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Hopf algebras of dimension 16

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REPÚBLICA ARGENTINA

HOPF ALGEBRAS OF DIMENSION 16

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Abstract. We complete the classification of Hopf algebras of dimension 16 over an algebraically closed field of characteristic zero. We show that a non-semisimple Hopf algebra of dimension 16, has either the Chevalley property or its dual is pointed.

1. Introduction

Let k be an algebraically closed field of characteristic 0. In 1975, I. Kaplansky posed the question of classifying all Hopf algebras over k of a fixed dimension. Since the only semisimple and pointed Hopf algebras are the group algebras, we shall adopt the convention that 'pointed' means 'pointed non-semisimple'. Many results have been found, dealing mainly with the semisimple or pointed cases.

Concerning Kaplansky´s question, there are very few general results. The Kac-Zhu Theorem [Z], states that a Hopf algebra of prime dimension is isomorphic to a group algebra. S-H. Ng [Ng] proved that in dimension p^2, the only Hopf algebras are the group algebras and the Taft algebras, using previous results in [AS1], [Ma3]. It is a common belief that a Hopf algebra of dimension pq, where p and q are distinct prime numbers, is semisimple. Hence, it should be isomorphic to a group algebra or a dual group algebra by [GW], [EG], [Ma2]. This conjecture was verified for some particular values of p and q, see [AN, BD, EG2, Ng2, Ng3]. In particular, it is known that Hopf algebras of dimension 14 and 15 are group algebras or dual group algebras [BD], [AN].

In fact, all Hopf algebras of dimension ≤ 15 are classified: for dimension ≤ 11 the problem was solved by [W]; an alternative proof appears in [Ş]. The classification for dimension 12 was done by [F] in the semisimple case and then completed by [N] in the general case.

It turns out that any Hopf algebra of dimension ≤ 15 is either semisimple or pointed or its dual is pointed. On the other hand, there exist Hopf algebras of dimension 16 that are non-semisimple, non-pointed and their duals are also non-pointed. Nevertheless, these Hopf algebras satisfies a certain property which we call the *Chevalley property*.

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Recall that a tensor category $\mathcal{C}$ over k has the Chevalley property if the tensor product of any two simple objects is semisimple. We shall say that a Hopf algebra H has the *Chevalley property* if the category $\mathrm{Comod}(H)$ of H-comodules does.

Remarks 1.1. (i) The notion of the Chevalley property in the setting of Hopf algebras was introduced by [AEG]: it is said in *loc. cit.* that a Hopf algebra has the Chevalley property if the category $\mathrm{Rep}(H)$ of H-modules does.

(ii) Unlike [AEG], in [CDMM, Section 1], the authors refer the Chevalley property to the category of H-comodules; this definition is the one we adopt. Note that it is equivalent to say that the coradical of H is a Hopf subalgebra.

(iii) If H is semisimple or pointed then it has the Chevalley property.

Here is the main result of the present paper.

Theorem 1.2. *Let H be a Hopf algebra of dimension 16. If H does not have the Chevalley property then H^* is pointed.*

As a consequence of Theorem 1.2, we obtain the classification of Hopf algebras of dimension 16.

Theorem 1.3. *Let H be a Hopf algebra of dimension 16. Then H is isomorphic to one and only one of the Hopf algebras in the following list.*

(i) The group algebras of groups of order 16 *and their duals.*
(ii) The semisimple Hopf algebras listed in [K, Thm. 1.2].
(iii) The pointed Hopf algebras listed in [CDR, Section 2.5].
(iv) The duals of the pointed Hopf algebras listed in [B, Sec. 4.2, Table 2].
(v) The two non-semisimple non-pointed self-dual Hopf algebras with the Chevalley property listed in [CDMM, Thm. 5.1].

Proof. Let H be a Hopf algebra of dimension 16. If H is semisimple, then H is either a group algebra, or a dual of a group algebra or is one of the list given in [K, Thm. 1.2]. Suppose now that H is non-semisimple. If it is pointed, then H is one of the Hopf algebras given in [CDR, Section 2.5]. If H is non-pointed and has the Chevalley property, then it must be one of the two Hopf algebras given by [CDMM, Thm. 5.1]. Then, the result follows from Theorem 1.2. □

The paper is organized as follows. In Section 2 we recall the definition and some known facts about Hopf algebra extensions. We give a detailed description of the cleft extensions of the Sweedler algebra T_{-1} in Subsection 2.2, a result from [Ma]– we follow the exposition in [DT]. As a consequence we show in Lemma 2.8, that a Hopf algebra which is an extension of T_{-1} by T_{-1} is isomorphic to the tensor product $T_{-1} \otimes T_{-1}$. In Section 3 we recall the classification of non-semisimple Hopf algebras of dimension 8 given by [Ş], since it is used several times in the proof of our main theorem. In Section 4 we discuss some consequences of results of [N] and [Ş] concerning Hopf algebras H generated by a simple subcoalgebra of dimension 4 stable by the antipode. In particular, we show in Theorem 4.5 that under certain assumptions H^* must be pointed. Finally we prove our main theorem in Section 5. We first describe all possible coradicals of a Hopf algebra of dimension 16 which does not have the Chevalley property. It turns out that there are 6 possible coradicals. This leads us to do the proof case by case according to the type of the coradical. The most difficult cases are those where the coradical has two simple subcoalgebras C and D of dimension 4, since one does not know whether they are stable by the antipode. The problem is solved by looking at the subalgebra generated by C, which is indeed a Hopf subalgebra, in the case that both C and D are stable by the antipode. In the other case, one assumes that H^* is non-pointed and then one gets a contradiction by looking at the Hopf subalgebras of dimension 8 contained in it.

If H is a Hopf algebra over k then Δ, ε, S denote respectively the comultiplication, the counit and the antipode; $G(H)$ denotes the group of group-like elements of H; $(H_n)_{n\in\mathbb{N}}$ denotes the coradical filtration of H; L_h (resp. R_h) is the left (resp. right) multiplication in H by h. The left and right adjoint action $\mathrm{ad}_\ell, \mathrm{ad}_r : H \to \mathrm{End}(H)$, of H on itself are given, in Sweedler notation, by:

$$\mathrm{ad}_\ell(h)(x) = \sum h_1 x S(h_2), \qquad \mathrm{ad}_r(h)(x) = \sum S(h_1) x h_2,$$

for all $h, x \in H$. The set of (g,h)-*primitives* (with $h, g \in G(H)$) and *skew-primitives* are:

$$\begin{aligned} \mathcal{P}_{g,h}(H) &:= \{x \in H \mid \Delta(x) = x \otimes h + g \otimes x\}, \\ \mathcal{P}(H) &:= \textstyle\sum_{h,g\in G(H)} \mathcal{P}_{h,g}(H). \end{aligned}$$

We say that $x \in k \cdot (h-g)$ is a *trivial* skew-primitive; otherwise, it is *non-trivial.*

Let K be a coalgebra with a distinguished group-like 1. If M is a right K-comodule via δ, then the space of *right coinvariants* is

$$M^{\mathrm{co}\,\delta} = \{x \in M \mid \delta(x) = x \otimes 1\}.$$

In particular, if $\pi : H \to K$ is a morphism of Hopf algebras, then H is a right K-comodule via $(1 \otimes \pi)\Delta$ and in this case $H^{\mathrm{co}\,\pi} := H^{\mathrm{co}(1\otimes\pi)\Delta}$.

Let $\mathcal{M}^*(n,k)$ denote the simple coalgebra of dimension n^2, dual to the matrix algebra $\mathcal{M}(n,k)$. A basis $\{e_{ij} \mid 1 \le i,j \le n\}$ of $\mathcal{M}^*(n,k)$ such that $\Delta(e_{ij}) = \sum_{l=1}^n e_{il} \otimes e_{lj}$ and $\varepsilon(e_{ij}) = \delta_{ij}$ is called a *comatrix basis.*

2. Extensions

Our references for the theory of extensions of algebras and Hopf algebras are [AD] and [M].

2.1. **Extensions of Hopf algebras.**

Definition 2.1. Let $A \subset C$ be an extension of k-algebras and B be a Hopf algebra. $A \subset C$ is a B-*cleft extension* if C is a (right) B-comodule algebra via δ with $C^{\mathrm{co}\,\delta} = A$ and there is $\gamma : B \to C$ a morphism of B-comodules which is convolution invertible.

It is known that any cleft extension arises as a *crossed product* $A\#_{\rightharpoonup,\sigma}B$, and conversely any crossed product is a cleft extension [M, Thm. 7.2.2]. Here $\rightharpoonup : B \otimes A \to A$ is a *weak action* and $\sigma : B \otimes B \to A$ is a 2-*cocycle* satisfying certain compatibility conditions, so that $A \otimes B$ becomes an associative algebra with a new product and unit $1 \otimes 1$. The multiplication is given by:

$$(a\#b)(a'\#b') = a(b_1 \rightharpoonup a')\sigma(b_2, b'_1)\#b_3b'_2, \tag{1}$$

for all $a, a' \in A$ and $b, b' \in B$. See [AD, Section 2] or [M, Section 7] for details.

Definition 2.2. [AD]. Let $A \overset{\imath}{\hookrightarrow} C \overset{\pi}{\twoheadrightarrow} B$ be a sequence of Hopf algebras morphisms. We shall say that it is *exact* and C is an *extension of A by B* if:

(i) $\imath$ is injective (then we identify A with its image);
(ii) π is surjective;
(iii) $\pi\imath = \varepsilon$;
(iv) $\ker \pi = A^+C$ (A^+ is the kernel of the counit);
(v) $A = C^{\mathrm{co}\,\pi}$.

The following statement condenses some known results and is useful to find exact sequences.

Lemma 2.3. *Let C be a finite-dimensional Hopf algebra. If $\pi : C \to B$ is an epimorphism of Hopf algebras then* $\dim C = \dim C^{\mathrm{co}\,\pi} \dim B$. *Moreover, if $A = C^{\mathrm{co}\,\pi}$ is a Hopf subalgebra then the sequence $A \overset{\imath}{\hookrightarrow} C \overset{\pi}{\twoheadrightarrow} B$ is exact.*

Proof. The equality of dimension follows from [S, Thm. 2.4. (1.b)]. Moreover, if $A = C^{\mathrm{co}\,\pi}$ then $\pi_{|A} = \varepsilon_{|A}$ and therefore $A^+C \subseteq \ker \pi$. It follows from [S, Thm. 2.4. (2.a)] that $\dim B = \dim(C/A^+C)$. Therefore $A^+C = \ker \pi$, and the lemma follows. □

Exact sequences of finite-dimensional Hopf algebras are cleft by [S, Thm. 2.2] so by the results in [AD, Subsection 3.2] we have the following. Recall the definition of Hopf datum [AD, Def. 2.26] and the corresponding Hopf algebra $A^{\rho,\tau}\#_{\rightharpoonup,\sigma}B$ associated to it.

Theorem 2.4. *Let A and B be finite-dimensional Hopf algebras.*

(i) Let $A \overset{\imath}{\hookrightarrow} C \overset{\pi}{\twoheadrightarrow} B$ be an exact sequence of Hopf algebras. Then C is finite-dimensional and there exists a Hopf datum $(\rightharpoonup, \sigma, \rho, \tau)$ such that $C \simeq A^{\rho,\tau}\#_{\rightharpoonup,\sigma}B$ as Hopf algebras.

(ii) *Conversely, if* $(\rightharpoonup, \sigma, \rho, \tau)$ *is a Hopf datum over* A *and* B*, then the maps* $\imath(a) = a\#1$ *and* $\pi(a\#b) = \varepsilon(a)b$ *are morphisms of Hopf algebras and give rise to an exact sequence of Hopf algebras*

$$A \overset{\imath}{\hookrightarrow} A^{\rho,\tau}\#_{\rightharpoonup,\sigma}B \overset{\pi}{\twoheadrightarrow} B.$$

(iii) *Let* $\phi : B \to A$ *be a convolution-invertible linear map such that* $\phi(1) = 1$ *and* $\varepsilon \circ \phi = \varepsilon$*. Then* $A^{\rho,\tau}\#_{\rightharpoonup,\sigma}B \simeq A^{\rho^{\phi^{-1}},\tau^{\phi^{-1}}}\#_{{}^{\phi}\rightharpoonup,{}^{\phi}\sigma}B$ *for any Hopf datum* $(\rightharpoonup, \sigma, \rho, \tau)$*.* $\square$

In particular, the last part of the Theorem says that $A\#_{{}^{\phi}\rightharpoonup,{}^{\phi}\sigma}B \simeq A\#_{\rightharpoonup,\sigma}B$ as cleft extensions.

2.2. **Cleft extensions of the Sweedler algebra T_{-1}.** Given an algebra A and a Hopf algebra B, in general, it is not easy to find a compatible pair $(\rightharpoonup, \sigma)$ giving rise to a crossed product $A\#_{\rightharpoonup,\sigma}B$. However, the classification given by [DT] and [Ma] provides a way to construct all compatible pairs $(\rightharpoonup, \sigma)$ when $B = T_{-1}$, the Sweedler algebra of dimension 4. Explicitly,

$$\begin{aligned} T_{-1} &= k\langle g, x \mid g^2 = 1, x^2 = 0, xg = -gx\rangle, \\ \Delta(g) &= g \otimes g \quad \text{and} \quad \Delta(x) = x \otimes g + 1 \otimes x. \end{aligned} \tag{2}$$

Definition 2.5. [DT, Def. 2.4], [Ma, Def. 3.1]. Let A be an algebra. A 5-tuple $\mathfrak{D} = (F, D, \alpha, \beta, \gamma)$, where $F, D \in \mathrm{End}(A)$, $\alpha \in \mathcal{U}(A)$ (the units of A) and $\beta, \gamma \in A$ is called a T_{-1}*-cleft datum over* A if it satisfies:

($\mathfrak{D}$1) F is an algebra morphism, ($\mathfrak{D}$2) $D(aa') = aD(a') + D(a)F(a')$,
($\mathfrak{D}$3) $F^2(a)\alpha = \alpha$, ($\mathfrak{D}$4) $(FD(a) + DF(a))\alpha = \gamma a - F(a)\gamma$,
($\mathfrak{D}$5) $D(a)\gamma + D^2(a)\alpha = \beta a - a\beta$, ($\mathfrak{D}$6) $F(\alpha) = \alpha$, ($\mathfrak{D}$7) $D(\beta) = 0$,
($\mathfrak{D}$8) $D(\alpha) = \gamma - F(\gamma)$, ($\mathfrak{D}$9) $D(\gamma) = \beta - F(\beta)$,

for all $a, a' \in A$.

Definition 2.6. [DT, Thm. 2.3, Def. 2.4], [Ma, Prop. 3.4]. If $\mathfrak{D} = (F, D, \alpha, \beta, \gamma)$ is a T_{-1}-cleft datum over A, then $C_{\mathfrak{D}} := A\#_{\rightharpoonup,\sigma}T_{-1}$ is an associative algebra where $\rightharpoonup : T_{-1} \otimes A \to A$ is the weak action given by:

$$1 \rightharpoonup a = a,\ g \rightharpoonup a = F(a),\ x \rightharpoonup a = D(a),\ (gx) \rightharpoonup a = FD(a)\alpha,$$

and $\sigma : T_{-1} \otimes T_{-1} \to T_{-1}$ is the 2-cocycle given by the following table:

σ	1	g	x	gx
1	1	1	0	0
g	1	α	0	0
x	0	γ	β	$-F(\beta)$
gx	0	$F(\gamma)$	$F(\beta)$	$-\alpha\beta$

The T_{-1}-cleft data classify all T_{-1}-cleft extensions:

Theorem 2.7. [DT, Cor. 2.5, Thm. 2.7], [Ma, Prop. 3.4].

(i) *If $A \subset C$ is a T_{-1}-cleft extension, then it is isomorphic to $C_{\mathfrak{D}}$ for some T_{-1}-cleft datum $\mathfrak{D}$ over A.*

(ii) *Let $\mathfrak{D} = (F, D, \alpha, \beta, \gamma)$ and $\mathfrak{D}' = (F', D', \alpha', \beta', \gamma')$ be T_{-1}-cleft data over an algebra A. Then $C_{\mathfrak{D}} \simeq C_{\mathfrak{D}'}$ as T_{-1}-extensions if only if there exists element $s \in \mathcal{U}(A)$ and $t \in A$ such that for all $a \in A$:*

$$
\begin{aligned}
&(C_{\mathfrak{D}}1) \quad F'(a) = sF(a)s^{-1}, && (C_{\mathfrak{D}}2) \quad D'(a) = (tF(a) + D(a) - at)s^{-1},\\
&(C_{\mathfrak{D}}3) \quad \alpha' = sF(s)\alpha, && (C_{\mathfrak{D}}4) \quad \beta' = \beta + t\gamma + (tF(t) + D(t))\alpha,\\
&(C_{\mathfrak{D}}5) \quad \gamma' = s\gamma + (tF(s) + D(s) + sF(t))\alpha. && \square
\end{aligned}
$$

Moreover, there is a linear map $\phi : T_{-1} \to A$ such that $({}^{\phi}\rightharpoonup, {}^{\phi}\sigma) = (\rightharpoonup', \sigma')$, see for example [AD, Prop. 3.2.12]. It is given explicitly by:

$$\phi(1) = 1, \quad \phi(g) = s, \quad \phi(x) = t \quad \text{and} \quad \phi(gx) = sF(t)\alpha. \tag{3}$$

Next we list some properties of T_{-1} that will be useful in the sequel.

($T1$) $\operatorname{Rad} T_{-1} = k \cdot x \oplus k \cdot gx$ and $hh' = 0 \ \forall\ h, h' \in \operatorname{Rad} T_{-1}$.

($T2$) $\mathcal{U}(T_{-1}) = \{a + bg + h \mid a, b \in k,\ a^2 - b^2 \neq 0,\ h \in \operatorname{Rad} T_{-1}\}$ (multiply by $a - bg + h$ and use that $h^2 = 0$).

($T3$) $\{t \in T_{-1} \mid t^2 = 1\} = \{\pm 1, \pm g + h \mid h \in \operatorname{Rad} T_{-1}\}$.

($T4$) $\forall\ h \in k[G(T_{-1})]$ there exists $s \in k[G(T_{-1})]$ such that $s^2 = h$ (write the necessary equations to find s and solve them– that is possible because k is an algebraically closed field of characteristic zero).

We end this section by proving a theorem which determines all possible extensions of T_{-1} by T_{-1} (up to isomorphisms).

Lemma 2.8. *If $T_{-1} \overset{\imath}{\hookrightarrow} H \overset{\pi}{\twoheadrightarrow} T_{-1}$ is an exact sequence of Hopf algebras then $H \simeq T_{-1} \otimes T_{-1}$.*

Proof. By 2.4, $H \simeq T_{-1}\,{}^{\rho,\tau}\#_{\rightharpoonup,\sigma} T_{-1}$ for some Hopf datum $(\rightharpoonup, \sigma, \rho, \tau)$. In particular, $T_{-1} \subset H$ is a T_{-1}-cleft extension. So $T_{-1}\#_{\rightharpoonup,\sigma}T_{-1} \simeq C_{\mathfrak{D}}$ as algebras, where $\mathfrak{D}$ is a T_{-1}-cleft datum over T_{-1}.

Our aim is to change the initial T_{-1}-cleft datum $\mathfrak{D}$ by another equivalent but more appropriate, using 2.7, in such a way that we still have an exact sequence of Hopf algebras. Because of (3) and 2.4 (iii), this is possible if the following conditions for $s \in \mathcal{U}(T_{-1})$ and $t \in T_{-1}$ are satisfied:

$$\varepsilon(s) = 1, \qquad \varepsilon(t) = 0 \quad \text{and} \quad \varepsilon(F(t)) = 0. \tag{4}$$

Let $\mathfrak{D} = (F, D, \alpha, \beta, \gamma)$ be a T_{-1}-cleft datum over T_{-1}. By ($\mathfrak{D}$1) and ($\mathfrak{D}$3), F is an algebra automorphism of T_{-1}. Then by (T3), $F(g) = \pm g + h$ for some $h \in \operatorname{Rad} T_{-1}$. Actually, $F(g) = g + h$. In fact,

$$(1\#g)(g\#1) = (g \rightharpoonup g)\sigma(g, 1) \otimes g = F(g)\#g, \tag{5}$$

the last equality follows from 2.6. If we apply π, the Hopf algebra morphism defined in 2.4, we find that $\varepsilon(F(g)) = \varepsilon(g)$. Therefore $F(g) = g + h$.

Let $s = g + \frac{h}{2}$, $t = 0$ and $\phi : T_{-1} \to A$ as in (3). Then the algebra automorphism F' corresponding to the new cleft datum $\mathfrak{D}'$ equivalent to $\mathfrak{D}$ satisfies

$$F'(g) = g \tag{6}$$

by $(C_{\mathfrak{D}}1)$; and we still have an exact sequence of Hopf algebras by (4). For simplicity, we still write $\mathfrak{D}$ for $\mathfrak{D}'$.

We now perform a second change of datum. By $(\mathfrak{D}3)$ with $a = g$, we have that $\alpha \in k[G(T_{-1})]$. Moreover, $\varepsilon(\alpha) = 1$ since

$$(1\#g)(1\#g) = (g \rightharpoonup 1)\sigma(g, g) \otimes 1 = \alpha\#1, \tag{7}$$

the last equality by 2.6. Applying π, it follows that $\varepsilon(\alpha) = 1$. By $(T4)$, we can pick $s \in k[G(T_{-1})]$ such that $s^2 = \alpha^{-1}$; note that $F(s) = s$. Moreover, we may assume that $\varepsilon(s) = 1$ since $(-s)^2 = \alpha^{-1}$. Let also $t = 0$ and $\phi : T_{-1} \to A$ as in (3). Then the new cleft datum given as in 2.7 (ii) has

$$\alpha' = 1,\ F'(g) = g, \tag{8}$$

by $(C_{\mathfrak{D}}1)$ and $(C_{\mathfrak{D}}3)$; and by (4), we still have an exact sequence of Hopf algebras. Again, we write $\mathfrak{D}$ instead of $\mathfrak{D}'$.

We now perform a further change of datum. Let $s = 1$, $t = \frac{g}{2}D(g)$ and $\phi : T_{-1} \to A$ as in (3). By $(\mathfrak{D}2)$, $D(1) = 0$ and therefore $0 = gD(g) + D(g)g$ also by $(\mathfrak{D}2)$. Then $D(g) \in \operatorname{Rad} T_{-1}$. Thus, using $(C_{\mathfrak{D}}2)$, the new cleft datum defined as in 2.7 (ii) has

$$D(g) = 0,\ F(g) = g,\ \alpha = 1 \tag{9}$$

and we still have an exact sequence of Hopf algebras (note that $F(t) \in \operatorname{Rad} T_{-1}$ since $t \in \operatorname{Rad} T_{-1}$).

We perform still another change of datum, corresponding to $s = 1$ and $t = -\frac{1}{2}\gamma$. Indeed, note that $0 = \gamma g - g\gamma$, by $(\mathfrak{D}4)$ with $a = g$ and (9). Then $\gamma \in k[G(T_{-1})]$. We claim moreover that $\varepsilon(\gamma) = 0$. In fact,

$$\begin{aligned}(1\#g)(1\#x) &= (g \rightharpoonup 1)\sigma(g, x) \otimes 1 + (g \rightharpoonup 1)\sigma(g, 1) \otimes gx \\ &= \gamma\#1 + 1\#gx;\end{aligned} \tag{10}$$

the last equality follows from 2.6. Applying π, it follows that $\varepsilon(\gamma) = 0$. Then the new cleft datum has

$$\gamma = 0,\ F(g) = g,\ D(g) = 0,\ \alpha = 1 \tag{11}$$

by $(C_{\mathfrak{D}}5)$; and we still have an exact sequence of Hopf algebras– note that $F(\gamma) = \gamma$.

We perform the last change of datum, corresponding to $s = g$ and $t = 0$. Since F is an algebra morphism, there exists $a, b \in k$ such that

$$F_{|\operatorname{Rad} T_{-1}} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

on the basis $\{x, gx\}$. As $\alpha = 1$, $F^2 = \mathrm{id}$ by ($\mathfrak{D}3$). Then either $a = \pm 1$ and $b = 0$ or $a = 0$ and $b = \pm 1$. By ($C_{\mathfrak{D}}1$), the new cleft datum has either

$$F = \mathrm{id}, \qquad D(g) = 0, \qquad \alpha = 1 \quad \text{and } \gamma = 0 \quad \text{or} \tag{12}$$

$$F(g) = g, \quad F(x) = gx, \quad D(g) = 0, \qquad \alpha = 1 \quad \text{and } \gamma = 0. \tag{13}$$

In both cases, we still have exact sequences of Hopf algebras.

We next claim that $D = 0$ and $\beta = 0$. In (12), $D = 0$ by ($\mathfrak{D}4$); hence $\beta \in k$ (the center of T_{-1}) by ($\mathfrak{D}5$). In (13), $0 = xD(x) + D(x)gx$ by ($\mathfrak{D}2$), hence $D(x)xg = xD(x)$. If we write $D(x) = c + dg + h$ with $c, d \in k$ and $h \in \operatorname{Rad} T_{-1}$, then

$$(c + dg)xg = x(c + dg) \Rightarrow d = c = -d.$$

Therefore $D(x) = h \in \operatorname{Rad} T_{-1}$. Moreover, since $D(g) = 0$, $D(gx) = gD(x)$. Now, since $\alpha = 1$, $\gamma = 0$ and $F(h) = gh$ for all $h \in \operatorname{Rad} T_{-1}$, we see from ($\mathfrak{D}4$) that

$$0 = FD(x) + DF(x) = F(h) + D(gx) = gh + gD(x) = gh + gh = 2gh.$$

Therefore $D = 0$, and β must belong to k too by ($\mathfrak{D}5$). In both cases, we see by 2.6 that $x \rightharpoonup a = D(a) = 0$, $\sigma(x, 1) = \sigma(1, x) = x^2 = 0$ and

$$(1\#x)(1\#x) = \sigma(x, x) \otimes 1 = \beta \otimes 1. \tag{14}$$

Applying π, since $\beta \in k$, it follows that $\beta = 0$.

We define the algebra morphism $\overset{\frown}{F}$ by $\overset{\frown}{F}(g) := g$ and $\overset{\frown}{F}(x) := gx$. Then H must be isomorphic as algebra to $C_{\mathfrak{D}}$ where $\mathfrak{D}$ is one of the following cleft data:

$$\mathfrak{D}_0 := (\mathrm{id}, 0, 1, 0, 0) \text{ or} \tag{15}$$

$$\overset{\frown}{\mathfrak{D}} := (\overset{\frown}{F}, 0, 1, 0, 0). \tag{16}$$

Our next aim is to show that

$$\operatorname{Rad}(T_{-1}) \otimes T_{-1} + T_{-1} \otimes \operatorname{Rad}(T_{-1}) \subseteq \operatorname{Rad} H. \tag{17}$$

If $\mathfrak{D} = \mathfrak{D}_0$, then $H \simeq T_{-1} \otimes T_{-1}$ as algebras and (17) follows. If $\mathfrak{D} = \overset{\frown}{\mathfrak{D}}$ then $H \simeq T_{-1} \#_{\rightharpoonup, \sigma} T_{-1}$. Here $(\rightharpoonup, \sigma)$ is given by

$$1 \rightharpoonup h = h, \quad g \rightharpoonup h = \begin{cases} h & h \in k[G(T_{-1})] \\ gh & h \in \operatorname{Rad} T_{-1} \end{cases}, \quad x \rightharpoonup h = gx \rightharpoonup h = 0; \tag{18}$$

$$\sigma(h, h') = \varepsilon(h)\varepsilon(h'), \tag{19}$$

for all $h, h' \in T_{-1}$, by 2.6. By explicit calculations, we have that $\operatorname{Rad} T_{-1} \otimes T_{-1}$ and $T_{-1} \otimes \operatorname{Rad} T_{-1}$ are nilpotent ideals of H. Then (17) follows.

Now (17) implies that

$$\dim(H^*)_0 \leq \dim(H/(\operatorname{Rad}(T_{-1}) \otimes T_{-1} + T_{-1} \otimes \operatorname{Rad}(T_{-1}))) = 16 - 12 = 4.$$

Therefore, any simple representation of H is one-dimensional, i.e., H^* is pointed. Moreover, since $(T_{-1})^* \simeq T_{-1}$, H^* is also an extension of T_{-1} by T_{-1}. Therefore $(H^*)^* \simeq H$ is pointed too.

Summarizing, H and H^* are pointed, the groups $G(H)$ and $G(H^*)$ have order ≤ 4 and both contain a normal Sweedler Hopf subalgebra. By inspection in the classification list of pointed Hopf algebras of dimension 16 given in [CDR], we see that H must be isomorphic to $T_{-1} \otimes T_{-1}$. □

3. Non-semisimple Hopf algebras of dimension 8

We shall need the classification of the non-semisimple Hopf algebras of dimension 8 [Ş]. We give this list, including the defining relations of the algebra structure and the comultiplication in terms of the generators. Let i be a primitive 4-root of 1.

$$\begin{aligned}\mathcal{A}_2 := \quad & k\langle g, x, y \mid g^2 - 1 = x^2 = y^2 = gx + xg = gy + yg = xy + yx = 0\rangle, \\ & \Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes g + 1 \otimes x, \quad \Delta(y) = y \otimes g + 1 \otimes y.\end{aligned}$$

$$\begin{aligned}\mathcal{A}'_4 := k\langle g, x \mid g^4 - 1 = x^2 = gx + xg = 0\rangle, \\ \Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes g + 1 \otimes x;\end{aligned}$$

$$\begin{aligned}\mathcal{A}''_4 := k\langle g, x \mid g^4 - 1 = x^2 - g^2 + 1 = gx + xg = 0\rangle, \\ \Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes g + 1 \otimes x;\end{aligned}$$

$$\begin{aligned}\mathcal{A}'''_{4,i} : k\langle g, x \mid g^4 - 1 = x^2 = gx - ixg = 0\rangle, \\ \Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes g^2 + 1 \otimes x;\end{aligned}$$

$$\begin{aligned}\mathcal{A}_{2,2} := k\langle g, h, x \mid g^2 = h^2 = 1,\ x^2 = gx + xg = hx + xh = gh - hg = 0\rangle, \\ \Delta(g) = g \otimes g, \quad \Delta(h) = h \otimes h, \quad \Delta(x) = x \otimes g + 1 \otimes x.\end{aligned}$$

Remarks 3.1. There are the following isomorphisms: $\mathcal{A}_2 \simeq (\mathcal{A}_2)^*$, $\mathcal{A}'''_{4,i} \simeq \mathcal{A}'''_{4,-i} \simeq (\mathcal{A}'_4)^*$ and $\mathcal{A}_{2,2} \simeq (\mathcal{A}_{2,2})^*$ [Ş]. Moreover, one can check case-by-case that all these Hopf algebras have Hopf subalgebras isomorphic to T_{-1}.

3.1. **The unique Hopf algebra of dimension 8 which does not have the Chevalley property.** By [Ş], $\mathcal{A} := (\mathcal{A}''_4)^*$ is the unique Hopf algebra of dimension 8 neither semisimple nor pointed; its coradical is $\mathcal{A}_0 = k[C_2] \oplus \mathcal{M}^*(2, k)$ and $\mathcal{A}$ is generated as an algebra by $\mathcal{M}^*(2, k)$.

We next compute explicitly the multiplication of the elements of a comatrix basis of $\mathcal{M}^*(2, k)$. For this, we first describe the simple representations of $\mathcal{A}''_4$. Let g and x be the generators of $\mathcal{A}''_4$.

Lemma 3.2. *The simple one-dimensional representations of $\mathcal{A}''_4$ are ε and $\alpha : \mathcal{A}''_4 \longmapsto k$, where*

$$\alpha(g) = -1, \quad \alpha(x) = 0. \tag{20}$$

The unique (up to isomorphisms) simple representation of dimension 2 *of* $\mathcal{A}''_4$ *is* $\rho : \mathcal{A}^* \longmapsto \mathcal{M}(2,k)$,

$$\rho(g) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho(x) = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}. \tag{21}$$

Proof. For simplicity, if (V, ϱ) is simple representation of $\mathcal{A}''_4$, we write a instead of $\varrho(a) \in \operatorname{End} V$.

In case that $\dim V = 1$, from relations $g^4 = 1$ and $xg = -gx$ it follows that g is a 4-root of 1 and $x = 0$. Then we have $g^2 = 1$, by the relation $x^2 - g^2 + 1 = 0$. So that either $g = 1$ or $g = -1$. This defines ε and α respectively.

In case that $\dim V = 2$, by $g^4 = \mathrm{id}$, we can choose a basis of V consisting of eigenvectors of g. Then the eigenvalues of g are different 4-roots of 1. In fact, if they are equal then $x = 0$ (by $xg = -gx$) but the representation $\varrho(g) = i^j \cdot \mathrm{id}$, $0 \leq j \leq 3$, $\varrho(x) = 0$ is not simple. Let ξ and η be the eigenvalues of g. Then

$$xg = \begin{pmatrix} \xi x_{11} & \eta x_{12} \\ \xi x_{21} & \eta x_{22} \end{pmatrix} = \begin{pmatrix} -\xi x_{11} & -\xi x_{12} \\ -\eta x_{21} & -\eta x_{22} \end{pmatrix} = -gx, \tag{22}$$

so $x_{11} = x_{22} = 0$. Moreover, $x_{12} \neq 0 \neq x_{21}$. Indeed, both $x_{12} \neq 0$ and $x_{21} \neq 0$, because the representation is simple. Therefore $\eta = -\xi$. By $0 = x^2 - g^2 + \mathrm{id}$, we have that

$$0 = \begin{pmatrix} x_{12}x_{21} - \xi^2 + 1 & 0 \\ 0 & x_{12}x_{21} - \xi^2 + 1 \end{pmatrix}. \tag{23}$$

Since $x_{12} \neq 0 \neq x_{21}$, it follows that $\xi^2 \neq 1$. Therefore ξ is a primitive 4-root of 1 and $x_{12}x_{21} = -2$. Taking $x_{12} = 2$ and $x_{21} = -1$, we find ρ.

Since $(\mathcal{A}''_4)^*_0 = \mathcal{A}_0 = k[C_2] \oplus \mathcal{M}^*(2,k)$, the lemma follows. □

Let (k^2, ρ) be the 2-dimensional representation given by 3.2. Let $\{E_{ij} \mid 1 \leq i,j \leq 2\}$ be the coordinate functions of $\mathcal{M}(2,k)$. If $e_{ij} := E_{ij} \circ \rho$, then $\mathcal{E}_{\mathcal{A}} := \{e_{ij} \mid 1 \leq i,j \leq 2\}$ is a comatrix basis of the simple subcoalgebra of $\mathcal{A}$ isomorphic to $\mathcal{M}^*(2,k)$.

Lemma 3.3. *The elements of* $\mathcal{E}_{\mathcal{A}}$ *satisfy:*

$$S(e_{11}) = e_{22}, \quad S(e_{22}) = e_{11}, \quad S(e_{12}) = -\xi e_{12}, \quad S(e_{21}) = \xi e_{21}, \tag{24}$$

$$e_{11}^2 = e_{22}^2 = \alpha, \quad e_{12}^2 = e_{21}^2 = 0, \tag{25}$$

$$e_{11}e_{22} = e_{22}e_{11} = \varepsilon, \quad e_{12}e_{21} = e_{21}e_{12} = 0, \tag{26}$$

$$e_{12}e_{11} = \xi e_{11}e_{12}, \quad e_{21}e_{11} = \xi e_{11}e_{21}, \tag{27}$$

$$e_{12}e_{22} = -\xi e_{22}e_{12}, \quad e_{21}e_{22} = -\xi e_{22}e_{21}. \tag{28}$$

In particular we have that

$$\Delta(e_{11}e_{12}) = e_{11}e_{12} \otimes \varepsilon + \alpha \otimes e_{11}e_{12}, \tag{29}$$

$$\Delta(e_{11}e_{21}) = e_{11}e_{21} \otimes \alpha + \varepsilon \otimes e_{11}e_{21}, \tag{30}$$

i.e., $e_{11}e_{12}$ *and* $e_{11}e_{21}$ *are the non-trivial skew-primitives of* $\mathcal{A}$.

Proof. Since $\mathcal{A} = (\mathcal{A}_4'')^*$, the multiplication of $\mathcal{A}$ is given by the convolution product and the antipode of $\mathcal{A}$ by $S(a) = a \circ \mathcal{S}$ for all $a \in \mathcal{A}$, with $\mathcal{S}$ the antipode of $\mathcal{A}_4''$.

Note that $\{g^n x^m \mid 0 \le n \le 3, 0 \le m \le 1\}$ is a basis of $\mathcal{A}_4''$, $\mathcal{S}(g) = g^{-1}$ and $\mathcal{S}(x) = -xg^{-1}$ [Ş]. Then, by 3.2, we have

$$S(e_{11})(g^n) = e_{11}(\mathcal{S}(g^n)) = e_{11}(g^{-n}) = \xi^{-n} = (-\xi)^n = e_{22}(g^n)$$

$$\text{and } S(e_{11})(g^n x) = e_{11}(\mathcal{S}(g^n x)) = e_{11}(-xg^{-n-1}) = 0 = e_{22}(g^n x),$$

then $S(e_{11}) = e_{22}$. Similarly, we prove $S(e_{22}) = e_{11}$. Clearly, $S(e_{12})(g^n) = S(e_{21})(g^n) = 0$ for all n. Moreover, by 3.2,

$$\begin{aligned} S(e_{12})(g^n x) &= e_{12}(\mathcal{S}(g^n x)) = e_{12}(-xg^{-n-1}) \\ &= -2(-\xi)^{-n-1} = -2\xi\xi^n = -\xi e_{12}(g^n x), \end{aligned}$$

then $S(e_{12}) = -\xi e_{12}$. Similarly, we prove $S(e_{21}) = \xi e_{21}$ and (24) follows.

Since g is a group-like, we have

$$e_{11}^2(g^n) = (e_{11}(g^n))^2 = \xi^{2n} = (-1)^n = \alpha(g^n) \text{ and}$$

$$e_{12}^2(g^n) = (e_{12}(g^n))^2 = 0,$$

for all n. Since x is a $(1, g)$-primitive, then

$$e_{11}^2(g^n x) = e_{11}(g^n x)e_{11}(g^{n+1}) + e_{11}(g^n)e_{11}(g^n x) = 0 = \alpha(g^n x) \text{ and}$$

$$e_{12}^2(g^n x) = e_{12}(g^n x)e_{12}(g^{n+1}) + e_{12}(g^n)e_{12}(g^n x) = 0.$$

Therefore $e_{11}^2 = \alpha$ and $e_{12}^2 = 0$. Analogously, we prove $e_{22}^2 = \alpha$ and $e_{21}^2 = 0$ and (25) follows.

From similar calculations it follows that $e_{12}e_{21} = 0 = e_{21}e_{12}$. Then, by (24) and the definition of antipode, we get $e_{11}e_{22} = \varepsilon = e_{22}e_{11}$ and (26) follows.

If (27) holds, (28) also holds. In fact, (28) follows from (24) and (27). Since $g \in G(\mathcal{A}_4'')$, $e_{11}e_{12}(g^n) = e_{12}e_{11}(g^n) = 0$. On the other hand

$$\begin{aligned} e_{11}e_{12}(g^n x) &= e_{11}(g^n x)e_{12}(g^{n+1}) + e_{11}(g^n)e_{12}(g^n x) \\ &= e_{11}(g^n)e_{12}(g^n x) = \xi^n \xi^n 2 = (-1)^n 2 \quad \text{and} \\ e_{12}e_{11}(g^n x) &= e_{12}(g^n x)e_{11}(g^{n+1}) + e_{12}(g^n)e_{11}(g^n x) \\ &= e_{12}(g^n x)e_{11}(g^{n+1}) = \xi^n 2\xi^{n+1} = (-1)^n 2\xi, \end{aligned}$$

then $e_{12}e_{11} = \xi e_{11}e_{12}$. Also $e_{11}e_{21}(g^n) = e_{21}e_{11}(g^n) = 0$ and

$$\begin{aligned} e_{11}e_{21}(g^n x) &= e_{11}(g^n x)e_{21}(g^{n+1}) + e_{11}(g^n)e_{21}(g^n x) \\ &= e_{11}(g^n)e_{21}(g^n x) = -\xi^n(-\xi)^n = -1, \\ e_{21}e_{11}(g^n x) &= e_{21}(g^n x)e_{11}(g^{n+1}) + e_{21}(g^n)e_{11}(g^n x) \\ &= e_{21}(g^n x)e_{11}(g^{n+1}) = -(-\xi)^n \xi^{n+1} = -\xi, \end{aligned}$$

then $e_{21}e_{11} = \xi e_{11}e_{21}$ and (27) follows.

Since Δ is an algebra morphism, $e_{11}e_{12}$ and $e_{11}e_{21}$ are skew-primitive by (25) and (26). Since $(\varepsilon - \alpha)(g) \neq 0$, they also are non-trivial. □

Let T be the Hopf subalgebra of $\mathcal{A}$ generated by α and $y := e_{11}e_{21}$. Note that it is isomorphic to T_{-1}. Let $C_2 = \langle c \rangle$ be the cyclic group of order two. We end this section with the following lemma that will be needed later.

Lemma 3.4. *(i) If $\pi : \mathcal{A} \to T_{-1}$ is a morphism of Hopf algebras, then $\pi(\mathcal{A}) \subseteq k[G(T_{-1})]$ and $T \subseteq \mathcal{A}^{\operatorname{co}\pi}$.*

(ii) $\mathcal{A}$ fits into an exact sequence of Hopf algebras $T \overset{\imath}{\hookrightarrow} \mathcal{A} \overset{\psi}{\twoheadrightarrow} k[C_2]$.

Proof. The unique group-like of order two of $\mathcal{A}_4''$ is central. Then $\mathcal{A}_4''$ cannot have a Hopf subalgebra isomorphic to T_{-1}; implying that π is not an epimorphism. Hence $\pi(\mathcal{A}) \subseteq k[G(T_{-1})]$.

Clearly, $k[G(T_{-1})]$ does not have nilpotent elements. Then $\pi(e_{12}) = \pi(e_{21}) = 0$, by (25). Therefore $\pi(e_{11}), \pi(e_{22}) \in G(T_{-1})$ and by (25), $\pi(\alpha) = 1$. In particular, it follows that $T \subseteq \mathcal{A}^{co\pi}$.

Let $\psi : \mathcal{A} \to k[C_2]$ be the Hopf algebra epimorphism induced by the inclusion of the Hopf subalgebra of $\mathcal{A}_4''$ generated by the central group-like element g^2. By the paragraph above and 2.3, (ii) follows. □

4. Hopf algebras generated by simple coalgebras

In this section, we discuss some consequences of results of [N] and [Ş]. The following theorem will be used later.

Theorem 4.1. [Ş, Thm. 1.4. b)] *Let f be a coalgebra automorphism of $C = \mathcal{M}^*(2, k)$ of finite order n. Then there is a comatrix basis $\{e_{ij} \mid 1 \leq i, j \leq 2\}$ of C and a root of unity ω such that $f(e_{ij}) = \omega^{i-j} e_{ij}$ and $\operatorname{ord} \omega = n$.*

Lemma 4.2. *Let $\pi : H \to K$ be a morphism of finite-dimensional Hopf algebras such that $\pi(g) = 1$ for some $g \in G(H)$, $g \neq 1$. Suppose that H is generated by C and 1 as an algebra, where C is a simple subcoalgebra of dimension 4 stable by L_g. Then $\pi(H) \subseteq k[G(K)]$.*

The same holds true with R_g instead of L_g; or with $\operatorname{ad}_\ell(g)$ or $\operatorname{ad}_r(g)$ if $g \notin \mathcal{Z}(H)$.

Proof. First, we claim that $L_{g|C} \neq \operatorname{id}_C$. Indeed, let $\{e_{ij} \mid 1 \leq i, j \leq 2\}$ be a comatrix basis of C, then $1 = e_{11}S(e_{11}) + e_{12}S(e_{21})$. If $L_{g|C} = \operatorname{id}_C$, multiplying on both sides of the equality we get that $g = 1$, a contradiction. Since $L_{g|C}$ is a coalgebra automorphism of C, applying 4.1, we get $\{e_{ij} \mid 1 \leq i, j \leq 2\}$ a comatrix basis of C such that

$$L_g(e_{ij}) = ge_{ij} = \omega^{i-j} e_{ij}, \quad \text{with} \quad \omega \in k, \operatorname{ord}(\omega) = \operatorname{ord}(L_{g|C}) > 1. \tag{31}$$

Applying π on both sides of (31), we get $\pi(e_{12}) = \pi(e_{21}) = 0$. Then $\pi(e_{11}), \pi(e_{22}) \in G(K)$ and therefore $\pi(H) \subseteq k[G(K)]$.

The proof for R_g, $\operatorname{ad}_\ell(g)$ and $\operatorname{ad}_r(g)$ is similar. Note that $\operatorname{ad}_\ell(g)$ and $\operatorname{ad}_r(g)$ cannot be the identity because $g \notin \mathcal{Z}(H)$. □

Lemma 4.3. *Let $\pi : H \to K$ be an epimorphism of finite-dimensional Hopf algebras and assume that K is non-semisimple. Suppose that H is generated by C and 1 as an algebra, where C is a simple subcoalgebra of H of dimension 4 stable by S_H^2. Then* $\operatorname{ord} S_H^2 = \operatorname{ord} S_K^2$.

Proof. By 4.1, there is a comatrix basis $\{e_{ij} \mid 1 \leq i, j \leq 2\}$ of C such that

$$S_H^2(e_{ij}) = \omega^{i-j} e_{ij}, \quad \text{with} \quad \omega \in k, \operatorname{ord}(\omega) = \operatorname{ord}(S_{H|C}^2). \tag{32}$$

Applying π on both sides of (32), we get $S_K^2(\pi(e_{ij})) = \omega^{i-j}\pi(e_{ij})$, that is, at least one of the numbers $\omega^{\pm 1}$ is an eigenvalue of S_K^2. Indeed, $\pi(e_{12}) \neq 0$ or $\pi(e_{21}) \neq 0$ since otherwise $\pi(e_{11}), \pi(e_{22}) \in G(K)$ and K would be semisimple, because H is generated by C and 1 as an algebra. Then $\operatorname{ord}(\omega) = \operatorname{ord}(S_{H|C}^2) = \operatorname{ord} S_H^2$ divides $\operatorname{ord} S_K^2$.

Finally, since $K^* \hookrightarrow H^*$, $(S_K^2)^{\operatorname{ord} S_H^2} = 1$ and therefore they must be equal. □

The following proposition is due to Natale. It is the key step for the proof of the last result of this section.

Proposition 4.4. [N, Prop. 1.3]. *Let H be a finite-dimensional non-semisimple Hopf algebra. Suppose that H is generated by a simple subcoalgebra of dimension 4 which is stable by the antipode. Then H fits into an central exact sequence $k^G \overset{\imath}{\hookrightarrow} H \overset{\pi}{\twoheadrightarrow} A$, where G is a finite group and A^* is a pointed non-semisimple Hopf algebra.* □

Theorem 4.5. *Let H be a non-semisimple Hopf algebra such that* $\dim H$ *is odd or equal to $p^a q^b$, with p, q primes. Suppose that H is generated by a simple subcoalgebras of dimension 4 which is stable by the antipode. If*

$$H_0 = G(H) \oplus \mathcal{M}^*(2, k) \quad \text{or} \quad G(H) \cap \mathcal{Z}(H) = 1$$

then H^* is pointed.

Proof. By 4.4, H fits into an central exact sequence $k^G \overset{\imath}{\hookrightarrow} H \overset{\pi}{\twoheadrightarrow} A$, where G is a finite group and A^* is a pointed non-semisimple Hopf algebra.

Suppose that $G \neq 1$. Since $|G|$ divides $\dim H$ by [NZ], G is solvable by the Feit-Thompson Theorem in the case that $\dim H$ is odd, and the Burnside Theorem in the other case. Thus k^G has at least one non-trivial group-like. Let $\alpha \in G(k^G) \subseteq G(H)$ be non-trivial.

Suppose that $H_0 = G(H) \oplus \mathcal{M}^*(2, k)$. Since L_α is a coalgebra automorphism of H, L_α fixes $\mathcal{M}^*(2, k)$. As $\pi(\alpha) = 1$, by 4.2 it follows that A is generated by group-likes. In particular, A is semisimple which is a contradiction. Therefore $G = 1$, that is, $H = A$ and H^* is pointed.

If $G(H) \cap \mathcal{Z}(H) = 1$ then clearly $G = 1$ and H^* is pointed. □

5. Proof of the Main Theorem

Our first step to prove Theorem 1.2 is to describe the possible coradical of a Hopf algebra of dimension 16 which does not have the Chevalley property.

It turns out that there are 6 possible coradicals. This leads us to do the proof case by case according to the type of the coradical.

Definition 5.1. We say that a Hopf algebra H is *of type* $(n_1, n_2, \dots, n_t)$ if the coradical of H is $H_0 \simeq k^{n_1} \oplus \mathcal{M}^*(2,k)^{n_2} \cdots \oplus \mathcal{M}^*(t,k)^{n_t}$.

Remark 5.2. Let H be a pointed Hopf algebra of dimension 16. Then by [B, Section 4.2], H^* is pointed or it is of type (2,1), (2,2), (2,3) or (4,2).

Remark 5.3. Let H be a non-semisimple non-pointed Hopf algebra of dimension 16 which has the Chevalley property. Then by [CDMM, Thm. 5.1], H is self-dual and of type $(4,1)$.

Proposition 5.4. *Let H be Hopf algebra of dimension* 16 *which does not have the Chevalley property. Then H is of type: (1,2), (2,1), (2,2), (2,3), (4,1) or (4,2).*

Proof. If $G(H) = 1$, then by [BD, Prop. 7.1] we know that H must be of type (1,2).

Now suppose that H is of type $(|G(H)|, n_2, n_3)$ with $|G(H)| > 1$. By [NZ], $|G(H)|$ divides 16. Moreover by [AN, Lemma 2.1], it divides

$$\dim H_0 = |G(H)| + 4 \cdot n_2 + 9 \cdot n_3.$$

Thus $n_3 = 0$. If $|G(H)| = 2$, then $n_2 = 1, 2$ or 3 and if $|G(H)| = 4$, then $n_2 = 1$ or 2. □

We next give some properties of Hopf algebra of dimension 16 which does not have the Chevalley property. We recall first the following statement due to Beattie and Dascalescu.

Proposition 5.5. [BD, Cor. 4.3]. *Let H be a finite-dimensional non-cosemisimple Hopf algebra with $H_0 \simeq k[G] \oplus \mathcal{M}^*(n_1,k) \oplus \cdots \oplus \mathcal{M}^*(n_t,k)$ with t a positive integer, $2 \le n_1 \le \cdots \le n_t$, and such that H has no non-trivial skew-primitives. Then*

$$\dim H > \dim H_1 = \dim H_0 + \dim P_1 \ge (1 + 2n_1)|G| + \sum_{i=1}^{t} n_i^2. \qquad (33)$$ □

Lemma 5.6. *Let H be a Hopf algebra of dimension 16.*

- *(i) If H is of type $(4,1)$ or $(4,2)$ then H has a pointed Hopf subalgebra K of dimension 8 such that $G(H) = G(K)$.*
- *(ii) If H is of type $(2,2)$ or $(2,3)$ then H has a Hopf subalgebra isomorphic to T_{-1}.*
- *(iii) If H is of type $(2,1)$ and H^* is non-pointed then H has a Hopf subalgebra isomorphic to $\mathcal{A}$ (see Subsection 3.1). In particular, it contains a Hopf subalgebra isomorphic to T_{-1}.*
- *(iv) If H is of type $(2,n)$ with $1 \le n \le 3$, then $G(H) \cap \mathcal{Z}(H) = 1$. If it is of type $(4,n)$ with $1 \le n \le 2$, then $|G(H) \cap \mathcal{Z}(H)| \le 2$.*

Proof. If H is of type $(2,2)$, $(2,3)$, $(4,1)$ or $(4,2)$ then H has a non-trivial skew-primitive. Otherwise, we can apply 5.5 to H and we obtain a contradiction by (33). Let K be the Hopf subalgebra of H generated by $G(H)$ and $\mathcal{P}(H)$. By [M, Lemma 5.5.1], K is pointed and $\dim K > |G(H)|$.

If $|G(H)| = 4$, then $\dim K = 8$ by [NZ]. This proves (i).

If $|G(H)| = 2$, by [NZ] and [Ş] K is isomorphic to T_{-1} or $\mathcal{A}_2$ (see Section 3). But $\mathcal{A}_2$ has a Hopf subalgebra isomorphic to T_{-1} by 3.1. This proves (ii).

Let H be as in (iii) and let C be the unique simple subcoalgebra of H of dimension 4. The Hopf subalgebra K generated by C has dimension 8 or 16. We claim that $\dim K = 8$ and therefore $K \simeq \mathcal{A}$ by [Ş]. In fact, if $K = H$, then H^* is pointed by 4.5. But this cannot happen, since H^* is non-pointed by hypothesis; then $\dim K = 8$.

Finally, we prove (iv). If H is of type $(2,n)$ with $1 \leq n \leq 3$, then by (ii) and (iii) it contains a Sweedler subalgebra T_{-1}. In particular, $G(T_{-1}) = G(H)$ and the claim follows since $G(T_{-1}) \cap \mathcal{Z}(T_{-1}) = 1$. If H is of type $(4,n)$ with $1 \leq n \leq 2$, then the assertion follows by (i) and Section 3. □

We assume for the rest of the paper that H is a Hopf algebra of dimension 16 *which has not the Chevalley property.*

Note that by 5.3, if H^* is non-pointed, then it does not have the Chevalley property either.

In the next subsections we prove that H cannot be of type $(1,2)$ – see 5.8; also, if H is of type (s,t) then H^* has the Chevalley property, for each (s,t) with $s > 1$, according to 5.4 – see 5.9; 5.13; 5.14 and 5.15. Then the Theorem 1.2 is proved.

5.1. **Type** $(1,2)$.

Remark 5.7. Let H be a finite-dimensional Hopf algebra generated by two simple subcoalgebras C and D such that $S(C) = D$. Then C and 1 generate H as an algebra.

Indeed, the subalgebra A of H generated by C and 1, is a sub-bialgebra. Since $\dim H < \infty$, A is a Hopf subalgebra and then $D = S(C) \subseteq A$.

Proposition 5.8. *H cannot be of type (1,2).*

Proof. Suppose that H is of type $(1,2)$. Then H^* is not cosemisimple and hence it is non-semisimple by [LR]. Moreover, it must be non-pointed by 5.2 and H^* does not have the Chevalley property by 5.3. Thus we can apply 5.4 to H^*.

Let C and D be the simple subcoalgebras of H of dimension 4. If C (and hence D) is stable by S, then the Hopf subalgebra K generated by C is H. Otherwise, K should be isomorphic to $\mathcal{A}$ or semisimple by the classification of 8-dimensional Hopf algebras. In both cases we would have $1 \neq G(K) \subseteq G(H)$. Hence by 4.5, H^* is pointed, which is a contradiction.

Therefore S permutes C with D, and so H is generated by C and D as an algebra by [NZ]. In particular, C and 1 generate H as an algebra by 5.7.

We claim now that $S^4 = \mathrm{id}$. Indeed, if $G(H^*) = 1$ the claim follows from Radford's formula for S^4. If $G(H^*) \neq 1$ then by 5.6, H^* has a Hopf subalgebra K such that K^* is non-semisimple and $S_K^4 = \mathrm{id}_K$. Then there exists an epimorphism of Hopf algebras $\pi : H \to K^*$ and by 4.3, the claim follows.

Therefore, by [BD, Prop. 5.3] H has a simple subcoalgebras of dimension 4 stable by S, which is a contradiction to the fact that S permutes the simple subcoalgebras. □

5.2. **Type** $(4, 1)$.

Proposition 5.9. *H cannot be of type (4,1).*

Proof. Let K be the Hopf subalgebra of H generated by C, the simple subcoalgebra of H of dimension 4. Note that $\dim K \neq 16$ since otherwise H^* would be pointed by 4.5, and this would contradict 5.2. Hence $\dim K = 8$ and K is non-pointed. If K is semisimple, then by [LR] K is cosemisimple and $K = H_0$ by counting, a contradiction to the hypothesis on H. Hence $K \simeq \mathcal{A}$.

As $G(\mathcal{A}) = C_2$, there exists $g \in G(H) - G(K)$. Since C is the unique simple subcoalgebra of H of dimension 4, C is stable by L_g, which is a coalgebra automorphism of H. Let B be the Hopf subalgebra generated by K and g. Since the multiplication is associative and C generates K as an algebra, we have that $B = K + k[g]$ as a vector spaces; in particular $8 < \dim B < 16$. This is impossible by [NZ]. □

We finish this subsection with a criterion that helps us to know when H^* is pointed. The key of argument comes from the proof of [G, Thm. 2.1].

Lemma 5.10. *Suppose that H fits into an exact sequence $K \overset{\imath}{\hookrightarrow} H \overset{\pi}{\twoheadrightarrow} k[C_2]$, where K^* is pointed. Then H^* is pointed.*

Proof. By subsection 2.1, H is isomorphic as algebra to a crossed product $K\#_{\rightharpoonup,\sigma}k[C_2]$. Denote by g the generator of C_2. Then the weak action $l(g) : K \mapsto K, a \mapsto (g \rightharpoonup a)$ is an isomorphism of algebras. In particular $\operatorname{Rad} K$ is stable by $l(g)$ and therefore $\operatorname{Rad} K\#_{\rightharpoonup,\sigma}k[C_2]$ is a nilpotent C_2-graded ideal. This implies that $\operatorname{Rad} K\#_{\rightharpoonup,\sigma}k[C_2] \subseteq \operatorname{Rad} H$. Besides $H/(\operatorname{Rad} K\#_{\rightharpoonup,\sigma}k[C_2]) \simeq (K/\operatorname{Rad} K)\#_{\overline{\rightharpoonup},\overline{\sigma}}k[C_2]$ is a semisimple algebra, by [M, Thm. 7.4.2]. Then $\operatorname{Rad} H \subseteq \operatorname{Rad} K\#_{\rightharpoonup,\sigma}k[C_2]$, and hence $H/\operatorname{Rad} H \simeq (K/\operatorname{Rad} K)\#_{\overline{\rightharpoonup},\overline{\sigma}}k[C_2]$.

We conclude the proof examining the dimension of $(H^*)_0$. Since K^* is pointed, $\dim(K^*)_0 = \dim(K/\operatorname{Rad} K) = 2, 4$ or 8 and therefore $\dim(H^*)_0 = \dim(H/\operatorname{Rad} H) = 4, 8$ or 16. If $\dim(H^*)_0 = 4$, H^* is clearly pointed. If $\dim(H^*)_0 = 8$, H^* is pointed by 5.4, 5.3 and 5.9. Since H is non-semisimple, $\dim(H^*)_0 = 16$ cannot occur. □

5.3. **Type** $(4, 2)$**.** Throughout this subsection C and D are the two simples subcoalgebras of H of dimension 4. We show in a series of lemmata that the dual of a Hopf algebra of type $(4, 2)$ is pointed. First we compute the order of S. Note that S^2 preserves C and D.

Lemma 5.11. *If H is of type (4,2) then* $\operatorname{ord} S^2_{|C} = \operatorname{ord} S^2_{|D} = 2$. *Moreover,* $\operatorname{ord} S = 4$.

Proof. Let K be the pointed Hopf subalgebra of H of dimension 8 given in 5.6. (i). Then $H = K \oplus C \oplus D$ is a direct sum of S^2-stable subspaces. Since H and K are non-semisimple, $\operatorname{Tr}(S^2) = \operatorname{Tr}(S^2_{|K}) = 0$. Moreover, by [LR, Lemma 3.2], we have that $\operatorname{Tr}(S^2_{|\mathcal{M}^*(2,k)}) \geq 0$, hence $\operatorname{Tr}(S^2_{|D}) = \operatorname{Tr}(S^2_{|C}) = 0$.

Let $\{e_{ij} \mid 1 \leq i, j \leq 2\}$ be a comatrix basis of C such that: $S^2(e_{ij}) = \omega^{i-j} e_{ij}$ with $\omega \in k$ and $\operatorname{ord} \omega = \operatorname{ord} S^2_{|C} = n$ by 4.1. Since $0 = \operatorname{Tr}(S^2_{|C}) = 2 + \omega + \omega^{-1}$, multiplying by ω on both sides we get $0 = 1 + 2\omega + \omega^2 = (1+\omega)^2$. Hence $\omega = -1$ and hence $\operatorname{ord} S^2_{|C} = 2$. The same holds true for D instead of C.

Finally, $\operatorname{ord} S = 4$ since by [Ş], $\operatorname{ord} S_{|K} = 4$. □

Lemma 5.12. *Let H be of type (4,2) and suppose that there exists $g \in G(H) \cap \mathcal{Z}(H)$ and H is generated as an algebra by C and 1. Then H^* is pointed.*

Proof. By 5.6 (iv), the order of g is 2. Then H fits into an exact sequence $k[C_2] \overset{\imath}{\hookrightarrow} H \overset{\pi}{\twoheadrightarrow} K$, where $K = H/k[C_2]^+H$. Since H is non-semisimple, K must be non-semisimple by [M, Thm. 7.4.2]. If K is pointed, then H^* is pointed by 5.10. If K is not pointed, then $K \simeq \mathcal{A}$ by the classification given by [Ş], see Section 3.

Suppose that $G(H) = \langle c \rangle$ is a cyclic group of order 4. Then $L_g = L_{c^2} = L_c^2$ must fix C and $\pi(g) = 1$ because $G(\mathcal{A}) \cap \mathcal{Z}(\mathcal{A}) = 1$. Then by 4.2, we obtain a contradiction. Therefore $G(H) \simeq C_2 \times C_2$. Hence, by 5.6 (i) and [Ş] H has a Hopf subalgebra isomorphic to $\mathcal{A}_{2,2}$.

Since $\mathcal{A}_{2,2}$ and $\mathcal{A}$ are not isomorphic, it follows that $\pi(\mathcal{A}_{2,2}) \subsetneq T_{-1} \subsetneq \mathcal{A}$.

Let $T_{-1} \overset{\imath}{\hookrightarrow} \mathcal{A} \overset{\psi}{\twoheadrightarrow} k[C_2]$ be the exact sequence given in 3.4. Then $\psi \circ \pi : H \to k[C_2]$ is an epimorphism of Hopf algebras and $\mathcal{A}_{2,2} \subseteq H^{\operatorname{co}(\psi\circ\pi)}$. Then by 2.3, H fits into the exact sequence $\mathcal{A}_{2,2} \overset{\imath}{\hookrightarrow} H \overset{\psi\circ\pi}{\twoheadrightarrow} k[C_2]$. Since $\mathcal{A}_{2,2}$ is self-dual, H^* is pointed by 5.10. □

Proposition 5.13. *Let H be of type (4,2). Then H^* is pointed.*

Proof. We divide the proof in two cases, according to the action of S on $\{C, D\}$.

<u>Case 1</u>: C and D are stable by S.

Let K be the Hopf subalgebra of H generated by C. First, suppose that $\dim K = 8$. Then $K \simeq \mathcal{A}$ because K is non-semisimple by 5.11 and [LR]. Let $g \in G(H) - G(K)$. We claim that K is a normal Hopf subalgebra and hence

H^* is pointed by 5.10. Indeed, if L_g and R_g fix C we get a contradiction as in the proof of 5.9. Thus we may assume that $L_g(C) = D$ and hence $L_g(D) = C$. Applying S to the second equality, it follows that $R_{g^{-1}}(D) = C$. Then C and therefore K are stable by $\mathrm{ad}_\ell(g)$. Since $\mathrm{ord}\, g < \infty$, K also is stable by $\mathrm{ad}_r(g)$. Since by [NZ], K and g generate H as an algebra, it follows that K is normal.

Now, suppose that $K = H$. Then by 4.4, H fits into the central exact sequence $k^G \overset{\imath}{\hookrightarrow} H \overset{\pi}{\twoheadrightarrow} A$, where G is a finite group and A^* is pointed non-semisimple. Since A is non-semisimple, $|G| \neq 8, 16$. Moreover, by 5.6 (iv) and [Ş], $|G| \neq 4$. If $|G| = 2$, then H^* is pointed by 5.12. If $|G| = 1$, then $H = A$ and hence H^* is pointed.

Case 2: C and D are permuted by S.

Note that by [NZ], C and D generate H as an algebra. Hence by 5.7, C and 1 generate H as an algebra.

Suppose that H^* is non-pointed. Then, by 5.6, there exists $\pi : H \twoheadrightarrow B$ an epimorphism Hopf algebras, where we can assume that B is isomorphic to: T_{-1}, $\mathcal{A}$ or $\mathcal{A}'''_{4,i}$. Indeed, H^* cannot be of type $(4, 1)$ by 5.9. If it is of type $(4, 2)$, then it contains a pointed Hopf subalgebra L of dimension 8 isomorphic to $\mathcal{A}'_4 = (\mathcal{A}'''_{4,i})^*$ or $\mathcal{A}''_4 = \mathcal{A}^*$, or L contains a Sweedler algebra, see 3.1. Finally, if H^* is of type $(2, n)$, then by 5.6 it contains a Sweedler algebra and the claim follows.

We first assume that $G(H)$ is cyclic. Say $G(H) = \langle g\rangle$. Then $L_{g^2}(C) = C$. If $\mathrm{ord}(\pi(g)) \leq 2$, then $\pi(g^2) = 1$ and by 4.2 $\pi(H) \subseteq k[G(B)]$, which is impossible because π is an epimorphism and B is non-semisimple. Hence $\mathrm{ord}\, g = 4$. Since $|G(T_{-1})| = |G(\mathcal{A})| = 2$, B must be isomorphic to $\mathcal{A}'''_{4,i}$ and $\pi(g)$ generates $G(B)$. We now proceed as in the proof of [N, Lemma 2.7]: let B^+ and B^- denote the following subspaces of B

$$B^+ := \{b \in B \mid S^2(b) = b\} \quad \text{and} \quad B^- := \{b \in B \mid S^2(b) = -b\}.$$

From the definition of $\mathcal{A}'''_{4,i}$ (see subsection 3), it follows that $B^+ = k[G(B)]$.

Let $\{e_{ij} \mid 1 \leq i, j \leq 2\}$ be a comatrix basis of C such that $S^2(e_{ij}) = (-1)^{i-j} e_{ij}$ by 4.1. Then $\pi(e_{11}), \pi(e_{22}) \in B^+ = k[G(B)]$ and hence

$$\Delta(\pi(e_{11})) = \pi(e_{11}) \otimes \pi(e_{11}) + \pi(e_{12}) \otimes \pi(e_{21}) \in B^+ \otimes B^+.$$

Since $\pi(e_{11})\otimes\pi(e_{11}) \in B^+\otimes B^+$ then $\pi(e_{12})\otimes\pi(e_{21}) \in B^+\otimes B^+$. But $\pi(e_{12})$, $\pi(e_{21}) \in B^-$, which forces $\pi(e_{12}) = 0$ or $\pi(e_{21}) = 0$. Then $\pi(e_{11}), \pi(e_{22}) \in G(B)$. We may assume that $\pi(e_{21}) = 0$. Let $n \in \mathbb{N}$ such that $\pi(g^n e_{11}) = 1$. From

$$\Delta(g^n e_{11}) = g^n e_{11} \otimes g^n e_{11} + g^n e_{12} \otimes g^n e_{21},$$
$$\Delta(g^n e_{21}) = g^n e_{21} \otimes g^n e_{11} + g^n e_{22} \otimes g^n e_{21},$$

it follows that $1, g^n e_{11}, g^n e_{21} \in H^{co\pi}$ and then $2 = \dim H^{co\pi} \geq 3$ (the equality follows from [S, Thm. 2.4]), which is impossible. Then H^* is pointed if $G(H) \simeq C_4$.

If $G(H) \simeq C_2 \times C_2$, there exists $g \in G(H)$, $g \neq 1$ such that $\pi(g) = 1$, because $G(B)$ is cyclic. We claim that $\mathrm{ad}_\ell(g)(C) = C$. Indeed, note that either

$$L_g(C) = C \Leftrightarrow L_g(D) = D \,(\text{and applying } S) \Leftrightarrow R_g(C) = C \quad \text{or} \tag{34}$$

$$L_g(C) = D \,(\text{and applying } S) \Leftrightarrow R_g(D) = C. \tag{35}$$

In any case we obtain that $\mathrm{ad}_\ell(g)(C) = C$. If $g \in \mathcal{Z}(H)$, then H^* is pointed by 5.12. If $g \notin \mathcal{Z}(H)$, then H^* is pointed by 4.2. In both cases we get a contradiction to the assumption that H^* is non-pointed.

□

5.4. **Type** $(2, n)$**.** In this last subsection we prove that if H is of type $(2, n)$, $1 \leq n \leq 3$, then H^* is pointed.

Proposition 5.14. *(i) Let H be of type (2,1) or (2,3). Then H^* is pointed.*
(ii) If H is of type (2,2) and has a simple subcoalgebra C of dimension 4 *stable by the antipode, then H^* is pointed.*

Proof. If H is a Hopf algebra satisfying the hypothesis of (i), then it contains a simple subcoalgebra of dimension 4 stable by the antipode. This is clear when H is of type $(2, 1)$ and when H is of type $(2, 3)$, the claim follows since $\operatorname{ord} S$ is a power of 2 by Radford's formula. We denote by C such a simple subcoalgebra.

We prove (i) and (ii) simultaneously. Let K be the Hopf subalgebra generated by C. By [NZ], $K = H$ or $K \simeq \mathcal{A}$, since K is non-pointed by construction and non-semisimple because $|G(K)| \leq |G(H)| = 2$.

If $K = H$, then H^* is pointed by 4.5, and the claim is proved.

Now let $K \simeq \mathcal{A}$ and assume that H^* is non-pointed. Since H^* does not have the Chevalley property, H^* must be of type $(2, n)$ by 5.4, 5.8, 5.9 and 5.13. Then by 5.6, there exists $\pi : H \to T_{-1}$ an epimorphism of Hopf algebras. Now we restrict π to $K \simeq \mathcal{A}$. Then by 3.4 (i), $K^{\mathrm{co}\,\pi}$ contains a copy of T_{-1}. Hence by 2.3, $H^{\mathrm{co}\,\pi} \simeq T_{-1}$ and H fits into an exact sequence $T_{-1} \overset{\imath}{\hookrightarrow} H \overset{\pi}{\twoheadrightarrow} T_{-1}$. But this cannot occur, since by 2.8 H should be pointed. Then H^* must be pointed. □

Proposition 5.15. *Let H be of type (2,2). Then H^* is pointed.*

Proof. Suppose that H^* is non-pointed. Then H^* also is of type $(2, 2)$, by 5.4, 5.8, 5.9, 5.13 and 5.14.

Let C and D be the two simple subcoalgebras of H of dimension 4. By 5.14 (ii), S permutes them and by [NZ], H is generated as an algebra by C and D; in particular H is also generated as an algebra by C and 1 by 5.7.

We split the proof of the proposition into several claims.

<u>*Claim*</u> 5.16. *(i) There exists $\pi : H \twoheadrightarrow T_{-1}$ an epimorphism of Hopf algebras.*
(ii) Let $1 \neq g \in G(H)$. Then $\pi(g) \neq 1$.
(iii) $S^2 = \mathrm{ad}_\ell(g)$.

Indeed, (i) follows from 5.6 (ii) applied to H^*. Using (34) and (35), it follows that $\mathrm{ad}_\ell(g)$ fixes C and D. Since π is an epimorphism and by 5.6 (iv) $g \notin \mathcal{Z}(H)$, we have by 4.2 that $\pi(g) \neq 1$, and (ii) follows.

We next prove (iii). By 4.1 there is a comatrix basis $\{e_{ij} \mid 1 \leq i, j \leq 2\}$ of C and $\omega \in k$ such that $gS^2(e_{ij})g = \omega^{i-j} e_{ij}$. Applying π when $i = 2$ and $j = 1$, we get

$$\omega\pi(e_{21}) = \pi(gS^2(e_{21})g) = \pi(g)S^2(\pi(e_{21}))\pi(g) = \pi(e_{21}).$$

The last equality follows from (ii) and the definition of the antipode of T_{-1}. The same holds true with e_{12} instead of e_{21}. Now, if $\omega \neq 1$ then $\pi(e_{12}) = \pi(e_{21}) = 0$, and hence $\pi(H) \subseteq k[G(T_{-1})]$. A contradiction, since π is an epimorphism. Thus $\omega = 1$ and (iii) follows. The claim is proved.

Let $\mathcal{E} := \{e_{ij} \mid 1 \leq i, j \leq 2\}$ denote a comatrix basis of C such that $S^2(e_{ij}) = ge_{ij}g = (-1)^{i-j} e_{ij}$ given by 4.1. Then, as in the proof of [N, Lemm. 2.7], the elements of $\mathcal{E}$ satisfy

$$\pi(e_{12}) = 0 \neq \pi(e_{21}) \in \mathcal{P}(T_{-1}) \quad \text{or} \quad \pi(e_{21}) = 0 \neq \pi(e_{12}) \in \mathcal{P}(T_{-1}). \tag{36}$$

and

$$\pi(e_{11}) = \pi(g) \text{ and } \pi(e_{22}) = 1 \quad \text{or} \quad \pi(e_{11}) = 1 \text{ and } \pi(e_{22}) = \pi(g), \tag{37}$$

The following claim is inspired by the proof of [BD, Prop. 5.3].

<u>Claim</u> 5.17. *If $f_{ij} := S(e_{ji})$ for $1 \leq i, j \leq 2$ then*

$$e_{11}f_{22} = f_{22}e_{11} = e_{22}f_{11} = f_{11}e_{22} = g \quad \textit{and} \tag{38}$$

$$e_{12}f_{21} = f_{21}e_{12} = e_{21}f_{12} = f_{12}e_{21} = 0. \tag{39}$$

Indeed, as in [BD, Prop. 5.3], we define

$$E_{11} := e_{11}f_{22},\ E_{12} := e_{12}f_{21},\ E_{21} := e_{21}f_{12} \text{ and } E_{22} = e_{22}f_{11}$$

$$F_{11} := f_{11}e_{22},\ F_{12} := f_{12}e_{21},\ F_{21} := f_{21}e_{12} \text{ and } F_{22} = f_{22}e_{11}.$$

Note that, as in [BD, Prop. 5.3], the coalgebra E generated by the E_{ij}'s is stable by S. The same holds true for F, the coalgebra generated by the F_{ij}'s. Since S permutes C and D, $\dim E$ and $\dim F$ are less than 4. We claim that neither $1 \in E$ nor $1 \in F$. Indeed, if $1 = \sum_{ij} a_{ij} E_{ij}$ with $a_{ij} \in k$ then we get a contradiction by writing:

$$1 = \pi(1) = \sum_{ij} a_{ij}\pi(E_{ij}) = (a_{11} + a_{22})\pi(g),$$

where the last equality follows from (37) and (36). The same holds true if we suppose $1 \in F$.

Then by [BD, Thm. 2.1], $E = F = k \cdot g$ and hence (38) and (39) hold. The claim is proved.

<u>Claim</u> 5.18. *There exists a Hopf subalgebra of H isomorphic to $\mathcal{A}_2$ (see Subsection 3).*

Indeed, let $x := f_{11}e_{12}$. Since $0 = \varepsilon(e_{12}) = f_{11}e_{12} + f_{21}e_{22}$, it follows that $x = -f_{21}e_{22}$. Then

$$\begin{aligned}
&\Delta(x) = \Delta(f_{11})\Delta(e_{12})\\
&= f_{11}e_{11} \otimes f_{11}e_{12} + f_{12}e_{11} \otimes f_{21}e_{12} + f_{11}e_{12} \otimes f_{11}e_{22} + f_{12}e_{12} \otimes f_{21}e_{22}\\
&= f_{11}e_{11} \otimes x + f_{12}e_{12} \otimes 0 + x \otimes g + f_{12}e_{12} \otimes (-x) \quad \text{[by (39) and (38)]}\\
&= (f_{11}e_{11} - f_{12}e_{12}) \otimes x + x \otimes g\\
&= 1 \otimes x + x \otimes g \qquad\qquad \text{[by } 1 = \varepsilon(f_{11}) = m(\mathrm{id} \otimes S)\Delta(f_{11})].
\end{aligned}$$

Moreover, since f_{11} is invertible and $e_{12} \neq 0$, it follows that $x \neq 0$.

Also, let $y := f_{22}e_{21}$. Since $0 = \varepsilon(e_{21}) = f_{12}e_{11} + f_{22}e_{21}$, it follows that $y = -f_{12}e_{11}$. Then

$$\begin{aligned}
&\Delta(y) = \Delta(f_{22})\Delta(e_{21})\\
&= f_{21}e_{21} \otimes f_{12}e_{11} + f_{22}e_{21} \otimes f_{22}e_{11} + f_{21}e_{22} \otimes f_{12}e_{21} + f_{22}e_{22} \otimes f_{22}e_{21}\\
&= f_{21}e_{21} \otimes (-y) + y \otimes g + f_{21}e_{22} \otimes 0 + f_{22}e_{22} \otimes y \quad \text{[by (38) and (39)]}\\
&= (f_{22}e_{22} - f_{21}e_{21}) \otimes y + y \otimes g\\
&= 1 \otimes y + y \otimes g \qquad\qquad \text{[by } 1 = \varepsilon(f_{22}) = m(\mathrm{id} \otimes S)\Delta(f_{22})].
\end{aligned}$$

Since f_{22} is invertible and $e_{21} \neq 0$, it follows that $y \neq 0$.

If $\{1 - g, x, y\}$ are linearly independent then the claim follows. In fact, the Hopf subalgebra generated by $\{g, x, y\}$ must be of dimension 8 by [NZ], and by Subsection 3 it must be isomorphic to $\mathcal{A}_2$.

We next prove that $\{1 - g, x, y\}$ are linearly independent. Let $a, b, c \in k$ such that $0 = a(1 - g) + bx + cy$. Appliying π we get that $-a(1 - \pi(g)) = b\pi(x) + c\pi(y)$. But by the Claim 5.16 (ii), we have that $\pi(g) \neq 1$ and by (36) and (37), $\pi(x) = 0$ or $\pi(y) = 0$. Hence, $\pi(g)$ is the group-like element of T_{-1} and $\pi(x)$ or $\pi(y)$ is the skew-primitive. Then $a = 0$ and $b = 0$ or $c = 0$. But if $b \neq 0$ or $c \neq 0$, then $x = 0$ or $y = 0$; a contradiction. Hence $a = b = c = 0$ that is, $\{1 - g, x, y\}$ are linearly independent. Hence H contains a Hopf subalgebra isomorphic to $\mathcal{A}_2$. This proves the claim.

Now, since $\mathcal{A}_2 \simeq (\mathcal{A}_2)^*$, applying 5.18 to H^*, we have that there exists $\Pi : H \to \mathcal{A}_2$ an epimorphism of Hopf algebras. Denote by $\Pi\mathcal{E}$ the subcoalgebra of $\mathcal{A}_2$ generated by the $\Pi(e_{ij})$'s. We will get a contradiction by trying to find the dimension of $\Pi\mathcal{E}$. Since $(\mathcal{A}_2)_0 \simeq C_2$, $\dim \Pi\mathcal{E} < 4$ and since Π is an epimorphism, $\dim \Pi\mathcal{E} \neq 0$. Next we apply [BD, Thm. 2.1] to $\Pi\mathcal{E}$: if $\dim \Pi\mathcal{E} \leq 2$ then $\Pi\mathcal{E} = \pi(C) \subseteq G(\mathcal{A}_2)$ by [BD, Thm. 2.1], and hence $\pi(H) \subseteq k[G(\mathcal{A}_2)]$; a contradiction. If $\dim \Pi\mathcal{E} = 3$, by [BD, Thm. 2.1], $\Pi\mathcal{E}$ is the linear span of two group-likes and a skew-primitive. Thus $\Pi\mathcal{E} = \Pi(C)$ is contained in a Hopf subalgebra of $\mathcal{A}_2$ isomorphic to T_{-1}. Then $\Pi(H) \subsetneq \mathcal{A}_2$, a contradiction.

Summarizing, H^* cannot have a Hopf subalgebra isomorphic to $\mathcal{A}_2$. Then H^* indeed is pointed. □

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