentiable, for each $\omega\in\Omega$ the function $\boldsymbol{\phi}\mapsto\mathbf{F}_M(\omega,\boldsymbol{\phi})$ is continuously differentiable on Θ, for every M.

Let $\delta_0>0$ such that $\Lambda=\{\boldsymbol{\phi}\in\Theta\colon\ |\boldsymbol{\phi}-\boldsymbol{\phi}_0|<\delta_0\}\subset\Theta$; hence by (4), we have that

$$b=\sup\left\{L\sup_{1\leq p\leq L}\left|\phi_p\right|\ :\ \boldsymbol{\phi}\in\Lambda\right\}<1. \tag{19}$$

Remark 5. *Let $\boldsymbol{\phi}\in\Lambda$. Then*

$$\left|\sum_{j=0}^{\infty}\sum_{\widetilde{\mathbf{k}}\in\mathbb{L}^j}I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi})\right|\leq\sum_{j=0}^{\infty}b^j.$$

In fact, $\left|\sum_{j=0}^{\infty}\sum_{\widetilde{\mathbf{k}}\in\mathbb{L}^j}I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi})\right|\leq\sum_{j=0}^{\infty}\sum_{\widetilde{\mathbf{k}}\in\mathbb{L}^j}\left|I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi})\right|=\sum_{j=0}^{\infty}\left(\sum_{p=1}^{L}|\phi_p|\right)^j\leq\sum_{j=0}^{\infty}b^j$. Consider also $0<\delta_1<\delta_0$, and let $C=B'(\delta_1)=\left\{\boldsymbol{\phi}\in\mathbb{R}^L\colon\ |\boldsymbol{\phi}-\boldsymbol{\phi}_0|\leq\delta_1\right\}$.

6.1 Proof of the Consistency of the RA estimator

The following lemmas and propositions are necessary to prove Theorem 1. This proof is given at the end of this subsection.

Lemma 1. *Let $\mathbf{I}_\infty(\boldsymbol{\phi}_0)$ be as in (14) and (15) and let $\Sigma(\mathbf{X}_T)$ be the covariance matrix of $\mathbf{X}_T=\left(X_{\underline{t}_1},\ldots,X_{\underline{t}_L}\right)'$. Then $\Sigma(\mathbf{X}_T)$ is positive definite and*

$$\Sigma(\mathbf{X}_T)=\sigma_\varepsilon^2\mathbf{I}_\infty(\boldsymbol{\phi}_0).$$

Let us prove that the covariance matrix is positive definite. Note first that, since $E(X_{\underline{t}_p})=0$, for all $1\leq p\leq L$, then

$$\Sigma(\mathbf{X}_T)=\left[E(X_{\underline{t}_p}X_{\underline{t}_q})\right]_{1\leq p,q\leq L}.$$

As $\Sigma(\mathbf{X}_T)$ is non negative definite, by contradiction we suppose that $\Sigma(\mathbf{X}_T)$ is not positive definite. Then there exists a vector $\mathbf{a}=(a_1,\ldots,a_L)'\neq(0,\ldots 0)$ such that

$$0=\mathbf{a}\Sigma(\mathbf{X}_T)\mathbf{a}'=\sum_{p=1}^{L}\sum_{q=1}^{L}a_pa_qE(X_{\underline{t}_p}X_{\underline{t}_q})=\sum_{p=1}^{L}\sum_{q=1}^{L}\langle a_pX_{\underline{t}_p},a_qX_{\underline{t}_q}\rangle_{L^2}=\left\|\sum_{p=1}^{L}a_pX_{\underline{t}_p}\right\|_{L^2}^2.$$

Hence

$$\sum_{p=1}^{L}a_pX_{\underline{t}_p}=0, \tag{20}$$

in L^2. Without loss of generality we may assume that the set $T=\{\underline{t}_1,\ldots,\underline{t}_L\}$ satisfies

$$\left\{\begin{array}{ll} t_{p,1}\geq t_{q,1} & 1\leq p<q\leq L\\ t_{p,2}>t_{q,2} & \text{if } t_{p,1}=t_{q,1}, 1\leq p<q\leq L.\end{array}\right. \tag{21}$$

Considering that $\mathbf{a}$ is different from the null vector and (20) then there exists

$$1 \leq p_1 < \cdots < p_r \leq L, \quad 2 \leq r \leq L, \tag{22}$$

such that

$$a_{p_i} \neq 0, \text{ for all } i = 1, \ldots, r. \tag{23}$$

For every $U \subset L^2(\Omega, F, P)$ we denote with $\mathbb{H}(U)$ the closed vector subspace of $L^2(\Omega, F, P)$ generated by U. So, using Proposition 1, we have that

$$X_{\underline{m}} \in \mathbb{H}\left(\left\{\varepsilon_{\underline{m}}\right\} \cup \left\{\varepsilon_{\underline{m}-s(j,\widetilde{\mathbf{k}})} : j \geq 1, \widetilde{\mathbf{k}} \in \mathbb{L}^j\right\}\right), \text{ for all } \underline{m} \in \mathbb{Z}^2.$$

Then, by (20), (22) and (23) we conclude that

$$\varepsilon_{\underline{t}_{p_1}} \in \mathbb{H}\left(\left\{\varepsilon_{\underline{t}_{p_2}}, \cdots, \varepsilon_{\underline{t}_{p_r}}\right\} \cup \left\{\varepsilon_{\underline{t}_{p_i}-s(j,\widetilde{\mathbf{k}})} : 1 \leq i \leq r, j \geq 1, \widetilde{\mathbf{k}} \in \mathbb{L}^j\right\}\right). \tag{24}$$

Now, using (21), (22) and the definition of $s(j, \widetilde{\mathbf{k}})$, $j \geq 1, \widetilde{\mathbf{k}} \in \mathbb{L}^j$, we establish that

$$\varepsilon_{\underline{t}_{p_1}} \notin \left(\left\{\varepsilon_{\underline{t}_{p_2}}, \cdots, \varepsilon_{\underline{t}_{p_r}}\right\} \cup \left\{\varepsilon_{\underline{t}_{p_i}-s(j,\widetilde{\mathbf{k}})} : 1 \leq i \leq r, j \geq 1, \widetilde{\mathbf{k}} \in \mathbb{L}^j\right\}\right).$$

Since $\widetilde{\boldsymbol{\varepsilon}} = \left(\varepsilon_{\underline{m}}\right)_{\underline{m} \in \mathbb{Z}^2}$ is a white noise, (24) is impossible. By this contradiction we conclude that the covariance matrix is positive definite.

The equality $\Sigma\left(\mathbf{X}_T\right) = \sigma_\varepsilon^2 \mathbf{I}_\infty(\boldsymbol{\phi}_0)$ follows from a straightforward calculation.

Lemma 2. *For each $\underline{m} \in \mathbb{Z}^2$, let $\mathbf{Z}_{\underline{m}}$ be the random vector with values in $\mathbb{R}^{L+1}$ given by*

$$\mathbf{Z}_{\underline{m}} = \left(X_{\underline{m}}, X_{\underline{m}-\underline{t}_1}, \ldots, X_{\underline{m}-\underline{t}_L}\right)'.$$

Let $c_0 \colon \Lambda \to \mathbb{R}$ be a continuous function. For each $j \geq 1$ and $\widetilde{\mathbf{k}} \in \mathbb{L}^j$ let $c_{j,\widetilde{\mathbf{k}}} \colon \Lambda \to \mathbb{R}$ be a continuous function. Assume that

c0) *There exists a sequence of positive real numbers, $\left(b_j\right)_{j \geq 0}$, such that*

$$\sum_{j=0}^{\infty} b_j < \infty,$$
$$\sup_{\boldsymbol{\phi} \in \Lambda} \left|c_0\left(\boldsymbol{\phi}\right)\right| \leq b_0, \text{ and}$$
$$\sup_{\boldsymbol{\phi} \in \Lambda} \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} \left|c_{j,\widetilde{\mathbf{k}}}\left(\boldsymbol{\phi}\right)\right| \leq b_j, \forall j \geq 1.$$

Let $W : \mathbb{R}^{L+1} \times \mathbb{R}^{L+1} \times \Lambda \to \mathbb{R}$ be a function satisfying the following assumptions:

w1) *There exists $K_W < \infty$ such that for each $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{L+1} \times \mathbb{R}^{L+1}$*

$$\sup_{\boldsymbol{\phi} \in \Lambda} \left|W(\mathbf{x}, \mathbf{y}, \boldsymbol{\phi})\right| \leq K_W \left|\rho\left(\mathbf{x}, \mathbf{y}\right)\right|,$$

where ρ is the function $\rho \colon \mathbb{R}^{L+1} \times \mathbb{R}^{L+1} \longrightarrow \mathbb{R}$ such that there exists a constant κ satisfying

$$E\left(\left|\rho\left(\mathbf{Z}_{\underline{m}}, \mathbf{Z}_{\underline{n}}\right)\right|\right) \leq \kappa \left\|X_{\underline{0}}\right\|_{L^2}, \tag{25}$$

for all $\underline{m}$, $\underline{n} \in \mathbb{Z}^2$.

w2) *Given $\mathbb{K} \subset \mathbb{R}^{L+1}$ a compact set and $d > 0$, there exists $d' > 0$ such that*

$$|\boldsymbol{\phi} - \boldsymbol{\phi}^*| < d', \boldsymbol{\phi} \text{ and } \boldsymbol{\phi}^* \in \Lambda \Longrightarrow |W(\mathbf{x}, \mathbf{y}, \boldsymbol{\phi}) - W(\mathbf{x}, \mathbf{y}, \boldsymbol{\phi}^*)| < d, \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{K}.$$

Let $\underline{n}_1 \in \mathbb{Z}^2$. For each $1 \leq N$ integer and $\underline{n} \in \mathbb{Z}^2$ let $\Psi_{\underline{n},N} \colon \Omega \times \Lambda \to \mathbb{R}$ be given by

$$\Psi_{\underline{n},N}(\omega, \boldsymbol{\phi}) = c_0(\boldsymbol{\phi}) W\left(\mathbf{Z}_{\underline{n}}(\omega), \mathbf{Z}_{\underline{n}-\underline{n}_1}(\omega), \boldsymbol{\phi}\right) + \sum_{j=1}^{N-1} \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} c_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}) W\left(\mathbf{Z}_{\underline{n}}(\omega), \mathbf{Z}_{\underline{n}-\underline{n}_1 - s(j,\widetilde{\mathbf{k}})}(\omega), \boldsymbol{\phi}\right).$$

Then

1) $\left(\Psi_{\underline{0},N}(\boldsymbol{\phi})\right)_{N\geq 1}$ *converges uniformly on $\boldsymbol{\phi} \in \Lambda$ in $L^2(\Omega, \mathbb{F}, P)$.*
Let us denote the limit as

$$\Psi_{\underline{0},\infty}(\omega, \boldsymbol{\phi}) = c_0(\boldsymbol{\phi}) W\left(\mathbf{Z}_{\underline{0}}(\omega), \mathbf{Z}_{\underline{0}-\underline{n}_1}(\omega), \boldsymbol{\phi}\right) + \sum_{j=1}^{\infty} \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} c_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}) W\left(\mathbf{Z}_{\underline{0}}(\omega), \mathbf{Z}_{\underline{0}-\underline{n}_1 - s(j,\widetilde{\mathbf{k}})}(\omega), \boldsymbol{\phi}\right).$$

2) *Writing*

$$T_{M,N}(\omega, \boldsymbol{\phi}) = \frac{1}{\#(W_M)} \sum_{\underline{m} \in W_M} \Psi_{\underline{m},N}(\omega, \boldsymbol{\phi})$$

there exists a strictly increasing sequence $(J(N))_N$ such that for all N there exists $\Omega_N \subseteq \Omega$ with $P(\Omega_N) = 1$ satisfying

$$\limsup_{M\to\infty} \sup_{\boldsymbol{\phi}\in C} \left|T_{M,M}(\omega, \boldsymbol{\phi}) - E\left(\Psi_{\underline{0},\infty}(\boldsymbol{\phi})\right)\right| \leq \limsup_{M\to\infty} \sup_{\boldsymbol{\phi}\in C} \left|T_{M,J(N)}(\omega, \boldsymbol{\phi}) - E\left(\Psi_{\underline{0},J(N)}(\boldsymbol{\phi})\right)\right| + \frac{1}{N},$$

if $\omega \in \Omega_N$.

3) *For each N there exists Ω'_N with $P(\Omega'_N) = 1$, satisfying $\lim_{M\to\infty} \sup_{\boldsymbol{\phi}\in C} \left|T_{M,N}(\omega, \boldsymbol{\phi}) - E\left(\Psi_{\underline{0},N}(\boldsymbol{\phi})\right)\right| = 0$ if $\omega \in \Omega'_N$.*

4) *There exists a subset Ω_0 of Ω with $P(\Omega_0) = 1$, such that*

$$\Omega_0 \subset \left\{\omega \in \Omega : \lim_{M\to\infty} \sup_{\boldsymbol{\phi}\in C} \left|\frac{1}{\#(W_M)} \sum_{\underline{m} \in W_M} \Psi_{\underline{m},M}(\omega, \boldsymbol{\phi}) - E\left(\Psi_{\underline{0},\infty}(\boldsymbol{\phi})\right)\right| = 0\right\}.$$

PROOF: The proof of 1) is an inmediate consequence of c0), w1) and (25). To prove 2), let $J(N)$ be the minimum positive integer such that for each N,

$$K_W.\kappa. \left\|X_{\underline{0}}\right\|_{L^2} \sum_{j=J(N)}^{\infty} b_j < \frac{1}{2N}.$$

Then

$$\begin{aligned}
&E\left(\sup_{\boldsymbol{\phi}\in C} \sum_{j=J(N)}^{\infty} \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} \left|c_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}) W\left(\mathbf{Z}_{\underline{0}}, \mathbf{Z}_{\underline{0}-\underline{n}_1 - s(j,\widetilde{\mathbf{k}})}, \boldsymbol{\phi}\right)\right|\right) \\
&\leq K_W. \sup_{\boldsymbol{\phi}\in C} \sum_{j=J(N)}^{\infty} \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} \left|c_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi})\right| E\left(\left|\rho\left(\mathbf{Z}_{\underline{0}}(\omega), \mathbf{Z}_{\underline{0}-\underline{n}_1 - s(j,\widetilde{\mathbf{k}})}(\omega)\right)\right|\right) \\
&\leq K_W.\kappa. \left\|X_{\underline{0}}\right\|_{L^2} \sup_{\boldsymbol{\phi}\in C} \sum_{j=J(N)}^{\infty} \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} \left|c_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi})\right| \\
&\leq K_W.\kappa. \left\|X_{\underline{0}}\right\|_{L^2} \sum_{j=J(N)}^{\infty} b_j \\
&< \frac{1}{2N}.
\end{aligned}$$

Hence, for each $\omega \in \Omega$ and $M > J(N)$,

$$
\begin{aligned}
&\sup_{\phi \in C} \left| T_{M,M}(\omega, \phi) - E\left(\Psi_{\underline{0},\infty}(\phi)\right)\right| \\
&\leq \sup_{\phi \in C} \left| T_{M,J(N)}(\omega, \phi) - E\left(\Psi_{\underline{0},J(N)}(\phi)\right)\right| \\
&+ \frac{1}{\#(W_M)} \sum_{\underline{m} \in W_M} \left(\sup_{\phi \in C} \sum_{j=J(N)}^{M-1} \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} \left| c_{j,\widetilde{\mathbf{k}}}(\phi)\, W\left(\mathbf{Z}_{\underline{m}}(\omega), \mathbf{Z}_{\underline{m}-\underline{n}_1-s(j,\widetilde{\mathbf{k}})}(\omega), \phi\right)\right| \right) \\
&+ E\left(\sup_{\phi \in C} \sum_{j=J(N)}^{\infty} \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} \left| c_{j,\widetilde{\mathbf{k}}}(\phi)\, W\left(\mathbf{Z}_{\underline{0}}, \mathbf{Z}_{\underline{0}-\underline{n}_1-s(j,\widetilde{\mathbf{k}})}, \phi\right)\right| \right).
\end{aligned}
$$

By the ergodicity, there exists $\Omega_N \subseteq \Omega$ with $P(\Omega_N) = 1$, such that if $\omega \in \Omega_N$, then

$$
\begin{aligned}
&\limsup_{M\to\infty} \frac{1}{\#(W_M)} \sum_{\underline{m} \in W_M} \left(\sup_{\phi \in C} \sum_{j=J(N)}^{\infty} \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} \left| c_{j,\widetilde{\mathbf{k}}}(\phi)\, W\left(\mathbf{Z}_{\underline{m}}(\omega), \mathbf{Z}_{\underline{m}-\underline{n}_1-s(j,\widetilde{\mathbf{k}})}(\omega), \phi\right)\right| \right) \\
&= E\left(\sup_{\phi \in C} \sum_{j=J(N)}^{\infty} \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} \left| c_{j,\widetilde{\mathbf{k}}}(\phi)\, W\left(\mathbf{Z}_{\underline{0}}, \mathbf{Z}_{\underline{0}-\underline{n}_1-s(j,\widetilde{\mathbf{k}})}, \phi\right)\right| \right).
\end{aligned}
$$

Thus, for each $\omega \in \Omega_N$ we have that

$$
\limsup_{M\to\infty} \sup_{\phi \in C} \left| T_{M,M}(\omega, \phi) - E\left(\Psi_{\underline{0},\infty}(\phi)\right)\right| \leq \limsup_{M\to\infty} \sup_{\phi \in C} \left| T_{M,J(N)}(\omega, \phi) - E\left(\Psi_{\underline{0},J(N)}(\phi)\right)\right| + 2\left(\frac{1}{2N}\right).
$$

Let us prove 3). Consider $N \geq 1$ and, for each $\phi \in C$, let $\Omega''_N(\phi)$ with $P(\Omega''_N(\phi)) = 1$, such that

$$
\Omega''_N(\phi) \subseteq \left\{ \omega \in \Omega : \lim_{M\to\infty} \left| T_{M,N}(\omega, \phi) - E\left(\Psi_{\underline{0},N}(\phi)\right)\right| = 0 \right\}.
$$

Let $n \geq 1$. Since $\left|\rho\left(\mathbf{Z}_{\underline{0}}, \mathbf{Z}_{\underline{0}-\underline{n}_1-s(j,\widetilde{\mathbf{k}})}\right)\right|$ is integrable for all $\widetilde{\mathbf{k}} \in \mathbb{L}^j, 1 \leq j \leq N-1$, there exists $0 < K_{N,n} < \infty$, such that

$$
\sup_{\widetilde{\mathbf{k}} \in \mathbb{L}^j, 1\leq j \leq N-1} \left(E\left(1_{A(K_{N,n})^c} \left|\rho\left(\mathbf{Z}_{\underline{0}}, \mathbf{Z}_{\underline{0}-\underline{n}_1}\right)\right|\right), E\left(1_{A(K_{N,n})^c} \left|\rho\left(\mathbf{Z}_{\underline{0}}, \mathbf{Z}_{\underline{0}-\underline{n}_1-s(j,\widetilde{\mathbf{k}})}\right)\right|\right)\right) < \frac{1}{6nK_W\left(\sum_{j=0}^{\infty} b_j\right)}, \tag{26}
$$

where

$$
A(K_{N,n}) = \left\{ \omega \in \Omega : \sup_{\underline{s} \in S_N} \left|\mathbf{Z}_{\underline{s}}(\omega)\right| \leqslant K_{N,n} \right\},
$$

$$
S_N = \{\underline{0}, \underline{0} - \underline{n}_1\} \cup \left\{ \underline{0} - \underline{n}_1 - s(j, \widetilde{\mathbf{k}}) : \widetilde{\mathbf{k}} \in \mathbb{L}^j, 1 \leq j \leq N-1 \right\}.
$$

Let $\phi \in C$. Considering that the functions c_0 and $c_{j,\widetilde{\mathbf{k}}}$, $\widetilde{\mathbf{k}} \in \mathbb{L}^j, 1 \leq j \leq N-1$, are continuous and also c0), w1), w2), the definition of $A(K_{N,n})$ and (26) we conclude that there exists a neighborhood, $V_{\boldsymbol{\phi},n}$, of $\boldsymbol{\phi}$ contained in Λ such that:

$$
\begin{aligned}
& E\left(\sup_{\boldsymbol{\phi}^*\in C\cap V_{\boldsymbol{\phi},n}} \left|\Psi_{\underline{0},N}(\boldsymbol{\phi}^*) - \Psi_{\underline{0},N}(\boldsymbol{\phi})\right|\right) \\
& \leq E\left(\sup_{\boldsymbol{\phi}^*\in C\cap V_{\boldsymbol{\phi},n}} \left|\Psi_{\underline{0},N}(\boldsymbol{\phi}^*)1_{A(K_{N,n})} - \Psi_{\underline{0},N}(\boldsymbol{\phi})1_{A(K_{N,n})}\right|\right) \\
& + E\left(\sup_{\boldsymbol{\phi}^*\in C\cap V_{\boldsymbol{\phi},n}} \left|\Psi_{\underline{0},N}(\boldsymbol{\phi}^*)1_{A(K_{N,n})^c} - \Psi_{\underline{0},N}(\boldsymbol{\phi})1_{A(K_{N,n})^c}\right|\right) \\
& \leq E\left(\sup_{\boldsymbol{\phi}^*\in C\cap V_{\boldsymbol{\phi},n}} \left|\Psi_{\underline{0},N}(\boldsymbol{\phi}^*)1_{A(K_{N,n})} - \Psi_{\underline{0},N}(\boldsymbol{\phi})1_{A(K_{N,n})}\right|\right) \\
& + E\left(\sup_{\boldsymbol{\phi}^*\in C\cap V_{\boldsymbol{\phi},n}} \left|\Psi_{\underline{0},N}(\boldsymbol{\phi}^*)1_{A(K_{N,n})^c}\right|\right) + E\left(\left|\Psi_{\underline{0},N}(\boldsymbol{\phi})1_{A(K_{N,n})^c}\right|\right) \\
& < \frac{1}{6n} + \frac{1}{6n} + \frac{1}{6n} = \frac{1}{2n}.
\end{aligned}
$$

Now, for each $\omega \in \Omega$,

$$
\begin{aligned}
& \sup_{\boldsymbol{\phi}^*\in C\cap V_{\boldsymbol{\phi},n}} \left|T_{M,N}(\omega, \boldsymbol{\phi}^*) - E\left(\Psi_{\underline{0},N}(\boldsymbol{\phi}^*)\right)\right| \\
& \leq \left|T_{M,N}(\omega, \boldsymbol{\phi}) - E\left(\Psi_{\underline{0},N}(\boldsymbol{\phi})\right)\right| \\
& + \frac{1}{\#(W_M)} \sum_{\underline{m}\in W_M} \sup_{\boldsymbol{\phi}^*\in C\cap V_{\boldsymbol{\phi},n}} \left|\Psi_{\underline{m},N}(\omega, \boldsymbol{\phi}^*) - \Psi_{\underline{m},N}(\omega, \boldsymbol{\phi})\right| \\
& + \sup_{\boldsymbol{\phi}^*\in C\cap V_{\boldsymbol{\phi},n}} \left|E\left(\Psi_{\underline{0},N}(\boldsymbol{\phi}^*)\right) - E\left(\Psi_{\underline{0},N}(\boldsymbol{\phi})\right)\right|.
\end{aligned} \tag{27}
$$

Also,

$$
\sup_{\boldsymbol{\phi}^*\in C\cap V_{\boldsymbol{\phi},n}} \left|E\left(\Psi_{\underline{0},N}(\boldsymbol{\phi}^*)\right) - E\left(\Psi_{\underline{0},N}(\boldsymbol{\phi})\right)\right| \leq E\left(\sup_{\boldsymbol{\phi}^*\in C\cap V_{\boldsymbol{\phi},n}} \left|\Psi_{\underline{0},N}(\boldsymbol{\phi}^*) - \Psi_{\underline{0},N}(\boldsymbol{\phi}^*)\right|\right) < \frac{1}{2n}. \tag{28}
$$

Because of the definition of $\Omega''_N(\boldsymbol{\phi})$, it follows that for all $\omega \in \Omega''_N(\boldsymbol{\phi})$:

$$
\lim_{M\to\infty} \left|T_{M,N}(\omega, \boldsymbol{\phi}) - E\left(\Psi_{\underline{0},N}(\boldsymbol{\phi})\right)\right| = 0. \tag{29}
$$

By ergodicity, there exists $\Omega_N(\boldsymbol{\phi}, n) \subset \Omega''_N(\boldsymbol{\phi})$ with $P\left(\Omega_N(\boldsymbol{\phi}, n)\right) = 1$ such that, if $\omega \in \Omega_N(\boldsymbol{\phi}, n)$, then

$$
\lim_{M\to\infty} \left|\frac{1}{\#(W_M)} \sum_{\underline{m}\in W_M} \sup_{\boldsymbol{\phi}^*\in C\cap V_{\boldsymbol{\phi},n}} \left|\Psi_{\underline{m},N}(\omega, \boldsymbol{\phi}^*) - \Psi_{\underline{m},N}(\omega, \boldsymbol{\phi})\right|\right| = E\left(\sup_{\boldsymbol{\phi}^*\in C\cap V_{\boldsymbol{\phi},n}} \left|\Psi_{\underline{0},N}(\boldsymbol{\phi}^*) - \Psi_{\underline{0},N}(\boldsymbol{\phi})\right|\right) < \frac{1}{2n}.
$$

Hence, by (27), (28), (29) and (6.1),

$$
\limsup_{M\to\infty} \sup_{\boldsymbol{\phi}^*\in C\cap V_{\boldsymbol{\phi},n}} \left|T_{M,N}(\omega, \boldsymbol{\phi}^*) - E\left(\Psi_{\underline{0},N}(\boldsymbol{\phi}^*)\right)\right| < \frac{1}{n}, \tag{30}
$$

for all $\omega \in \Omega_N(\boldsymbol{\phi}, n)$.

Since C is a compact set, there exists a finite set $\{\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_r\}$ contained in C such that

$$
C = \bigcup_{i=1}^{r} \left(C \cap V_{\boldsymbol{\phi}_i,n}\right).
$$

Thus, if we define $\Omega'_N(n) = \bigcap_{i=1}^{r} \Omega_N(\boldsymbol{\phi}_i, n)$, then $P(\Omega'_N(n)) = 1$. By (30),

$$\limsup_{M\to\infty} \sup_{\boldsymbol{\phi}^*\in C} \left| T_{M,N}(\omega, \boldsymbol{\phi}^*) - E\left(\Psi_{\underline{0},N}(\boldsymbol{\phi}^*)\right)\right| \leqslant \frac{1}{n}, \tag{31}$$

for all $\omega \in \Omega'_N(n)$. Setting $\Omega'_N = \bigcap_{n=1}^{\infty} \Omega'_N(n)$, then $P\left(\Omega'_N\right) = 1$, and because of (31) we have that

$$\lim_{M} \sup_{\boldsymbol{\phi}^*\in C} \left| T_{M,N}(\omega, \boldsymbol{\phi}^*) - E\left(\Psi_{\underline{0},N}(\boldsymbol{\phi}^*)\right)\right| = 0, \tag{32}$$

for all $\omega \in \Omega'_N$. The proof of 3) is completed. To prove 4), it is enough to set $\Omega_0 = \bigcap_{N\geq 1}\left(\Omega_N \cap \Omega'_N\right)$.■

Next lemma recalls a well known result and useful result in robustness.

Lemma 3 (The Zeros Lemma (Ruskin (1978)). *Let $U \subset \mathbb{R}^k$ be an open set, $\boldsymbol{\lambda}_0 \in U$, for each $n = 1, 2, \ldots$ let $\mathbf{q}_n : U \to \mathbb{R}^k$ and $\mathbf{q} : U \to \mathbb{R}^k$ be continuously differentiable functions. Assume that*

1) $\mathbf{q}(\boldsymbol{\lambda}_0) = \mathbf{0}$.

2) $\mathbb{D}\mathbf{q}(\boldsymbol{\lambda}_0)$ *is not zero.*

3) There exists $\gamma > 0$ such that $(\mathbf{q}_n)_n$ and $(\mathbb{D}\mathbf{q}_n)_n$ converge uniformly to $\mathbf{q}$ and $\mathbb{D}\mathbf{q}$ respectively on $B(\boldsymbol{\lambda}_0, \gamma) = \left\{\boldsymbol{\lambda} \in \mathbb{R}^k \colon \sup_{1\leq i\leq k} |\boldsymbol{\lambda}_i - \boldsymbol{\lambda}_{0i}| < \gamma\right\}$.

Then, there exist $n_0 \geq 1$ and a sequence $(\boldsymbol{\lambda}_n)_{n\geq n_0}$ in $B(\boldsymbol{\lambda}_0, \gamma)$ such that $(\boldsymbol{\lambda}_n)_n$ converges to $\boldsymbol{\lambda}_0$ and $\mathbf{q}_n(\boldsymbol{\lambda}_n) = 0$, for all $n \geq n_0$.

Proposition 2. *Let $\mathbf{f}$ be as in (17). Then $\mathbf{f}$ is continuously differentiable on Λ and satisfies*

1) For each $1 \leq p, q \leq L$ and for each $\boldsymbol{\phi} \in \Lambda$:

$$D_q\left(\left(\mathbf{f}(\boldsymbol{\phi})\right)_p\right) = A1_{p,q}(\boldsymbol{\phi}) + A2_{p,q}(\boldsymbol{\phi}) + \sum_{j=1}^{\infty}\sum_{\widetilde{\mathbf{k}}\in\mathbb{L}^j}\left(A3_{p,q}(j, \widetilde{\mathbf{k}}, \boldsymbol{\phi}) + A4_{p,q}(j, \widetilde{\mathbf{k}}, \boldsymbol{\phi}) + A5_{p,q}(j, \widetilde{\mathbf{k}}, \boldsymbol{\phi})\right),$$

where

$$\begin{aligned}
A1_{p,q}(\boldsymbol{\phi}) &= -\frac{1}{\sigma_\varepsilon} E\left(\eta_1\left(R_{\underline{0}}(\boldsymbol{\phi}), R_{-\underline{t}_p}(\boldsymbol{\phi})\right) X_{-\underline{t}_q}\right),\\
A2_{p,q}(\boldsymbol{\phi}) &= -\frac{1}{\sigma_\varepsilon} E\left(\eta_2\left(R_{\underline{0}}(\boldsymbol{\phi}), R_{-\underline{t}_p}(\boldsymbol{\phi})\right) X_{-\underline{t}_p-\underline{t}_q}\right),\\
A3_{p,q}(j, \widetilde{\mathbf{k}}, \boldsymbol{\phi}) &= D_q\left(I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi})\right) E\left(\eta\left(R_{\underline{0}}(\boldsymbol{\phi}), R_{-\underline{t}_p-s(j,\widetilde{\mathbf{k}})}(\boldsymbol{\phi})\right)\right),\\
A4_{p,q}(j, \widetilde{\mathbf{k}}, \boldsymbol{\phi}) &= -\frac{1}{\sigma_\varepsilon} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}) E\left(\eta_1\left(R_{\underline{0}}(\boldsymbol{\phi}), R_{-\underline{t}_p-s(j,\widetilde{\mathbf{k}})}(\boldsymbol{\phi})\right) X_{-\underline{t}_q}\right),\\
A5_{p,q}(j, \widetilde{\mathbf{k}}, \boldsymbol{\phi}) &= -\frac{1}{\sigma_\varepsilon} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}) E\left(\eta_2\left(R_{\underline{0}}(\boldsymbol{\phi}), R_{-\underline{t}_p-s(j,\widetilde{\mathbf{k}})}(\boldsymbol{\phi})\right) X_{-\underline{t}_p-\underline{t}_q-s(j,\widetilde{\mathbf{k}})}\right).
\end{aligned}$$

2) $\mathbb{D}\mathbf{f}(\boldsymbol{\phi}_0) = -\frac{1}{\sigma_\varepsilon} E\left(\eta_1\left(\frac{\varepsilon}{\sigma_\varepsilon}, \frac{\varepsilon'}{\sigma_\varepsilon}\right)\varepsilon'\right)\mathbf{I}_\infty(\boldsymbol{\phi}_0)$*, where ε and ε' are independent random variables with distribution F_ε, and $\mathbb{D}\mathbf{f}(\boldsymbol{\phi}_0)$ is not zero.*

PROOF: First we will prove that

$$\left(\mathbf{g}_N\right)_N \text{ converges uniformly to } \mathbf{f} \text{ on } \Lambda. \tag{33}$$

In fact, for any $\boldsymbol{\phi} \in \Lambda$, $1 \leq p \leq L$ and N we have

$$\left|\left(\mathbf{f}(\boldsymbol{\phi})\right)_p - \left(\mathbf{g}_N(\boldsymbol{\phi})\right)_p\right| = \sum_{j=N}^{\infty}\sum_{\widetilde{\mathbf{k}}\in\mathbb{L}^j} I_{j,\widetilde{\mathbf{k}}}(\phi) E\left(\left|\eta\left(R_{\underline{0}}(\boldsymbol{\phi}), R_{-\underline{t}_p-s(j,\widetilde{\mathbf{k}})}(\boldsymbol{\phi})\right)\right|\right) \leq K_\eta \sum_{j=N}^{\infty} b^j,$$

being b as in (19), this implies (33). Then, by a classical result in analysis, to prove 1) in Proposition 2, it is enough to show that

$$\left(D_q\left(\left(\mathbf{g}_N(\boldsymbol{\phi})\right)_p\right)\right)_N \text{ converges uniformly on } \Lambda. \tag{34}$$

Notice that for each $\boldsymbol{\phi} \in \Lambda$

$$D_q\left((\mathbf{g}_N(\boldsymbol{\phi}))_p\right) = A1_{p,q}(\boldsymbol{\phi}) + A2_{p,q}(\boldsymbol{\phi}) + \sum_{j=1}^{N-1} \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} \left(A3_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi}) + A4_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi}) + A5_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi})\right).$$

To prove (34), it is enough to show that for each $d > 0$, there exists a positive integer N such that for all positive integer n

$$\sup_{\boldsymbol{\phi} \in \Lambda} \left| D_q\left((\mathbf{g}_{N+n}(\boldsymbol{\phi}))_p\right) - D_q\left((\mathbf{g}_N(\boldsymbol{\phi}))_p\right)\right| < d. \tag{35}$$

Now,

$$\left|A3_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi})\right| = K_\eta \left|D_q\left(I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi})\right)\right|.$$

Then

$$\sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} \left|A3_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi})\right| \le K_\eta \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} \left|D_q\left(I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi})\right)\right|. \tag{36}$$

Since

$$\sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^h} I_{h,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}) = \left(\sum_{p=1}^{L} \phi_p\right)^h, \tag{37}$$

we have that

$$\sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} D_q\left(I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi})\right) = D_q\left(\left(\sum_{p'=1}^{L} \phi_{p'}\right)^j\right) = j\left(\sum_{p'=1}^{L} \phi_{p'}\right)^{j-1}.$$

Then, by (36), it follows that

$$\sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} \left|A3_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi})\right| \le K_\eta j \left(\sum_{p'=1}^{L} |\phi_{p'}|\right)^{j-1} \le K_\eta j\,(b)^{j-1}. \tag{38}$$

Assume that (a) of Assumption 2-(ii) holds. Then

$$\left|A4_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi})\right| \le \frac{1}{\sigma_\varepsilon}\left|I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi})\right| K_1 E\left(\left|X_{\underline{0}}\right|\right),$$

$$\sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} \left|A5_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi})\right| \le \frac{1}{\sigma_\varepsilon} K_2 E\left(\left|X_{\underline{0}}\right|\right)\left(\sum_{p'=1}^{L} |\phi_{p'}|\right)^j,$$

for all j.

Thus, for each N, n positive integers and $\boldsymbol{\phi} \in \Lambda$ we have

$$\begin{aligned}
&\left|D_q\left((\mathbf{g}_{N+n}(\boldsymbol{\phi}))_p\right) - D_q\left((\mathbf{g}_N(\boldsymbol{\phi}))_p\right)\right| \\
&\le \sum_{j=N}^{N+n} \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} \left(\left|A3_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi})\right| + \left|A4_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi})\right| + \left|A5_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi})\right|\right) \\
&\le \sum_{j=N}^{N+n} \left(K_\eta . j\,(b)^{j-1} + \frac{1}{\sigma_\varepsilon} K_1 E\left(\left|X_{\underline{0}}\right|\right) b^j + \frac{1}{\sigma_\varepsilon} K_2 E\left(\left|X_{\underline{0}}\right|\right) b^j\right).
\end{aligned}$$

Hence (35) is satisfied. Now, let us consider (b) of Assumption 2-(ii) instead of (a). By the Cauchy- Schwartz inequality

$$\left|A4_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi})\right| \le \frac{K_3}{\sigma_\varepsilon}\left|I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi})\right| \left(\sum_{p'=1}^{L} |\phi_{p'}|\right) E\left(\left|X_{\underline{0}}\right|^2\right).$$

Hence

$$\sum_{j=N}^{N+n}\sum_{\widetilde{\mathbf{k}}\in\mathbb{L}^j}\left|A4_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi})\right| \leq \frac{K_3}{\sigma_\varepsilon}E\left(\left|X_{\underline{0}}\right|^2\right)\sum_{j=N}^{N+n}\left(\sum_{p'=1}^{L}\left|\phi_{p'}\right|\right)^{j+1}, \tag{39}$$

for all N.
Similarly, considering (b) of Assumption 2-(ii) instead of (a) of that assumption:

$$\sum_{j=N}^{N+n}\sum_{\widetilde{\mathbf{k}}\in\mathbb{L}^j}\left|A5_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi})\right| \leq \frac{K_3}{\sigma_\varepsilon}E\left(\left|X_{\underline{0}}\right|^2\right)\sum_{j=N}^{N+n}\left(\sum_{p'=1}^{L}\left|\phi_{p'}\right|\right)^{j+1}, \tag{40}$$

for all N. Thus, using (38), (39) and (40), we have that

$$\left|D_q\left((\mathbf{g}_{N+n}(\boldsymbol{\phi}))_p\right)-D_q\left((\mathbf{g}_{N}(\boldsymbol{\phi}))_p\right)\right| \leq \sum_{j=N}^{N+n}\left(K_\eta j\,(b)^{j-1}+2\frac{K_3}{\sigma_\varepsilon}E\left(\left|X_{\underline{0}}\right|^2\right)b^j\right).$$

Hence the proof of (35) is completed. This finishes the proof of 1) in Proposition 2.

Now, let us prove 2) in Proposition 2.

$$\begin{aligned}
A1_{p,q}(\boldsymbol{\phi}_0) &= -\frac{1}{\sigma_\varepsilon}E\left(\eta_1\left(R_{\underline{0}}(\boldsymbol{\phi}_0),R_{-\underline{t}_p}(\boldsymbol{\phi}_0)\right)X_{-\underline{t}_q}\right)\\
&= -\frac{1}{\sigma_\varepsilon}E\left(\eta_1\left(\varepsilon_{\underline{0}},\varepsilon_{-\underline{t}_p}\right)X_{-\underline{t}_q}\right)\\
&= -\frac{1}{\sigma_\varepsilon}E\left(\eta_1\left(\varepsilon_{\underline{0}},\varepsilon_{-\underline{t}_p}\right)\left(\varepsilon_{-\underline{t}_q}+\sum_{j=1}^{\infty}\sum_{\widetilde{\mathbf{k}}\in\mathbb{L}^j}I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0)\varepsilon_{-\underline{t}_q-s(j,\widetilde{\mathbf{k}})}\right)\right).
\end{aligned}$$

Then

$$A1_{p,q}(\boldsymbol{\phi}_0)=\begin{cases} -\frac{1}{\sigma_\varepsilon}E\left(\eta_1\left(\frac{\varepsilon}{\sigma_\varepsilon},\frac{\varepsilon'}{\sigma_\varepsilon}\right)\varepsilon'\right) & \text{if } p=q\\ -\frac{1}{\sigma_\varepsilon}E\left(\eta_1\left(\frac{\varepsilon}{\sigma_\varepsilon},\frac{\varepsilon'}{\sigma_\varepsilon}\right)\varepsilon'\right)\sum_{(j,\widetilde{\mathbf{k}})\in(T-T)_1(\underline{t}_p-\underline{t}_q)}I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0) & \text{if } p\neq q\end{cases}, \tag{41}$$

where ε and ε' are independent random variables with distribution F_ε.

$$\begin{aligned}
A2_{p,q}(\boldsymbol{\phi}_0) &= -\frac{1}{\sigma_\varepsilon}E\left(\eta_2\left(R_{\underline{0}}(\boldsymbol{\phi}_0),R_{-\underline{t}_p}(\boldsymbol{\phi}_0)\right)X_{-\underline{t}_p-\underline{t}_q}\right)\\
&= -\frac{1}{\sigma_\varepsilon}E\left(\eta_2\left(\varepsilon_{\underline{0}},\varepsilon_{-\underline{t}_p}\right)\left(\varepsilon_{-\underline{t}_p-\underline{t}_q}+\sum_{j=1}^{\infty}\sum_{\widetilde{\mathbf{k}}\in\mathbb{L}^j}I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0)\varepsilon_{-\underline{t}_p-\underline{t}_q-s(j,\widetilde{\mathbf{k}})}\right)\right)=0.
\end{aligned} \tag{42}$$

Now

$$\begin{aligned}
A3_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi}_0) &= D_q\left(I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0)\right)E\left(\eta\left(R_{\underline{0}}(\boldsymbol{\phi}_0),R_{-\underline{t}_p-s(j,\widetilde{\mathbf{k}})}(\boldsymbol{\phi}_0)\right)\right)\\
&= D_q\left(I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0)\right)E\left(\eta\left(\frac{\varepsilon_{\underline{0}}}{\sigma_\varepsilon},\frac{\varepsilon_{-\underline{t}_p-s(j,\widetilde{\mathbf{k}})}}{\sigma_\varepsilon}\right)\right)=0,
\end{aligned} \tag{43}$$

by Assumption 2-(iii). Consider now

$$\begin{aligned}
A5_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi}_0) &= -\frac{1}{\sigma_\varepsilon}I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0)E\left(\eta_2\left(R_{\underline{0}}(\boldsymbol{\phi}_0),R_{-\underline{t}_p-s(j,\widetilde{\mathbf{k}})}(\boldsymbol{\phi}_0)\right)X_{-\underline{t}_p-s(j,\widetilde{\mathbf{k}})-\underline{t}_q}\right)\\
&= -\frac{1}{\sigma_\varepsilon}I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0)E\left(\eta_2\left(\frac{\varepsilon_{\underline{0}}}{\sigma_\varepsilon},\frac{\varepsilon_{-\underline{t}_p-s(j,\widetilde{\mathbf{k}})}}{\sigma_\varepsilon}\right)X_{-\underline{t}_p-s(j,\widetilde{\mathbf{k}})-\underline{t}_q}\right).
\end{aligned}$$

Now, since $\underline{t}_q\neq\underline{0}$ and (5) we have that $\varepsilon_{\underline{0}},\ \varepsilon_{-\underline{t}_p-s(j,\widetilde{\mathbf{k}})}$ and $X_{-\underline{t}_p-s(j,\widetilde{\mathbf{k}})-\underline{t}_q}$ are independent. Then, since $E\left(X_{-\underline{t}_p-s(j,\widetilde{\mathbf{k}})-\underline{t}_q}\right)=0$, it follows that

$$A5_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi}_0)=0. \tag{44}$$

Finally,

$$\begin{aligned}A4_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi}_0) &= -\frac{1}{\sigma_\varepsilon} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0) E\left(\eta_1\left(R_{\underline{0}}(\boldsymbol{\phi}_0), R_{-\underline{t}_p - s(j,\widetilde{\mathbf{k}})}(\boldsymbol{\phi}_0)\right) X_{-\underline{t}_q}\right)\\ &= -\frac{1}{\sigma_\varepsilon} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0) E\left(\eta_1\left(\frac{\varepsilon_{\underline{0}}}{\sigma_\varepsilon}, \frac{\varepsilon_{-\underline{t}_p - s(j,\widetilde{\mathbf{k}})}}{\sigma_\varepsilon}\right)\varepsilon_{-\underline{t}_q}\right) + \sum_{l=1}^{\infty}\sum_{\widetilde{\mathbf{h}}\in\mathbb{L}^l} F(j,\widetilde{\mathbf{k}},l,\widetilde{\mathbf{h}}),\end{aligned}$$

where

$$F(j,\widetilde{\mathbf{k}},l,\widetilde{\mathbf{h}}) = -\frac{1}{\sigma_\varepsilon} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0) I_{l,\widetilde{\mathbf{h}}}(\boldsymbol{\phi}_0) E\left(\eta_1\left(\frac{\varepsilon_{\underline{0}}}{\sigma_\varepsilon}, \frac{\varepsilon_{-\underline{t}_p - s(j,\widetilde{\mathbf{k}})}}{\sigma_\varepsilon}\right)\varepsilon_{-\underline{t}_q - s(l,\widetilde{\mathbf{h}})}\right).$$

Now, if $-\underline{t}_p - s(j,\widetilde{\mathbf{k}}) \neq -\underline{t}_q - s(l,\widetilde{\mathbf{h}})$, then

$$E\left(\eta_1\left(\frac{\varepsilon_{\underline{0}}}{\sigma_\varepsilon}, \frac{\varepsilon_{-\underline{t}_p - s(j,\widetilde{\mathbf{k}})}}{\sigma_\varepsilon}\right)\varepsilon_{-\underline{t}_q - s(l,\widetilde{\mathbf{h}})}\right) = 0,$$

Therefore

$$\sum_{j=1}^{\infty}\sum_{\widetilde{\mathbf{k}}\in\mathbb{L}^j} A4_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi}) = -\frac{1}{\sigma_\varepsilon} E\left(\eta_1\left(\frac{\varepsilon}{\sigma_\varepsilon}, \frac{\varepsilon'}{\sigma_\varepsilon}\right)\varepsilon'\right) A4^*_{p,q}, \tag{45}$$

where ε and ε' are independent random variables with distribution F_ε and

$$A4^*_{p,q} = \left(\sum_{(j,\widetilde{\mathbf{k}})\in(T-T)_1(\underline{t}_q - \underline{t}_p)} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0) + \sum_{(j,\widetilde{\mathbf{k}},l,\widetilde{\mathbf{h}})\in(T-T)_2(\underline{t}_p - \underline{t}_q)} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0) I_{l,\widetilde{\mathbf{h}}}(\boldsymbol{\phi}_0)\right).$$

By (41), (42), (43), (44), (45) and by the definition of $\mathbf{I}_\infty(\boldsymbol{\phi}_0)$ it follows that

$$\mathbb{D}\mathbf{f}(\phi_0) = -E\left(\eta_1\left(\frac{\varepsilon}{\sigma_\varepsilon}, \frac{\varepsilon'}{\sigma_\varepsilon}\right)\varepsilon'\right)\mathbf{I}_\infty(\boldsymbol{\phi}_0),$$

where ε and ε' are independent random variables with distribution F_ε.

Now, by the Lemmas 1 and Assumption 2-(iv), $\mathbb{D}\mathbf{f}(\phi_0)$, with the exception of a non zero constant, is equal to the covariance matrix of $(X_{\underline{m}})_{\underline{m}\in T}$ and, hence, it is invertible. This finishes the proof of the Proposition 2. ■

Proposition 3. *There exists a subset* Ω_0 *of* Ω *with* $P(\Omega_0) = 1$, *such that*

$$\Omega_0 \subset \left\{\omega\in\Omega : \lim_{M\to\infty}\sup_{\boldsymbol{\phi}\in C}|\mathbf{F}_M(\omega,\boldsymbol{\phi}) - \mathbf{f}(\boldsymbol{\phi})| = 0 \text{ and } \lim_{M\to\infty}\sup_{\boldsymbol{\phi}\in C}|\mathbb{D}\mathbf{F}_M(\omega,\boldsymbol{\phi}) - \mathbb{D}\mathbf{f}(\boldsymbol{\phi})| = 0\right\}.$$

PROOF: It is divided into two stages, a) and b).

a) There exists a subset Ω_{01} of Ω with $P(\Omega_{01}) = 1$, such that

$$\Omega_{01} \subset \left\{\omega\in\Omega : \lim_{M\to\infty}\sup_{\boldsymbol{\phi}\in C}|\mathbf{F}_M(\omega,\boldsymbol{\phi}) - \mathbf{f}(\boldsymbol{\phi})| = 0\right\}.$$

For all $1\le p\le L$, it is enough to apply Lemma 2 under the following setup

$$c_0(\boldsymbol{\phi}) = 1,\ c_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}) = I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}),$$

for all $j\ge 1$, $\widetilde{\mathbf{k}}\in\mathbb{L}^j$ and $\boldsymbol{\phi}\in\Lambda$.

$$W(\mathbf{x},\mathbf{y},\boldsymbol{\phi}) = \eta\left(\frac{x_1 - \sum_{p'=1}^{L}\phi_{p'}x_{p'+1}}{\sigma_\varepsilon}, \frac{y_1 - \sum_{p''=1}^{L}\phi_{p''}y_{p''+1}}{\sigma_\varepsilon}\right), \tag{46}$$

$$\rho(\mathbf{x},\mathbf{y}) = 1, \tag{47}$$

with $x=(x_1,\ldots,x_{L+1})'$, $y=(y_1,\ldots,y_{L+1})'$ and $\boldsymbol{\phi}=(\phi_1,\ldots,\phi_L)'$. Finally, set

$$\underline{n}_1=\underline{t}_p. \tag{48}$$

b) There exists a subset Ω_{02} of Ω with $P(\Omega_{02})=1$, such that

$$\Omega_{02}\subset\left\{\omega\in\Omega:\ \lim_{M\to\infty}\sup_{\boldsymbol{\phi}\in C}\left|\mathbb{D}\mathbf{F}_M(\omega,\boldsymbol{\phi})-\mathbb{D}\mathbf{f}(\boldsymbol{\phi})\right|=0\right\}.$$

For all $1\le p,q\le L$, the following notation will be used

$$\begin{aligned}
\left(a1_{\underline{n}}(\omega,\boldsymbol{\phi})\right)_{p,q} &= -\frac{1}{\sigma_\varepsilon}\eta_1\left(R_{\underline{n}}(\omega,\boldsymbol{\phi}),R_{\underline{n}-\underline{t}_p}(\omega,\boldsymbol{\phi})\right)X_{\underline{n}-\underline{t}_q}(\omega),\\
\left(a2_{\underline{n}}(\omega,\boldsymbol{\phi})\right)_{p,q} &= -\frac{1}{\sigma_\varepsilon}\eta_2\left(R_{\underline{n}}(\omega,\boldsymbol{\phi}),R_{\underline{n}-\underline{t}_p}(\omega,\boldsymbol{\phi})\right)X_{\underline{n}-\underline{t}_p-\underline{t}_q}(\omega),\\
\left(a3_{\underline{n},j,\widetilde{\mathbf{k}}}(\omega,\boldsymbol{\phi})\right)_{p,q} &= D_q\left(I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi})\right)\eta\left(R_{\underline{n}}(\omega,\boldsymbol{\phi}),R_{\underline{n}-\underline{t}_p-s(j,\widetilde{\mathbf{k}})}(\omega,\boldsymbol{\phi})\right),\\
\left(a4_{\underline{n},j,\widetilde{\mathbf{k}}}(\omega,\boldsymbol{\phi})\right)_{p,q} &= -\frac{1}{\sigma_\varepsilon}I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi})\eta_1\left(R_{\underline{n}}(\omega,\boldsymbol{\phi}),R_{\underline{n}-\underline{t}_p-s(j,\widetilde{\mathbf{k}})}(\omega,\boldsymbol{\phi})\right)X_{\underline{n}-\underline{t}_q}(\omega),\\
\left(a5_{\underline{n},j,\widetilde{\mathbf{k}}}(\omega,\boldsymbol{\phi})\right)_{p,q} &= -\frac{1}{\sigma_\varepsilon}I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi})\eta_2\left(R_{\underline{n}}(\omega,\boldsymbol{\phi}),R_{\underline{n}-\underline{t}_p-s(j,\widetilde{\mathbf{k}})}(\omega,\boldsymbol{\phi})\right)X_{\underline{n}-\underline{t}_p-s(j,\widetilde{\mathbf{k}})-\underline{t}_q}(\omega),
\end{aligned}$$

where $\omega\in\Omega$, $\underline{n}\in Z^2$, j is a positive integer $\widetilde{\mathbf{k}}\in\mathbb{L}^j$, $\boldsymbol{\phi}\in\Lambda$. Then

$$\begin{aligned}
D_q\left(\left(\mathbb{D}\mathbf{F}_M(\omega,\boldsymbol{\phi})\right)_p\right) &= \frac{1}{\#(W_M)}\sum_{\underline{m}\in W_M}\left(a1_{\underline{m}}(\omega,\boldsymbol{\phi})\right)_{p,q}+\frac{1}{\#(W_M)}\sum_{\underline{m}\in W_M}\left(a2_{\underline{m}}(\omega,\boldsymbol{\phi})\right)_{p,q}\\
&+\frac{1}{\#(W_M)}\sum_{\underline{m}\in W_M}\sum_{j=1}^{M-1}\sum_{\widetilde{\mathbf{k}}\in\mathbb{L}^j}\left(a3_{\underline{m},j,\widetilde{\mathbf{k}}}(\omega,\boldsymbol{\phi})\right)_{p,q}+\frac{1}{\#(W_M)}\sum_{\underline{m}\in W_M}\sum_{j=1}^{M-1}\sum_{\widetilde{\mathbf{k}}\in\mathbb{L}^j}\left(a4_{\underline{m},j,\widetilde{\mathbf{k}}}(\omega,\boldsymbol{\phi})\right)_{p,q}\\
&+\frac{1}{\#(W_M)}\sum_{\underline{m}\in W_M}\sum_{j=1}^{M-1}\sum_{\widetilde{\mathbf{k}}\in\mathbb{L}^j}\left(a5_{\underline{m},j,\widetilde{\mathbf{k}}}(\omega,\boldsymbol{\phi})\right)_{p,q}.
\end{aligned}$$

Thus, to prove b) it is enough to show the following five statements:

b1) There exists a subset Ω_G of Ω with $P(\Omega_G)=1$, such that for all $1\le p,q\le L$, Ω_G is contained in $\mathbb{G}$, where

$$\mathbb{G}=\left\{\omega\in\Omega:\ \lim_{M\to\infty}\sup_{\boldsymbol{\phi}\in C}\left|\frac{1}{\#(W_M)}\sum_{\underline{m}\in W_M}\left(a1_{\underline{m}}(\omega,\boldsymbol{\phi})\right)_{p,q}-A1_{p,q}(\boldsymbol{\phi})\right|=0\right\}.$$

b2) There exists a subset Ω_H of Ω with $P(\Omega_H)=1$, such that for each $\underline{s}$, $\underline{t}\in T$, Ω_H is contained in $\mathbb{H}$, where

$$\mathbb{H}=\left\{\omega\in\Omega:\ \lim_{M\to\infty}\sup_{\boldsymbol{\phi}\in C}\left|\frac{1}{\#(W_M)}\sum_{\underline{m}\in W_M}\left(a2_{\underline{m}}(\omega,\boldsymbol{\phi})\right)_{p,q}-A2_{p,q}(\boldsymbol{\phi})\right|=0\right\}.$$

b3) There exists a subset Ω_A of Ω with $P(\Omega_A)=1$, such that for each $\underline{s}$, $\underline{t}\in T$, Ω_A is included in $\mathbb{A}$, where

$$\mathbb{A}=\left\{\omega\in\Omega:\ \lim_{M\to\infty}\sup_{\boldsymbol{\phi}\in C}\left|\frac{1}{\#(W_M)}\sum_{\underline{m}\in W_M}\sum_{j=1}^{M-1}\sum_{\widetilde{\mathbf{k}}\in\mathbb{L}^j}\left(a3_{\underline{n},j,\widetilde{\mathbf{k}}}(\omega,\boldsymbol{\phi})\right)_{p,q}-\sum_{j=1}^{\infty}\sum_{\widetilde{\mathbf{k}}\in\mathbb{L}^j}A3_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi})\right|=0\right\}.$$

b4) There exists a subset Ω_B of Ω with $P(\Omega_B)=1$, such that for each $\underline{s}$, $\underline{t}\in T$, Ω_B is included in $\mathbb{B}$, where

$$\mathbb{B}=\left\{\omega\in\Omega:\ \lim_{M\to\infty}\sup_{\phi\in C}\left|\frac{1}{\#(W_M)}\sum_{\underline{m}\in W_M}\sum_{j=1}^{M-1}\sum_{\widetilde{\mathbf{k}}\in\mathbb{L}^j}\left(a4_{\underline{n},j,\widetilde{\mathbf{k}}}(\omega,\boldsymbol{\phi})\right)_{p,q}-\sum_{j=1}^{\infty}\sum_{\widetilde{\mathbf{k}}\in\mathbb{L}^j}A4_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi})\right|=0\right\}.$$

b5) There exists a subset Ω_D of Ω with $P(\Omega_D)=1$, such that for each $\underline{s}$, $\underline{t}\in T$, Ω_D is included in $\mathbb{D}$, where

$$\mathbb{D}=\left\{\omega\in\Omega:\ \lim_{M\to\infty}\sup_{\phi\in C}\left|\frac{1}{\#(W_M)}\sum_{\underline{m}\in W_M}\sum_{j=1}^{M-1}\sum_{\widetilde{\mathbf{k}}\in\mathbb{L}^j}\left(a5_{\underline{n},j,\widetilde{\mathbf{k}}}(\omega,\boldsymbol{\phi})\right)_{p,q}-\sum_{j=1}^{\infty}\sum_{\widetilde{\mathbf{k}}\in\mathbb{L}^j}A5_{p,q}(j,\widetilde{\mathbf{k}},\boldsymbol{\phi})\right|=0\right\}.$$

Proof of b1). If we suppose that (a) of Assumption 2-(ii) holds then we apply Lemma 2 under the following setup

$$c_0(\boldsymbol{\phi})=1,\ c_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi})=0,\forall j\geq 1,\ \widetilde{\mathbf{k}}\in\mathbb{L}^j,\ \boldsymbol{\phi}\in\Lambda, \tag{49}$$

$$W(\mathbf{x},\mathbf{y},\boldsymbol{\phi})=\eta_1\left(\frac{x_1-\sum_{p'=1}^{L}\phi_{p'}x_{p'+1}}{\sigma_\varepsilon},\frac{y_1-\sum_{p''=1}^{L}\phi_{p''}y_{p''+1}}{\sigma_\varepsilon}\right)x_{q+1}, \tag{50}$$

$$\rho(\mathbf{x},\mathbf{y})=x_{q+1}, \tag{51}$$

with $x=(x_1,\ldots,x_{L+1})'$, $y=(y_1,\ldots,y_{L+1})'$ and $\boldsymbol{\phi}=(\phi_1,\ldots,\phi_L)'$. Finally, $\underline{n}_1$ is considered as in (48). Suppose now that (b) of Assumption 2-(ii), then we replace (51) by

$$\rho(\mathbf{x},\mathbf{y})=\left|x_{q+1}\frac{y_1}{\sigma_\varepsilon}\right|+\frac{1}{L}\sum_{p'=1}^{L}\left|x_{q+1}\frac{y_{p'+1}}{\sigma_\varepsilon}\right|. \tag{52}$$

Proof of b2). Suppose that (a) of Assumption 2-(ii) holds. The proof consists of applying Lemma 2 under the following setup: c_0 and $c_{j,\widetilde{\mathbf{k}}}$ as in (49),

$$W(\mathbf{x},\mathbf{y},\boldsymbol{\phi})=\eta_2\left(\frac{x_1-\sum_{p'=1}^{L}\phi_{p'}x_{p'+1}}{\sigma_\varepsilon},\frac{y_1-\sum_{p''=1}^{L}\phi_{p''}y_{p''+1}}{\sigma_\varepsilon}\right)y_{q+1}, \tag{53}$$

$$\rho(\mathbf{x},\mathbf{y})=y_{q+1}, \tag{54}$$

with $x=(x_1,\ldots,x_{L+1})'$, $y=(y_1,\ldots,y_{L+1})'$ and $\boldsymbol{\phi}=(\phi_1,\ldots,\phi_L)'$. Finally, $\underline{n}_1$ is considered as in (48).

The proof is complete if we suppose that (b) of Assumption 2-(ii) holds, and change (54) by

$$\rho(\mathbf{x},\mathbf{y})=\left|y_{q+1}\frac{x_1}{\sigma_\varepsilon}\right|+\frac{1}{L}\sum_{p'=1}^{L}\left|y_{q+1}\frac{x_{p'+1}}{\sigma_\varepsilon}\right|. \tag{55}$$

Proof of b3). The proof consists in applying Lemma 2 under the following setup: $c_0(\boldsymbol{\phi})=0$, $c_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi})=D_q(I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}))$, $j\geq 1$, $\widetilde{\mathbf{k}}\in\mathbb{L}^j$ and $\phi\in\Lambda$, $W(x,y,\phi)$ as in (46), ρ as in (47) and $\underline{n}_1$ as in (48).

Proof of b4). Suppose that (a) of Assumption 2-(ii) holds. Then we apply Lemma 2 for

$$c_0(\boldsymbol{\phi})=0,\ c_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi})=-I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}), \tag{56}$$

for all $j\geq 1$, $\widetilde{\mathbf{k}}\in\mathbb{L}^j$ and $\boldsymbol{\phi}\in\Lambda$, $W(x,y,\boldsymbol{\phi})$ as in (50), ρ as in (51) and $\underline{n}_1$ as in (48). Suppose that (b) of Assumption 2-(ii) holds then consider ρ as in (52).

Proof of b5). Suppose (a) of Assumption 2-(ii) holds. Similarly, we apply Lemma 2 under the following conditions: c_0 and $c_{j,\widetilde{\mathbf{k}}}$ as in (56), $W(x,y,\boldsymbol{\phi})$ as in (53), ρ as in (54). Consider $\underline{n}_1$ as in (48). To complete the proof, under (b) of Assumption 2-(ii), consider ρ as in (55). This completes the proof of Proposition 3. ■

PROOF OF THEOREM 1

Let Ω_0 be as in Proposition 3. Let $\omega\in\Omega_0$. We consider the Zeros Lemma (Lemma 3) under the following conditions: $U=C^o$ (the interior of C), $\boldsymbol{\lambda}_0=\boldsymbol{\phi}_0$, for all M, $\mathbf{q}_M$ is the function $\boldsymbol{\phi}\mapsto\mathbf{F}_M(\omega,\boldsymbol{\phi})$, $\mathbf{q}$ is the function $\mathbf{f}$ defined in (17). Then, 1) in the Zeros Lemma follows from (18). 2) in the Zeros Lemma follows from 2) in Proposition 2. Finally, 3) in

Zeros Lemma follows from Proposition 3. Hence $\omega \in \Omega'''$. Setting $\Omega'' = \Omega_0$, the proof of the Theorem 1 is completed. ■

6.2 Proofs of the Asymptotic Normality of the RA Estimator

We will start introducing some additional notation.

Consider $1 \leq M, N \leq \infty$ integers, $\underline{n}, \underline{m} \in \mathbb{Z}^2$, $\omega \in \Omega$, $\boldsymbol{\phi} \in \Lambda$ and $1 \leq p \leq L$. Denoting

$$\begin{aligned} \left(\boldsymbol{\Upsilon}^{\eta}_{\underline{n},N}(\omega,\boldsymbol{\phi})\right)_p &= \sum_{j=N}^{\infty} \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}) \eta\left(R_{\underline{n}}(\omega,\boldsymbol{\phi}), R_{\underline{n}-\underline{t}_p - s(j,\widetilde{\mathbf{k}})}(\omega,\boldsymbol{\phi})\right), \\ \mathbf{Y}_{M,N}(\omega,\boldsymbol{\phi}) &= \frac{1}{\#(W_M)} \sum_{\underline{m} \in W_M} \boldsymbol{\Upsilon}^{\eta}_{\underline{m},N}(\omega,\boldsymbol{\phi}), \end{aligned}$$

one has

$$\sqrt{\#(W_M)}\mathbf{G}_{M,\infty}(\boldsymbol{\phi}) = \sqrt{\#(W_M)}\mathbf{G}_{M,N}(\boldsymbol{\phi}) + \sqrt{\#(W_M)}\mathbf{Y}_{M,N}(\boldsymbol{\phi}). \tag{57}$$

In particular,

$$\sqrt{\#(W_M)}\mathbf{G}_{M,\infty}\left(\widehat{\boldsymbol{\phi}}^{RA}_M\right) = \sqrt{\#(W_M)}\mathbf{F}_M\left(\widehat{\boldsymbol{\phi}}^{RA}_M\right) + \sqrt{\#(W_M)}\mathbf{Y}_{M,M}(\widehat{\boldsymbol{\phi}}^{RA}_M). \tag{58}$$

For each $N \geq 2$ integer, let $\mathbf{G}_N$ be the $L \times L$ matrix given by

$$\mathbf{G}_N = \mathbf{I}_N(\boldsymbol{\phi}_0) \cdot E\left(\eta\left(\frac{\varepsilon}{\sigma_\varepsilon}, \frac{\varepsilon'}{\sigma_\varepsilon}\right)^2\right),$$

where $\mathbf{I}_N(\boldsymbol{\phi}_0)$ is the $L \times L$ matrix given by

$$\mathbf{I}_N(\boldsymbol{\phi}_0)_{p,p} = 1 + \sum_{(j,\widetilde{\mathbf{k}},l,\widetilde{\mathbf{h}}) \in (T-T)^N_2(\underline{0})} I_{j,\widetilde{\mathbf{k}}}(\phi_0) I_{l,\widetilde{\mathbf{h}}}(\phi_0),$$

for all $1 \leq p \leq L$,

$$\begin{aligned} \mathbf{I}_N(\boldsymbol{\phi}_0)_{p,q} &= \sum_{(j,\widetilde{\mathbf{k}}) \in (T-T)^N_1(\underline{t}_p - \underline{t}_q)} I_{j,\widetilde{\mathbf{k}}}(\phi_0) + \sum_{(j,\widetilde{\mathbf{k}}) \in (T-T)^N_1(\underline{t}_q - \underline{t}_p)} I_{j,\widetilde{\mathbf{k}}}(\phi_0) \\ &\quad + \sum_{(j,\widetilde{\mathbf{k}},l,\widetilde{\mathbf{h}}) \in (T-T)^N_2(\underline{t}_p - \underline{t}_q)} I_{j,\widetilde{\mathbf{k}}}(\phi_0).I_{l,\widetilde{\mathbf{h}}}(\phi_0), \end{aligned}$$

for $p \neq q$ with $1 \leq p, q \leq L$, where

$$\begin{aligned} (T-T)^N_1(\underline{v}) &= \left\{(j,\widetilde{\mathbf{k}}) : N-1 \geq j \geq 1, \widetilde{\mathbf{k}} \in \mathbb{L}^j \text{ and } s(j,\widetilde{\mathbf{k}}) = \underline{v}\right\}, \\ (T-T)^N_2(\underline{v}) &= \left\{(j,\widetilde{\mathbf{k}},l,\widetilde{\mathbf{h}}) : N-1 \geq j,l \geq 1, \widetilde{\mathbf{k}} \in \mathbb{L}^j, \widetilde{\mathbf{h}} \in \mathbb{L}^l \text{ and } s(j,\widetilde{\mathbf{k}}) - s(l,\widetilde{\mathbf{h}}) = \underline{v}\right\}, \end{aligned}$$

with $\underline{v} \in (T-T)$ and $(T-T)$ as in (12).

Using the mean value theorem we have that

$$\sqrt{\#(W_M)}\mathbf{G}_{M,\infty}\left(\widehat{\boldsymbol{\phi}}^{RA}_M\right) = \sqrt{\#(W_M)}\mathbf{G}_{M,\infty}(\boldsymbol{\phi}_0) + \mathbb{D}\mathbf{G}_{M,\infty}\left(\widehat{\boldsymbol{\phi}}^{*}_M\right).\sqrt{\#(W_M)}\left(\widehat{\boldsymbol{\phi}}^{RA}_M - \boldsymbol{\phi}_0\right), \tag{59}$$

where, for each M, $\widehat{\boldsymbol{\phi}}^{*}_M$ is a random vector belonging to the segment in $\mathbb{R}^L$ that connects $\widehat{\boldsymbol{\phi}}^{RA}_M$ and $\boldsymbol{\phi}_0$.

The following lemmas and propositions are necessary to prove Theorem 2, which is proved at the end of this subsection.

Lemma 4. *Let $1 \leq M, N$ integers. For all $1 \leq p \leq L$ we have that*

$$\lim_{N\to\infty} \limsup_{M\to\infty} \left(\sqrt{\#(W_M)}\mathbf{Y}_{M,N}(\boldsymbol{\phi}_0)\right) = 0,$$

in probability.

PROOF: Let $M \geq 1$ be an integer and $\delta > 0$. By Chebyshev's inequality

$$\begin{aligned} P\left(\left|\left(\sqrt{\#(W_M)}\mathbf{Y}_{M,N}\left(\boldsymbol{\phi}_0\right)\right)_p\right| \geq \delta\right) &= P\left(\left|\left(\frac{1}{\sqrt{\#(W_M)}}\sum_{\underline{m}\in W_M}\boldsymbol{\Upsilon}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0)\right)_p\right| \geq \delta\right) \\ &\leq \frac{1}{\delta^2\#(W_M)}Var\left(\sum_{\underline{m}\in W_M}\left(\boldsymbol{\Upsilon}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0)\right)_p\right). \end{aligned}$$

Considering that,

$$\begin{aligned} \left(\boldsymbol{\Upsilon}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0)\right)_p &= \sum_{j=N}^{\infty}\sum_{\widetilde{\mathbf{k}}\in T^j} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0).\eta\left(R_{\underline{m}}(\boldsymbol{\phi}_0),R_{\underline{m}-\underline{t}_p-s(j,\widetilde{\mathbf{k}})}(\boldsymbol{\phi}_0)\right) \\ &= \sum_{j=N}^{\infty}\sum_{\widetilde{\mathbf{k}}\in \mathbb{L}^j} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0).\eta\left(\frac{\varepsilon_{\underline{m}}}{\sigma_\varepsilon},\frac{\varepsilon_{\underline{m}-\underline{t}_p-s(j,\widetilde{\mathbf{k}})}}{\sigma_\varepsilon}\right), \end{aligned}$$

we have that

$$E\left(\left(\boldsymbol{\Upsilon}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0)\right)_p\right) = 0.$$

and, therefore,

$$\begin{aligned} Var\left(\sum_{\underline{m}\in W_M}\left(\boldsymbol{\Upsilon}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0)\right)_p\right) &= E\left(\left(\sum_{\underline{m}\in W_M}\left(\boldsymbol{\Upsilon}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0)\right)_p\right)^2\right) \\ &= \sum_{\underline{m}\in W_M}\sum_{\underline{m'}\in W_M} E\left(\left(\boldsymbol{\Upsilon}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0)\right)_p\left(\boldsymbol{\Upsilon}^{\eta}_{\underline{m'},N}(\boldsymbol{\phi}_0)\right)_p\right). \end{aligned} \tag{60}$$

If $\underline{m} \neq \underline{m'}$, by the Assumption 3, it follows that

$$\begin{aligned} &E\left(\left(\boldsymbol{\Upsilon}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0)\right)_p\left(\boldsymbol{\Upsilon}^{\eta}_{\underline{m'},N}(\boldsymbol{\phi}_0)\right)_p\right) = \\ &= \sum_{j_1=N}^{\infty}\sum_{\widetilde{\mathbf{k}}_1\in\mathbb{L}^{j_1}}\sum_{j_2=N}^{\infty}\sum_{\widetilde{\mathbf{k}}_2\in\mathbb{L}^{j_2}} I_{j_1,\widetilde{\mathbf{k}}_1}(\boldsymbol{\phi}_0)I_{j_2,\widetilde{\mathbf{k}}_2}(\boldsymbol{\phi}_0)E\left(\eta\left(\frac{\varepsilon_{\underline{m}}}{\sigma_\varepsilon},\frac{\varepsilon_{\underline{m}-\underline{t}_p-s(j_1,\widetilde{\mathbf{k}}_1)}}{\sigma_\varepsilon}\right)\eta\left(\frac{\varepsilon_{\underline{m'}}}{\sigma_\varepsilon},\frac{\varepsilon_{\underline{m'}-\underline{t}_p-s(j_2,\widetilde{\mathbf{k}}_2)}}{\sigma_\varepsilon}\right)\right) \\ &= 0. \end{aligned}$$

Hence, for any $\underline{m}$

$$\begin{aligned} &E\left(\left(\left(\boldsymbol{\Upsilon}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0)\right)_p\right)^2\right) = \\ &= \sum_{j_1=N}^{\infty}\sum_{\widetilde{\mathbf{k}}_1\in\mathbb{L}^{j_1}}\sum_{j_2=N}^{\infty}\sum_{\widetilde{\mathbf{k}}_2\in\mathbb{L}^{j_2}} I_{j_1,\widetilde{\mathbf{k}}_1}(\boldsymbol{\phi}_0)I_{j_2,\widetilde{\mathbf{k}}_2}(\boldsymbol{\phi}_0)E\left(\eta\left(\frac{\varepsilon_{\underline{m}}}{\sigma_\varepsilon},\frac{\varepsilon_{\underline{m}-\underline{t}_p-s(j_1,\widetilde{\mathbf{k}}_1)}}{\sigma_\varepsilon}\right)\eta\left(\frac{\varepsilon_{\underline{m}}}{\sigma_\varepsilon},\frac{\varepsilon_{\underline{m}-\underline{t}_p-s(j_2,\widetilde{\mathbf{k}}_2)}}{\sigma_\varepsilon}\right)\right) \\ &= \sum_{(j_1,\widetilde{\mathbf{k}}_1,j_2,\widetilde{\mathbf{k}}_2)\in(T-T)_2^{N,\infty}(0)} I_{j_1,\widetilde{\mathbf{k}}_1}(\boldsymbol{\phi}_0)I_{j_2,\widetilde{\mathbf{k}}_2}(\boldsymbol{\phi}_0)E\left(\eta\left(\frac{\varepsilon_{\underline{m}}}{\sigma_\varepsilon},\frac{\varepsilon_{\underline{m}-\underline{t}_p-s(j_1,\widetilde{\mathbf{k}}_1)}}{\sigma_\varepsilon}\right)\eta\left(\frac{\varepsilon_{\underline{m}}}{\sigma_\varepsilon},\frac{\varepsilon_{\underline{m}-\underline{t}_p-s(j_2,\widetilde{\mathbf{k}}_2)}}{\sigma_\varepsilon}\right)\right), \end{aligned}$$

where

$$(T-T)_2^{N,\infty}(0) = \left\{(j_1,\widetilde{\mathbf{k}}_1,j_2,\widetilde{\mathbf{k}}_2) : j_1,j_2 \geq N, \widetilde{\mathbf{k}}_1 \in \mathbb{L}^{j_1}, \widetilde{\mathbf{k}}_2 \in \mathbb{L}^{j_2}, s(j_1,\widetilde{\mathbf{k}}_1) = s(j_2,\widetilde{\mathbf{k}}_2)\right\}.$$

So,

$$
\begin{aligned}
& Var\left(\sum_{\underline{m}\in W_M}\left(\boldsymbol{\Upsilon}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0)\right)_p\right) \\
= \;& \#(W_M)\sum_{(j_1,\widetilde{\mathbf{k}}_1,j_2,\widetilde{\mathbf{k}}_2)\in(T-T)_2^{N,\infty}(0)} I_{j_1,\widetilde{\mathbf{k}}_1}(\boldsymbol{\phi}_0)I_{j_2,\widetilde{\mathbf{k}}_2}(\boldsymbol{\phi}_0)E\left(\eta\left(\frac{\varepsilon}{\sigma_\varepsilon},\frac{\varepsilon'}{\sigma_\varepsilon}\right)^2\right),
\end{aligned}
$$

where ε and ε' are independent random variables with distribution F_ε.
Using this expression for the variance and taking lim sup in the Chebyshev inequality, we have that

$$
\begin{aligned}
& \limsup_{M\to\infty} P\left(\left|\left(\frac{1}{\sqrt{\#(W_M)}}\sum_{\underline{m}\in W_M}\boldsymbol{\Upsilon}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0)\right)_p\right|\geq\delta\right) \\
\leq \;& \frac{1}{\delta^2}\sum_{(j_1,\widetilde{\mathbf{k}}_1,j_2,\widetilde{\mathbf{k}}_2)\in(T-T)_2^{N,\infty}(0)} I_{j_1,\widetilde{\mathbf{k}}_1}(\boldsymbol{\phi}_0)I_{j_2,\widetilde{\mathbf{k}}_2}(\boldsymbol{\phi}_0)E\left(\eta\left(\frac{\varepsilon}{\sigma_\varepsilon},\frac{\varepsilon'}{\sigma_\varepsilon}\right)^2\right).
\end{aligned}
$$

Notice that $(T-T)_2^{N,\infty}(0)\subset(T-T)_2(0)$, $((T-T)_2^{N,\infty}(0))_N$ is a decreasing sequence of subsets of $(T-T)_2(0)$ such that $\bigcap_{N\geq1}(T-T)_2^{N,\infty}(0)=\emptyset$ and that $|\sum_{(j,\widetilde{\mathbf{k}},l,\widetilde{\mathbf{h}})\in(T-T)_2(\underline{0})} I_{j,\widetilde{\mathbf{k}}}(\phi_0).I_{l,\widetilde{\mathbf{h}}}(\phi_0)|<\infty$. Then, one has that $\sum_{(j_1,\widetilde{\mathbf{k}}_1,j_2,\widetilde{\mathbf{k}}_2)\in(T-T)_2^{N,\infty}(0)} I_{j_1,\widetilde{\mathbf{k}}_1}(\boldsymbol{\phi}_0)I_{j_2,\widetilde{\mathbf{k}}_2}(\boldsymbol{\phi}_0)\underset{N\to\infty}{\longrightarrow}0$. Thus,

$$
\lim_{N\to\infty}\limsup_{M\to\infty} P\left(\left|\left(\frac{1}{\sqrt{\#(W_M)}}\sum_{\underline{m}\in W_M}\boldsymbol{\Upsilon}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0)\right)_p\right|\geq\delta\right)=0.\blacksquare
$$

The following Central Limit Theorem for Random Fields is needed. The proof can be found in Guyon (1993), page 99.

Lemma 5 (Central Limit Theorem for Random Fields)**.** *Let $X=\{X_i\colon i\in\mathbb{Z}^d\}$ be a real valued random field such that $E(X_i)=0$, for all $i\in\mathbb{Z}^d$. For each $\mathbb{S}\subset\mathbb{Z}^d$, let $F(X,\mathbb{S})$ be the σ-algebra generated by $\{X_i^{-1}(C)\colon C$ is a Borel set $, i\in\mathbb{S}\}$; for each $k,\ l\in\mathbb{N}\cup\{\infty\}$ and $n\in\mathbb{N}$ let*

$$
\alpha_{k,l}(n)=\sup\{|P(A\cap B)-P(A)P(B)|:A,B\in\mathcal{C}_{k,l}(n)\}
$$

where

$$
\mathcal{C}_{k,l}(n)=\{A\in F(X,\mathbb{S}_1),B\in F(X,\mathbb{S}_2)\colon\#(\mathbb{S}_1)\leq k,\#(\mathbb{S}_2)\leq l,dist\,(\mathbb{S}_1,\mathbb{S}_2)\geq n\}.
$$

Let $(D_n)_{n\geq1}$ be a decreasing sequence of finite subsets of $\mathbb{Z}^d$ and $S_n=\sum_{i\in D_n}X_i$ with variance σ_n^2. If the following two conditions hold

(i) $\sum_{m\geq1}m^{d-1}\alpha_{k,l}(m)<\infty$ *if* $k+l\leq4$ *and* $\alpha_{1,\infty}(m)=o(m^{-d})$;

(ii) there exists $\delta>0$ *such that* $\sup_i\|X_i\|_{\delta+2}<\infty$ *and* $\sum_{m\geq1}m^{d-1}\left(\alpha_{1,1}(m)\right)^{\frac{\delta}{\delta+2}}<\infty$, *then*

$$
\limsup_n\frac{1}{\#(D_n)}\sum_{i,j\in D_n}|Cov(X_i,X_j)|<\infty.
$$

If, further, we assume that

(iii)

$$
\liminf_{n\to\infty}\frac{1}{\#(D_n)}\sigma_n^2>0,
$$

then

$$
\sigma_n^{-1}S_n\overset{D}{\longrightarrow}N(0,1).
$$

If, for each $i \in \mathbb{Z}^d$, X_i takes values in $\mathbb{R}^p$ and if (iii) is replaced by

$$\liminf_{n\to\infty} \frac{1}{\#(D_n)} \Sigma_n \geq I_0 > 0,$$

with I_0 a symmetric and positive definite matrix, then

$$\Sigma_n^{-1/2} S_n \xrightarrow{D} N(\mathbf{0}, \mathbf{I}_p),$$

where Σ_n is the covariance matrix of S_n and $\mathbf{I}_p$ is the identity matrix of order p.

Lemma 6. *There exists N_0 such that, for each $N \geq N_0$, $\sqrt{\#(W_M)} \mathbf{G}_{M,N}(\boldsymbol{\phi}_0) \xrightarrow{D} N(0, \mathbf{G}_N)$, as $M \to \infty$.*

PROOF: Let $N \geq 2$. For each $M \geq 1$, let us define

$$\mathbf{S}_{M,N} = \#(W_M) \mathbf{G}_{M,N}(\boldsymbol{\phi}_0) = \sum_{\underline{m} \in W_M} \boldsymbol{\Gamma}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0). \tag{61}$$

For each $\underline{m} \in \mathbb{Z}^2$, let

$$\boldsymbol{\Gamma}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0) = \left(\left(\boldsymbol{\Gamma}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0) \right)_1, \ldots, \left(\boldsymbol{\Gamma}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0) \right)_L \right)',$$

with

$$\begin{aligned} \left(\boldsymbol{\Gamma}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0) \right)_p &= \eta \left(R_{\underline{m}}(\boldsymbol{\phi}_0), R_{\underline{m}-\underline{t}_p}(\boldsymbol{\phi}_0) \right) + \sum_{j=1}^{N-1} \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0).\eta \left(R_{\underline{m}}(\boldsymbol{\phi}_0), R_{\underline{m}-\underline{t}_p - s(j,\widetilde{\mathbf{k}})}(\boldsymbol{\phi}_0) \right) \\ &= \eta \left(\frac{\varepsilon_{\underline{m}}}{\sigma_\varepsilon}, \frac{\varepsilon_{\underline{m}-\underline{t}_p}}{\sigma_\varepsilon} \right) + \sum_{j=1}^{N-1} \sum_{\widetilde{\mathbf{k}} \in \mathbb{L}^j} I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi}_0).\eta \left(\frac{\varepsilon_{\underline{m}}}{\sigma_\varepsilon}, \frac{\varepsilon_{\underline{m}-\underline{t}_p - s(j,\widetilde{\mathbf{k}})}}{\sigma_\varepsilon} \right). \end{aligned}$$

We will apply Lemma 5 to the $\mathbb{R}^L$-valued random field $\left(\boldsymbol{\Gamma}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0) \right)_{\underline{m} \in \mathbb{Z}^2}$. Since $(\varepsilon(\underline{m}))_{\underline{m} \in \mathbb{Z}^2}$ is a white noise, it is straightforward to verify that $\left(\boldsymbol{\Gamma}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0) \right)_{\underline{m} \in \mathbb{Z}^2}$ satisfies the conditions (i) and (ii) of this lemma.

Now, let us compute the covariance matrix of $\mathbf{S}_{M,N}$, $\Sigma(\mathbf{S}_{M,N})$. Since $E(\mathbf{S}_{M,N}) = \mathbf{0}$, then $\Sigma(\mathbf{S}_{M,N})$ is the $L \times L$ matrix given by

$$\left[\sum_{\underline{m} \in W_M} \sum_{\underline{m}' \in W_M} E \left(\left(\boldsymbol{\Gamma}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0) \right)_p \left(\boldsymbol{\Gamma}^{\eta}_{\underline{m}',N}(\boldsymbol{\phi}_0) \right)_q \right) \right]_{1 \leq p,q \leq L}.$$

By the Assumption 3, if $\underline{m} \neq \underline{m}'$, then $E \left(\left(\boldsymbol{\Gamma}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0) \right)_p \left(\boldsymbol{\Gamma}^{\eta}_{\underline{m}',N}(\boldsymbol{\phi}_0) \right)_q \right) = 0$, for any p and q such that $1 \leq p, q \leq L$. Hence

$$\Sigma(\mathbf{S}_{M,N}) = \left[\sum_{\underline{m} \in W_M} E \left(\left(\boldsymbol{\Gamma}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0) \right)_p \left(\boldsymbol{\Gamma}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0) \right)_q \right) \right]_{1 \leq p,q \leq L}. \tag{62}$$

Using Assumption 3, a simple calculation shows that

$$\left[E \left(\left(\boldsymbol{\Gamma}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0) \right)_p \left(\boldsymbol{\Gamma}^{\eta}_{\underline{m},N}(\boldsymbol{\phi}_0) \right)_q \right) \right]_{1 \leq p,q \leq L} = \mathbf{G}_N,$$

for all $\underline{m} \in W_M$. Therefore, by (62)

$$\frac{1}{\#(W_M)} \Sigma(\mathbf{S}_{M,N}) = \mathbf{G}_N. \tag{63}$$

Now, $\mathbf{I}_N(\boldsymbol{\phi}_0) \to_{N\to\infty} \mathbf{I}_\infty(\boldsymbol{\phi}_0)$ and $\mathbf{I}_\infty(\boldsymbol{\phi}_0)$ is positive definite. Then, there exists N_0 such that for each $N \geq N_0$ $\mathbf{I}_N(\boldsymbol{\phi}_0)$ is positive definite. Hence, by (63), for each $N \geq N_0$ (iii) of Lemma 5 is satisfied with $D_M = W_M$; hence

$$\left(\Sigma(\mathbf{S}_{M,N}) \right)^{-1/2} \mathbf{S}_{M,N} \xrightarrow[M\to\infty]{D} N(0, \mathbf{I}_L), \tag{64}$$

where $\mathbf{I}_L$ is the $L \times L$ identity matrix. Therefore by (63) and (61),

$$\left(\Sigma\left(\mathbf{S}_{M,N}\right)\right)^{-1/2}\mathbf{S}_{M,N} = \frac{1}{\sqrt{\#(W_M)}}\left(\mathbf{G}_N\right)^{-1/2}\#(W_M)\mathbf{G}_{M,N}\left(\boldsymbol{\phi}_0\right) = \left(\mathbf{G}_N\right)^{-1/2}\sqrt{\#(W_M)}\mathbf{G}_{M,N}\left(\boldsymbol{\phi}_0\right),$$

for all M and $N \geq N_0$. Consequently, by (64) the proof is completed. ■

Lemma 7.

$$\mathbb{D}\mathbf{G}_{M,\infty}\left(\widehat{\boldsymbol{\phi}}_M^*\right) \xrightarrow{P} \mathbb{D}\mathbf{f}(\boldsymbol{\phi}_0),\ \ \textit{as}\ \ M \to \infty,$$

where $\widehat{\boldsymbol{\phi}}_M^*$ *is as in (59).*

PROOF: Let δ and δ' be any positive numbers. From the proof of Proposition 3, we see that

$$\sup_{\boldsymbol{\phi}\in C}\left|\mathbb{D}\mathbf{G}_{M,\infty}\left(\boldsymbol{\phi}\right) - \mathbb{D}\mathbf{f}(\boldsymbol{\phi})\right| \xrightarrow{P} \mathbf{0},\ \text{as}\ M \to \infty.$$

Then, there exists M_1 such that

$$M \geq M_1 \Longrightarrow P\left(\Omega_M(\delta/2)\right) \geq 1 - \delta'/2,$$

where

$$\Omega_M(\delta/2) = \left\{\omega \in \Omega:\ \sup_{\boldsymbol{\phi}\in C}\left|\mathbb{D}\mathbf{G}_{M,\infty}\left(\omega,\boldsymbol{\phi}\right) - \mathbb{D}\mathbf{f}(\boldsymbol{\phi})\right| < \delta/2\right\}.$$

Hence, for each $\omega \in \Omega_M(\delta/2)$ and $M \geq M_1$,

$$\left|\mathbb{D}\mathbf{G}_{M,\infty}\left(\omega, \widehat{\boldsymbol{\phi}}_M^*\left(\omega\right)\right) - \mathbb{D}\mathbf{f}(\widehat{\boldsymbol{\phi}}_M^*\left(\omega\right))\right| < \delta/2. \tag{65}$$

Because of the continuity of the function $\boldsymbol{\phi} \mapsto \mathbb{D}\mathbf{f}(\boldsymbol{\phi})$, there exists $\delta'' > 0$, such that

$$\left|\boldsymbol{\phi} - \boldsymbol{\phi}_0\right| < \delta'' \Longrightarrow \left|\mathbb{D}\mathbf{f}(\boldsymbol{\phi}) - \mathbb{D}\mathbf{f}(\boldsymbol{\phi}_0)\right| < \delta/2. \tag{66}$$

By (16) and (6.2), it follows that there exists M_2 such that

$$M \geq M_2 \Longrightarrow P\left(\Omega'_M(\delta'')\right) \geq 1 - \delta'/2,$$

with

$$\Omega'_M(\delta'') = \left\{\omega \in \Omega:\ \left|\widehat{\boldsymbol{\phi}}_M^*\left(\omega\right) - \boldsymbol{\phi}_0\right| < \delta''\right\}.$$

Consequently

$$\Omega'_M(\delta'') \subseteq \left\{\omega \in \Omega:\ \left|\mathbb{D}\mathbf{f}\left(\widehat{\boldsymbol{\phi}}_M^*\left(\omega\right)\right) - \mathbb{D}\mathbf{f}(\boldsymbol{\phi}_\mathbf{0})\right| < \delta/2\right\}. \tag{67}$$

Then, for $M \geq \max(M_1, M_2)$, using (65) and (67) we have that

$$\left|\mathbb{D}\mathbf{G}_{M,\infty}\left(\omega, \widehat{\boldsymbol{\phi}}_M^*\left(\omega\right)\right) - \mathbb{D}\mathbf{f}(\boldsymbol{\phi}_\mathbf{0})\right| < \delta,\ \text{if}\ \omega \in \Omega_M(\delta/2) \cap \Omega'_M(\delta'').$$

Thus

$$P\left(\left\{\omega \in \Omega:\ \left|\mathbb{D}\mathbf{G}_{M,\infty}\left(\omega, \widehat{\boldsymbol{\phi}}_M^*\left(\omega\right)\right) - \mathbb{D}\mathbf{f}(\boldsymbol{\phi}_\mathbf{0})\right| < \delta\right\}\right) \geq 1 - \delta'.■$$

Proposition 4.

$$\sqrt{\#(W_M)}\mathbf{G}_{M,\infty}\left(\widehat{\boldsymbol{\phi}}_M^{RA}\right) \xrightarrow{P} \mathbf{0},\ \ \textit{as}\ \ M \to \infty.$$

PROOF: Since (16) and (58) it is enough to show that for each $1 \leq p \leq L$,

$$\left(\sqrt{\#(W_M)}\mathbf{Y}_{M,M}(\widehat{\boldsymbol{\phi}}_M^{RA})\right)_p \xrightarrow{P} 0, \text{as}\ M \to \infty.$$

Now, for each $\omega \in \Omega$ and $\boldsymbol{\phi} \in \Lambda$

$$
\begin{aligned}
&\left|\left(\sqrt{\#(W_M)}\mathbf{Y}_{M,M}(\omega,\boldsymbol{\phi})\right)_p\right| \\
&\leq \frac{1}{\sqrt{\#(W_M)}}\sum_{j=M}^{\infty}\sum_{\widetilde{\mathbf{k}}\in\mathbb{L}^j}\left|I_{j,\widetilde{\mathbf{k}}}(\boldsymbol{\phi})\right|\sum_{\underline{m}\in W_M}\left|\eta\left(R_{\underline{m}}(\omega,\boldsymbol{\phi}),R_{\underline{m}-\underline{t}_p-s(j,\widetilde{\mathbf{k}})}(\omega,\boldsymbol{\phi})\right)\right| \\
&\leq \frac{1}{\sqrt{\#(W_M)}}\sum_{j=M}^{\infty}b^j\#(W_M)K_\eta \\
&= K_\eta b\sqrt{\#(W_M)}\sum_{j=M}^{\infty}b^{j-1} = K_\eta b\frac{\sqrt{\#(W_M)}}{M}\sum_{j=M}^{\infty}Mb^{j-1} \\
&\leq K_\eta b\frac{\sqrt{\#(W_M)}}{M}\sum_{j=M}^{\infty}jb^{j-1} \to 0, \text{ as } M\to\infty,
\end{aligned}
$$

since $\frac{\sqrt{\#(W_M)}}{M} = \frac{\sqrt{(M+1)^2-1}}{M} = \sqrt{1+\frac{2}{M}}$, and $\sum_{j=1}^{\infty} jb^{j-1}$ is the derivative of $b \mapsto \frac{1}{1-b}$ with $0<b<1$.■

Lemma 8 (Theorem 3.2 in Billingsley (1999), page 27)**.** *Suppose that for each $N, M \in \mathbb{N}$, $(X_{N,M}, X_M)$ are random elements of $S\times S$ where (S,ρ) is a metric space. If for all N*

$$
\begin{aligned}
&X_{N,M}\xrightarrow{D}Z_N, \text{ as } M\to\infty \\
&Z_N\xrightarrow{D}X, \text{ as } N\to\infty \\
&\text{for all } \varepsilon>0\colon \lim_{N\to\infty}\limsup_{M\to\infty}P\left[\rho\left(X_{N,M},X_M\right)\geq\varepsilon\right]=0,
\end{aligned}
$$

then $X_M\xrightarrow{D}X$ as $M\to\infty$.

Proposition 5. *Let $\mathbf{G}$ be the $L\times L$ matrix given by*

$$
\mathbf{G} = \mathbf{I}_\infty(\boldsymbol{\phi}_0)\cdot E\left(\left(\eta\left(\frac{\varepsilon}{\sigma_\varepsilon},\frac{\varepsilon'}{\sigma_\varepsilon}\right)\right)^2\right), \tag{68}
$$

where ε and ε' are independent random variables with distribution F_ε. Then $\sqrt{\#(W_M)}\mathbf{G}_{M,\infty}(\boldsymbol{\phi}_0)\xrightarrow{D}N(\mathbf{0},\mathbf{G})$, as $M\to\infty$.

PROOF: Let N_0 be as in Lemma 6 and for each $N\geq N_0$, let

$$
\begin{aligned}
\mathbf{X}_{N,M} &= \sqrt{\#(W_M)}\mathbf{G}_{M,N}(\boldsymbol{\phi}_0), \\
\mathbf{X}_M &= \sqrt{\#(W_M)}\mathbf{G}_{M,\infty}(\boldsymbol{\phi}_0).
\end{aligned}
$$

For each $N\geq N_0$, let Z_N be a random vector variable with distribution $N(0,\mathbf{G}_N)$ and X a random vector variable with distribution $N(0,\mathbf{G})$. By Lemma 6, for each $N\geq N_0$,

$$
\mathbf{X}_{N,M}\xrightarrow{D}\mathbf{Z}_N, \text{ as } M\to\infty.
$$

Since $\mathbf{G}_N\longrightarrow\mathbf{G}$, as $N\to\infty$, then

$$
\mathbf{Z}_N\xrightarrow{D}\mathbf{X}, \text{ as } N\to\infty.
$$

By (57), $\mathbf{X}_M-\mathbf{X}_{N,M}=\sqrt{\#(W_M)}\mathbf{Y}_{M,N}(\boldsymbol{\phi})$ and, by Lemma 4,

$$
\lim_{N\to\infty}\limsup_{M\to\infty}\left(\sqrt{\#(W_M)}\mathbf{Y}_{M,N}(\boldsymbol{\phi}_0)\right)=0.
$$

So, the assumptions of Lemma 8 are satisfied; hence $X_M\xrightarrow{D}X$ as $M\to\infty$, that is,

$$
\sqrt{\#(W_M)}\mathbf{G}_{M,\infty}(\boldsymbol{\phi}_0)\xrightarrow{D}N(\mathbf{0},\mathbf{G}), \text{ as } M\to\infty.\blacksquare
$$

PROOF OF THEOREM 2: By Lemma 7, (59), Proposition 4, Proposition 5, and Slutzky's Theorem we have that

$$\sqrt{\#(W_M)}\left(\widehat{\boldsymbol{\phi}}_M^{RA} - \boldsymbol{\phi}_0\right) \xrightarrow{D} N\left(0, \mathbb{D}\mathbf{f}(\boldsymbol{\phi}_0)^{-1}\mathbf{G}\mathbb{D}\mathbf{f}(\boldsymbol{\phi}_0)^{-1}\right), \text{as } M \to \infty. \quad (69)$$

Now, notice (see (68)) that $\mathbb{D}\mathbf{f}(\boldsymbol{\phi}_0) = -\frac{1}{\sigma_\varepsilon}E\left(\eta_1\left(\frac{\varepsilon}{\sigma_\varepsilon}, \frac{\varepsilon'}{\sigma_\varepsilon}\right).\varepsilon'\right).\mathbf{I}_\infty(\boldsymbol{\phi}_0)$, where ε and ε' are independent random variables with distribution F_ε. From the results proved in the previous Section and (69) the proof is completed. ■

7 CONCLUDING REMARKS

The following comments give a brief summary of the results that we obtained in this paper.

- ✓ Under mild regularity conditions, we established the asymptotic normality and consistency of a class of robust estimators (the RA estimators) for the parameter $\boldsymbol{\phi}$ of a two dimensional autoregressive unilateral process.
- ✓ The results we proved extend the asymptotic theory of the RA estimators, available only for one-dimensional time series (see Bustos et al. (1984)).
- ✓ Although in the literature several reasonable classes of estimators for the parameter $\boldsymbol{\phi}$ have been proposed, such as M and GM estimators, their asymptotic behavior are still open problems. Moreover, the advantage of the RA estimator over the other classes is that they are less sensitive to the presence of additive outliers (see Ojeda et al. (2002)).

The following proposals outline some directions for future work:

- ✓ The extension of the results proved in this paper to the model with colored noise (that is, with non null autocorrelation), instead of white noise $\widetilde{\boldsymbol{\varepsilon}} = \left(\varepsilon_{\underline{m}}\right)_{\underline{m}\in\mathbb{Z}^2}$, would be of importance in signal and image processing.
- ✓ In the case of causal AR-2D processes of order two or three, algorithms for the computation of the RA estimators have been proposed (see Ojeda (1999), Ojeda et al. (2002) and Vallejos et al. (2006)). It would be important to develop efficient algorithms when the order of the process is greater than three.
- ✓ To analyze the behaviour of the RA estimator in combination with image restoration techniques.
- ✓ To study properties of RA estimator, in particular, and robust estimators, in general, as alternatives to the least squares estimators under not causal and semi causal AR-2D.

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