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Quantum subgroups of a simple quantum group at roots of 1

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REPÚBLICA ARGENTINA

QUANTUM SUBGROUPS OF A SIMPLE QUANTUM GROUP AT ROOTS OF 1

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Abstract. Let G be a connected, simply connected, simple complex algebraic group and let ϵ be a primitive ℓ-th root of 1, ℓ odd and $3 \nmid \ell$ if G is of type G_2. We determine all Hopf algebra quotients of the quantized coordinate algebra $\mathcal{O}_\epsilon(G)$.

1. Introduction and preliminaries

1.1. **Introduction.** The purpose of this paper is to determine all quantum subgroups of a quantum group at a root of one, or in equivalent terms, to determine all Hopf algebra quotients of a quantized coordinate algebra at a root of one (over the complex numbers). This problem was first considered by P. Podleś [P95] for quantum $SU(2)$ and $SO(3)$. The characterization of all *finite-dimensional* Hopf algebra quotients of the quantized coordinate algebra $\mathcal{O}_q(SL_N)$ was obtained by Eric Müller [M00]. Müller's approach is via explicit computations with matrix coefficients; this strategy does not apply to more general simple groups.

The present work can be viewed as a continuation of the long tradition of studying subgroups of a simple algebraic group. In fact, our main theorem assumes the knowledge of such subgroups, see Definition 1.1. Besides its intrinsical mathematical interest, our result would have implications in quantum harmonic analysis– see for example [L02]– and in the study of module categories over the tensor category of comodules over the Hopf algebra $\mathcal{O}_\epsilon(G)$– in the sense of [EO04].

An outcome of our main theorem is the construction of many new examples of finite-dimensional Hopf algebras. At the present time, all examples of finite-dimensional Hopf algebras, we are aware of, are:

- group algebras of finite groups,
- small quantum groups introduced by Lusztig [L90a, L90b], and variations thereof [AS],

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- other pointed Hopf algebras with abelian group arising from the Nichols algebras discovered in [Gñ00, He],
- a few examples of pointed Hopf algebras with non-abelian group [MS00, Gñ],
- combinations of the preceding via some standard operations (duals, twisting, Hopf subalgebras and quotients, extensions).

How to build examples of Hopf algebras via extensions of a group algebra by a dual group algebra is well understood– see for instance [Ma02]. Out of this, extensions can in principle be constructed by means of weak actions and coactions, and pairs of compatible 2-cocycles. However, very few explicit examples were presented in this way, to our knowledge no one in finite dimension, except for the trivial tensor product of two Hopf algebras. Our examples are indeed nontrivial extensions of finite quantum groups by finite groups, but it is not clear how they could be explicitly presented through actions, coactions and cocycles. A natural subsequent question is when the new examples of Hopf algebras are isomorphic with each other; this will be addressed in (the forthcoming new version of) [AG].

Furthermore, a result of Ştefan [St99, Thm. 1.5] says that a non-semi-simple finite-dimensional Hopf algebra generated by a simple 4-dimensional coalgebra stable by the antipode is a quotient of the quantized coordinate algebra of $SL(2)$ at a root of one. It is tempting to suggest that finite-dimensional quotients of more general quantized coordinate algebras might play a prominent role in the classification of Hopf algebras.

We notice that a different problem is sometimes referred to with a similar name: this is the classification of indecomposable module categories over fusion categories arising in conformal field theory, *e. g.* from the representation theory of finite quantum groups at roots of one. See [O02, KiO02]. There is no evident relation between these two problems.

1.2. **Statement of the main result.** Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{h} \subseteq \mathfrak{g}$ a fixed Cartan subalgebra, $\Pi = \{\alpha_1, \dots, \alpha_n\}$ a basis of the root system $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and $n = \operatorname{rk} \mathfrak{g}$.

Definition 1.1. A *subgroup datum* is a collection $\mathcal{D} = (I_+, I_-, N, \Gamma, \sigma, \delta)$ where

- $I_+ \subseteq \Pi$ and $I_- \subseteq -\Pi$. Let $\Psi_\pm = \{\alpha \in \Phi : \operatorname{Supp} \alpha \subseteq I_\pm\}$, $\mathfrak{l}_\pm = \sum_{\alpha \in \Psi_\pm} \mathfrak{g}_\alpha$ and $\mathfrak{l} = \mathfrak{l}_+ \oplus \mathfrak{h} \oplus \mathfrak{l}_-$; $\mathfrak{l}$ is an algebraic Lie subalgebra of $\mathfrak{g}$. Let L be the connected Lie subgroup of G with $\mathrm{Lie}(L) = \mathfrak{l}$.
- N is a subgroup of $\widehat{\mathbb{T}_{I^c}}$, see Remark 2.12 below.
- Γ is an algebraic group.
- $\sigma : \Gamma \to L$ is an injective homomorphism of algebraic groups.
- $\delta : N \to \widehat{\Gamma}$ is a group homomorphism.

If Γ is finite, we call $\mathcal{D}$ a *finite subgroup datum*. We parameterize with injective group homomorphisms rather than group inclusions for a better description of the isomorphism classes [AG]. An equivalence relation among subgroup data is defined in Subsection 2.4.

Our main result is the following.

Theorem 1. *There is a bijection between*

(*a*) *Hopf algebra quotients* $q : \mathcal{O}_\epsilon(G) \to A$.
(*b*) *Subgroup data up to equivalence.*

In Section 2, we carry out the construction of a quotient $A_{\mathcal{D}}$ of $\mathcal{O}_\epsilon(G)$ starting from a subgroup datum $\mathcal{D}$, see Theorem 2.17. In Subsection 2.4, we study the lattice of quotients $A_{\mathcal{D}}$. In Section 3, we attach a subgroup datum $\mathcal{D}$ to an arbitrary Hopf algebra quotient A and prove that $A_{\mathcal{D}} \simeq A$ as quotients of $\mathcal{O}_\epsilon(G)$. This concludes the proof of the Theorem 1. As an immediate corollary of Theorem 1, we get

Theorem 2. *There is a bijection between*

(*a*) *Hopf algebra quotients* $q : \mathcal{O}_\epsilon(G) \to A$ *such that* $\dim A < \infty$.
(*b*) *Finite subgroup data up to equivalence.*

Theorem 2 generalizes the main result of [M00].

1.3. **Conventions.** Let $C = (a_{ij})_{1\le i,j\le n}$ be the Cartan matrix of $\mathfrak{g}$ and suppose that $\mathfrak{g}$ is generated by the elements $\{h_i,\ e_i,\ f_i |\ 1 \le i \le n\}$ subject to the Chevalley-Serre relations. Let $Q = \mathbb{Z}\Phi = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$ be the root lattice, $\varpi_1, \dots, \varpi_n$ the fundamental weights, $P = \bigoplus_{i=1}^n \mathbb{Z}\varpi_i$ the weight lattice and W the Weyl group. Let P_+ be the cone of dominant weights and $Q_+ = P_+ \cap P$. Let $(-,-)$ be the positive definite symmetric bilinear form on $\mathfrak{h}^*$ induced by the Killing form of $\mathfrak{g}$. Let $d_i = \frac{(\alpha_i,\alpha_i)}{2} \in \{1, 2, 3\}$.

For $t,\ m \in \mathbb{N}_0$, $q \in \mathbb{C}$ and $u \in \mathbb{Q}(q) \smallsetminus \{0, \pm 1\}$ we denote:

$$[t]_u := \frac{u^t - u^{-t}}{u - u^{-1}}, \quad [t]_u! := [t]_u[t-1]_u \cdots [1]_u, \quad \begin{bmatrix} m \\ t \end{bmatrix}_u := \frac{[m]_u!}{[t]_u![m-t]_u!},$$
$$(t)_u := \frac{u^t - 1}{u - 1}, \quad (t)_u! := (t)_u(t-1)_u \cdots (1)_u, \quad \begin{pmatrix} m \\ t \end{pmatrix}_u := \frac{(m)_u!}{(t)_u!(m-t)_u!}.$$

1.4. **Definitions.** In this subsection we recall the definition of the quantized coordinate algebra of G. Let $R = \mathbb{Q}[q, q^{-1}]$, q an indeterminate. If $p_\ell(q) \in R$ denotes the ℓ-th cyclotomic polynomial, then $R/[p_\ell(q)R] \simeq \mathbb{Q}(\epsilon)$.

Definition 1.2. The *simply connected* quantized enveloping algebra $\check{U}_q(\mathfrak{g})$ of $\mathfrak{g}$ is the $\mathbb{Q}(q)$-algebra with generators $\{K_\lambda |\ \lambda \in P\}$, $E_1, \dots, E_n$ and $F_1, \dots, F_n$, satisfying the following relations for $\lambda,\ \mu \in P$ and $1 \le i, j \le n$:

$$
\begin{aligned}
K_0 = 1, \qquad & K_\lambda K_\mu = K_{\lambda+\mu}, \\
K_\lambda E_j K_{-\lambda} = q^{(\lambda,\alpha_j)} E_j, \qquad & K_\lambda F_j K_{-\lambda} = q^{-(\lambda,\alpha_j)} F_j, \\
E_i F_j - F_j E_i &= \delta_{ij} \frac{K_{\alpha_i} - K_{\alpha_i}^{-1}}{q_i - q_i^{-1}}, \\
\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i} & E_i^{1-a_{ij}-l} E_j E_i^l = 0 \qquad (i \neq j), \\
\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i} & F_i^{1-a_{ij}-l} F_j F_i^l = 0 \qquad (i \neq j).
\end{aligned}
$$

Definition 1.3. [DL94, Section 3.4] Let $q_i = q^{d_i}$, $1 \leq i \leq n$. The algebra $\Gamma(\mathfrak{g})$ is the R-subalgebra of $\check{U}_q(\mathfrak{g})$ generated by the elements

$$
\begin{aligned}
& K_{\alpha_i}^{-1} && (1 \leq i \leq n), \\
& \begin{pmatrix} K_{\alpha_i}; 0 \\ t \end{pmatrix} := \prod_{s=1}^{t} \left(\frac{K_{\alpha_i} q_i^{-s+1} - 1}{q_i^s - 1} \right) && (t \geq 1,\ 1 \leq i \leq n), \\
& E_i^{(t)} := \frac{E_i^t}{[t]_{q_i}!} && (t \geq 1,\ 1 \leq i \leq n), \\
& F_i^{(t)} := \frac{F_i^t}{[t]_{q_i}!} && (t \geq 1,\ 1 \leq i \leq n).
\end{aligned}
$$

Let $\mathcal{C}$ be the strictly full subcategory of $\Gamma(\mathfrak{g})$-mod whose objects are $\Gamma(\mathfrak{g})$-modules M such that M is a free R-module of finite rank and the operators K_{α_i} and $\begin{pmatrix} K_{\alpha_i}; 0 \\ t \end{pmatrix}$ are diagonalizable with eigenvalues q_i^m and $\binom{m}{t}_{q_i}$ respectively, for some $m \in \mathbb{N}$ and for all $1 \leq i \leq n$.

Definition 1.4. [DL94, Section 4.1] Let $R_q[G]$ denote the R-submodule of $\operatorname{Hom}_R(\Gamma(\mathfrak{g}), R)$ spanned by the coordinate functions t_i^j of representations M from $\mathcal{C}$: $< g, t_i^j > = < g \cdot m_i, m^j >$, where (m_i) is an R-basis of M, (m^j) is the dual basis of the dual module and $g \in \Gamma(\mathfrak{g})$. Since the subcategory $\mathcal{C}$ is a tensor one, $R_q[G]$ is a Hopf algebra.

Definition 1.5. [DL94, Section 6] The algebra $R_q[G]/[p_\ell(q) R_q[G]]$ is denoted by $\mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)}$ and is called the *quantized coordinate algebra* of G over $\mathbb{Q}(\epsilon)$ at the root of unity ϵ. In the same way as for $\mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)}$, we can form the $\mathbb{Q}(\epsilon)$-Hopf algebra $\Gamma_\epsilon(\mathfrak{g}) := \Gamma(\mathfrak{g})/[p_\ell(q)\Gamma(\mathfrak{g})]$.

We now relate the Hopf algebras $\mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)}$ and $\Gamma_\epsilon(\mathfrak{g})$.

Definition 1.6. A *Hopf pairing* between two Hopf algebras U and H over a ring $\mathcal{R}$ is a bilinear form $(-,-) : H \times U \to \mathcal{R}$ such that, for all $u,\ v \in U$

and $f,\ h \in H$,

$$(i) \quad (h, uv) = (h_{(1)}, u)(h_{(2)}, v); \qquad (iii) \quad (1, u) = \varepsilon(u);$$
$$(ii) \quad (fh, u) = (f, u_{(1)})(h, u_{(2)}); \qquad (iv) \quad (h, 1) = \varepsilon(h).$$

It follows that $(h, \mathcal{S}(u)) = (\mathcal{S}(h), u)$, for all $u \in U$, $h \in H$. Given a Hopf pairing, one has Hopf algebra maps $U \to H^\circ$ and $H \to U^\circ$, where H° and U° are the Sweedler duals. The pairing is called *perfect* if these maps are injections.

Proposition 1.7. [DL94, 4.1 and 6.1] *There exists a perfect Hopf pairing $R_q[G] \otimes_R \Gamma(\mathfrak{g}) \to R$, which induces a perfect Hopf pairing $\mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)} \otimes_{\mathbb{Q}(\epsilon)} \Gamma_\epsilon(\mathfrak{g}) \to \mathbb{Q}(\epsilon)$. In particular, $\mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)} \subseteq \Gamma_\epsilon(\mathfrak{g})^\circ$ and $\Gamma_\epsilon(\mathfrak{g}) \subseteq \mathcal{O}_\epsilon(G)^\circ_{\mathbb{Q}(\epsilon)}$.* $\square$

If k is any field containing $\mathbb{Q}(\epsilon)$, we denote $\mathcal{O}_\epsilon(G)_k := \mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)} \otimes_{\mathbb{Q}(\epsilon)} k$. When $k = \mathbb{C}$ we simply write $\mathcal{O}_\epsilon(G)$ for $\mathcal{O}_\epsilon(G)_{\mathbb{C}}$. The following two results imply by [Mo93, Prop. 3.4.3] that $\mathcal{O}_\epsilon(G)$ is a central extension of $\mathcal{O}(G)$ by a finite-dimensional Hopf algebra.

Theorem 1.8. (a) [DL94, Prop. 6.4] *$\mathcal{O}_\epsilon(G)$ contains a central Hopf subalgebra isomorphic to the coordinate algebra $\mathcal{O}(G)$ of G.*
(b) [BG, III.7.11] *$\mathcal{O}_\epsilon(G)$ is a free $\mathcal{O}(G)$-module of rank $\ell^{\dim G}$.* $\square$

We end this section by spelling out explicitly the quotient of $\mathcal{O}_\epsilon(G)$ by its central Hopf subalgebra $\mathcal{O}(G)$.

Let $\overline{\mathcal{O}_\epsilon(G)} = \mathcal{O}_\epsilon(G)/[\mathcal{O}(G)^+\mathcal{O}_\epsilon(G)]$ and denote by $\pi : \mathcal{O}_\epsilon(G) \to \overline{\mathcal{O}_\epsilon(G)}$ the quotient map. By Theorem 1.8 and [Mo93, Prop. 3.4.3], $\overline{\mathcal{O}_\epsilon(G)}$ is a Hopf algebra of dimension $\ell^{\dim G}$ which fits into the exact sequence

$$1 \to \mathcal{O}(G) \to \mathcal{O}_\epsilon(G) \to \overline{\mathcal{O}_\epsilon(G)} \to 1.$$

Let $\mathbf{u}_\epsilon(\mathfrak{g})$ be the *Frobenius-Lusztig kernel* of $\mathfrak{g}$ at ϵ; that is, the Hopf subalgebra of $\Gamma_\epsilon(\mathfrak{g})$ generated by the elements E_i, F_i and K_{α_i} for $1 \leq i \leq n$. See [BG] for details. We denote by

$$\mathbb{T} := \{K_{\alpha_1}, \dots, K_{\alpha_n}\} = G(\mathbf{u}_\epsilon(\mathfrak{g})) \tag{1}$$

the "finite torus" of group-like elements of $\mathbf{u}_\epsilon(\mathfrak{g})$.

Theorem 1.9. [BG, III.7.10] *$\overline{\mathcal{O}_\epsilon(G)} \simeq \mathbf{u}_\epsilon(\mathfrak{g})^*$ as Hopf algebras.* $\square$

Summarizing, the quantized coordinate algebra $\mathcal{O}_\epsilon(G)$ of G at ϵ fits into the central exact sequence

$$1 \to \mathcal{O}(G) \xrightarrow{\iota} \mathcal{O}_\epsilon(G) \xrightarrow{\pi} \mathbf{u}_\epsilon(\mathfrak{g})^* \to 1. \tag{2}$$

We shall need the following technical lemma.

Lemma 1.10. *There exists a surjective algebra map $\varphi : \Gamma_\epsilon(\mathfrak{g}) \to \mathbf{u}_\epsilon(\mathfrak{g})$ such that $\varphi|_{\mathbf{u}_\epsilon(\mathfrak{g})} = \mathrm{id}$.*

Proof. Since $\Gamma_\epsilon(\mathfrak{g}) = \Gamma(\mathfrak{g})/[p_\ell(q)\Gamma(\mathfrak{g})]$, we may define φ as a map from $\Gamma(\mathfrak{g})$ such that $\varphi(q) = \epsilon$. Let φ be the unique algebra map which takes the following values on the generators:

$$\varphi(E_i^{(m)}) = \begin{cases} E_i^{(m)} & \text{if } 1 \leq m < \ell \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi(F_i^{(m)}) = \begin{cases} F_i^{(m)} & \text{if } 1 \leq m < \ell \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi(\left(\begin{smallmatrix} K_{\alpha_i};0 \\ m \end{smallmatrix}\right)) = \begin{cases} \left(\begin{smallmatrix} K_{\alpha_i};0 \\ m \end{smallmatrix}\right) & \text{if } 1 \leq m < \ell \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi(K_{\alpha_i}^{-1}) = K_{\alpha_i}^{\ell-1}, \qquad \varphi(q) = \epsilon,$$

for all $1 \leq i \leq n$. Since φ is the identity on the generators of $\mathbf{u}_\epsilon(\mathfrak{g})$ and $E_i^\ell = 0 = F_i^\ell$, $K_{\alpha_i}^\ell = 1$ on $\mathbf{u}_\epsilon(\mathfrak{g})$, it follows from a direct computation that φ satisfies the relations given in [DL94, Section 3.4], see [G07, 4.1.17] for details. Hence φ is a well-defined algebra map whose image is $\mathbf{u}_\epsilon(\mathfrak{g})$. $\square$

1.5. **Hopf subalgebras of a pointed Hopf algebra.** We describe in this subsection Hopf subalgebras of pointed Hopf algebras. Let U be a Hopf algebra such that the coradical U_0 is a Hopf subalgebra. Let $(U_n)_{n\geq 0}$ be the coradical filtration of U, set $U_{-1} = 0$, $\operatorname{gr} U(n) = U_n/U_{n-1}$ and let $\operatorname{gr} U = \oplus_{n\geq 0} \operatorname{gr} U(n)$ be the associated graded Hopf algebra. Let $\iota : U_0 \to \operatorname{gr} U$ be the canonical inclusion and let $\pi : \operatorname{gr} U \to U_0$ be the homogeneous projection. Let $R = (\operatorname{gr} U)^{\operatorname{co} \pi}$ be the diagram of U; R is a graded braided Hopf algebra, that is, a Hopf algebra in the category ${}^{U_0}_{U_0}\mathcal{YD}$ of Yetter-Drinfeld modules over U_0. Its coalgebra structure is given by $\Delta_R(r) = \vartheta_R(r_{(1)}) \otimes r_{(2)}$, for all $r \in R$, where $\vartheta_R : \operatorname{gr} U \to R$ is the map defined by

$$\vartheta_R(a) = a_{(1)}\iota\pi(\mathcal{S}a_{(2)}), \qquad \forall\, a \in \operatorname{gr} U. \tag{3}$$

It can be easily shown that $\vartheta_R(rh) = r\epsilon(h)$, $\vartheta_R(hr) = h\cdot r$ for $r \in R$, $h \in U_0$. One has that $\operatorname{gr} U \simeq R\#U_0$, $R = \oplus_{n\geq 0} R(n)$, $R(0) \simeq \mathbb{C}$ and $R(1) = \mathcal{P}(R)$. We say that R is a *Nichols algebra* if R is generated as algebra by $R(1)$. See [AS02] for more details.

To state the following result, we need to introduce some terminology. Let A be a Hopf algebra, M a Yetter-Drinfeld module over A and B a Hopf subalgebra of A. We say that a vector subspace N of M is *B-compatible* if

(a) it is stable under the action of B, and
(b) it bears a B-comodule structure inducing the coaction of A.

In inaccurate but descriptive words, "N is a Yetter-Drinfeld submodule over B" (although M is not necessarily a Yetter-Drinfeld module over B).

Lemma 1.11. *Let Y be a Hopf subalgebra of U. Then the coradical Y_0 is a Hopf subalgebra and the diagram S of Y is a braided Hopf subalgebra of R.*

If $R = \mathfrak{B}(V)$ is a Nichols algebra with $\dim V < \infty$, then S is also a Nichols algebra. In this case, Hopf subalgebras of U are parameterized by pairs (Y_0, W) where Y_0 is a Hopf subalgebra of U_0 and $W \subset V = R(1)$ is Y_0-compatible.

Proof. The first claim follows since $Y_0 = Y \cap U_0$ and the intersection of two Hopf subalgebras is a Hopf subalgebra. By [Mo93, Lemma 5.2.12], the coradical filtration of Y is given by $Y_n = Y \cap U_n$; thus we have an injective homogeneous map of Hopf algebras $\gamma : \operatorname{gr} Y \hookrightarrow \operatorname{gr} U$ inducing the commutative diagram

$$\begin{tikzcd} \operatorname{gr} Y \arrow[r, hook, "\gamma"] \arrow[d, "\pi_Y"] & \operatorname{gr} U \arrow[d, "\pi"] \\ Y_0 \arrow[r, hook] & U_0. \end{tikzcd}$$

Thus $S = \{a \in \operatorname{gr} Y : (\mathrm{id} \otimes \pi_Y)\Delta(a) = a \otimes 1\}$ is a subalgebra, and also a braided vector subspace, of R. Note that $\gamma\vartheta_S = \vartheta_R\gamma$, *cf.* (3); thus S is a subcoalgebra of R. Assume now that $R \simeq \mathfrak{B}(V)$ is a Nichols algebra with $\dim V < \infty$. Taking graded duals, we have a surjective map of graded braided Hopf algebras $\wp : \mathfrak{B}(V^*) \to S^{\operatorname{gr dual}}$. Since $\mathfrak{B}(V^*)$ and $S^{\operatorname{gr dual}}$ are pointed irreducible coalgebras, by [Sw69, Thm. 9.1.4], $\wp$ maps the coradical filtration of the first onto the coradical filtration of the second; hence $\mathcal{P}(S^{\operatorname{gr dual}}) = S^{\operatorname{gr dual}}(1)$ and *a fortiori* S is generated in degree 1, *i. e.* is a Nichols algebra. Furthermore, Y is determined by Y_0 and $S(1)$, the last being Y_0-compatible. Conversely, if Y_0 is a Hopf subalgebra of U_0 and $W \subset R(1)$ is Y_0-compatible, then choose $(y_i)_{i\in I}$ in U_1 such that the classes $(\overline{y}_i)_{i\in I}$ in U_1/U_0 generate $W\#1$. Then the subalgebra Y of U generated by Y_0 and $(y_i)_{i\in I}$ is a actually a Hopf subalgebra giving rise to the pair (Y_0, W). □

The lemma above also holds if V is a locally finite braided vector space.

Let us now turn to Hopf subalgebras of pointed Hopf algebras. The notion of “compatibility” for groups reads as follows. Let G be a group and M a Yetter-Drinfeld module over the group algebra $\mathbb{C}[G]$. If F is a subgroup of G, a vector subspace N of M is *F-compatible* if

(a) it is stable under the action of F, and
(b) it is a $\mathbb{C}[G]$-subcomodule and $\operatorname{Supp} N := \{g \in G : N^g \neq 0\}$ is contained in F.

Corollary 1.12. *Let U be a pointed Hopf algebra whose diagram R is a Nichols algebra. Then Hopf subalgebras of U are parameterized by pairs (F, W) where F is a subgroup of $G(U)$ and $W \subset R(1)$ is F-compatible.* □

The Corollary reads even nicer if $G(U)$ is abelian and $\dim R(1)^g = 1$ for all $g \in \operatorname{Supp} R(1)$. Indeed, Hopf subalgebras of U are parameterized in this case by pairs (F, J) where F is a subgroup of $G(U)$ and $J \subset \operatorname{Supp} R(1)$ is contained in F. We recover in this way results from [CM96, M98].

Corollary 1.13. [M98, Thm. 6.3] *The Hopf subalgebras of* $\mathbf{u}_\epsilon(\mathfrak{g})$ *are parameterized by triples* (Σ, I_+, I_-)*, where* Σ *is a subgroup of* $\mathbb{T}$ *and* $I_+ \subseteq \Pi$, $I_- \subseteq -\Pi$ *such that* $K_{\alpha_i} \in \Sigma$ *if* $\alpha_i \in I_+ \cup -I_-$*.* □

1.6. **A five-lemma for extensions of Hopf algebras.** The following general lemma was kindly communicated to us by Akira Masuoka.

Lemma 1.14. *Let* H *be a bialgebra over an arbitrary commutative ring, and let* A*,* A' *be right* H*-Galois extensions over a common algebra* B *of* H*-coinvariants. Assume that* A' *is right* B*-faithfully flat. Then any* H*-comodule algebra map* $\theta : A \to A'$ *that is identical on* B *is an isomorphism.*

Proof. Let $\beta : A \otimes_B A \to A \otimes H$, $\beta(x \otimes y) = xy_{(0)} \otimes y_{(1)}$ and $\beta' : A' \otimes_B A' \to A' \otimes H$, $\beta'(x' \otimes y') = x'y'_{(0)} \otimes y'_{(1)}$ be the corresponding Galois maps, for $x, y \in A$, $x', y' \in A'$. Using the A-module structure of A' given by θ, we can extend β to an isomorphism

$$\alpha : A' \otimes_B A \simeq A' \otimes_A A \otimes_B A \xrightarrow{\mathrm{id} \otimes \beta} A' \otimes_A A \otimes H \simeq A' \otimes H.$$

Explicitly, $\alpha(a' \otimes a) = a'\theta(a_{(0)}) \otimes a_{(1)}$ for all $a' \in A'$, $a \in A$. Then α fits into the following commutative diagram

$$\begin{tikzcd} A' \otimes_B A \arrow[rr, "\mathrm{id} \otimes \theta"] \arrow[dr, "\simeq", "\alpha"'] & & A' \otimes_B A' \arrow[dl, "\simeq"', "\beta"] \\ & A' \otimes H & \end{tikzcd}$$

Hence $\mathrm{id} \otimes \theta$ is an isomorphism; since A' is right B-faithfully flat, θ is an isomorphism. □

The lemma applies to a commutative diagram of Hopf algebras

$$\begin{tikzcd} 1 \arrow[r] & B \arrow[r, "\iota"] \arrow[d, equal] & A \arrow[r, "\pi"] \arrow[d, "\theta"] & H \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & B \arrow[r, "\iota'"] & A' \arrow[r, "\pi'"] & H \arrow[r] & 1, \end{tikzcd} \tag{4}$$

where the rows are exact sequences of Hopf algebras, in the sense of [AD95]: $A^{\mathrm{co}\,\pi} = B$ and $\ker \pi = B^+A$; *ditto* for A'. If the top row is a cleft exact sequence, then θ is an isomorphism [AD95, Lemma 3.2.19]. Masuoka's Lemma 1.14 implies another version of the five-lemma: If A and A' are H-Galois over B, and A' is right B-faithfully flat, then θ is also an isomorphism.

Corollary 1.15. *Assume in* (4) *that* $\dim H$ *is finite,* A' *is noetherian and* B *is central in* A'*. Then* θ *is an isomorphism.*

Proof. As the rows are exact, the corresponding Galois maps β and β' are surjective; since $\dim H < \infty$, they are bijective [KT81, Thm. 1.7]. Thus A and A' are H-Galois over B. Now A' is B-faithfully flat by [S93, Thm. 3.3]. □

2. Constructing quantum subgroups

In this section we construct quotients of the quantized coordinate algebra $\mathcal{O}_\epsilon(G)$. We do this in three steps.

2.1. **First step.** We construct in this subsection a quotient of $\mathcal{O}_\epsilon(G)$ associated to a Hopf subalgebra of $\mathbf{u}_\epsilon(\mathfrak{g})$; it corresponds to a connected Lie subgroup L of G. Let $r : \mathbf{u}_\epsilon(\mathfrak{g})^* \to H$ be a surjective Hopf algebra morphism. Then we have an injective Hopf algebra map ${}^t r : H^* \to \mathbf{u}_\epsilon(\mathfrak{g})$ and by Corollary 1.13, the Hopf algebra H^* corresponds to a triple (Σ, I_+, I_-). We shall eventually show that this triple is part of a subgroup datum as in Definition 1.1.

2.1.1. *The Hopf subalgebra* $\Gamma_\epsilon(\mathfrak{l})$ *of* $\Gamma_\epsilon(\mathfrak{g})$.

Definition 2.1. For every triple (Σ, I_+, I_-) define $\Gamma(\mathfrak{l})$ to be the subalgebra of $\Gamma(\mathfrak{g})$ generated by the elements

$$\begin{aligned}
&K_{\alpha_i}^{-1} && (1 \leq i \leq n),\\
&\begin{pmatrix} K_{\alpha_i};0 \\ m \end{pmatrix} := \prod_{s=1}^{m} \left(\frac{K_{\alpha_i} q_i^{-s+1} - 1}{q_i^s - 1} \right) && (m \geq 1,\ 1 \leq i \leq n),\\
&E_j^{(m)} := \frac{E_j^m}{[m]_{q_j}!} && (m \geq 1,\ j \in I_+),\\
&F_k^{(m)} := \frac{F_k^m}{[m]_{q_k}!} && (m \geq 1,\ k \in I_-),
\end{aligned}$$

where $q_i = q^{d_i}$ for $1 \leq i \leq n$. Note that $\Gamma(\mathfrak{l})$ does not depend on Σ.

Choosing a reduced expression $s_{i_1} \cdots s_{i_N}$ of the longest element of the Weyl group one can order totally the positive part Φ_+ of the root system Φ with $\beta_1 = \alpha_{i_1}$, $\beta_2 = s_{i_1}\alpha_{i_2}, \ldots,$ $\beta_N = s_{i_1} \cdots s_{i_{N-1}} \alpha_{i_N}$. Then using the algebra automorphisms T_i introduced by Lusztig [L90b], one may define corresponding root vectors $E_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}} E_{i_k}$ and $F_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}} F_{i_k}$. Consider now the R-submodules of $\Gamma(\mathfrak{g})$ given by

$$J_\ell = R\Big\{ \prod_{\beta \geq 0} F_\beta^{(n_\beta)} \cdot \prod_{i=1}^{n} \begin{pmatrix} K_{\alpha_i};0 \\ t_i \end{pmatrix} K_{\alpha_i}^{\mathrm{Ent}(t_i/2)} \cdot \prod_{\alpha \geq 0} E_\alpha^{(m_\alpha)} : \\ \exists\ n_\beta, t_i, m_\alpha \not\equiv 0 \mod (\ell) \Big\}$$

$$\Gamma_\ell = R\Big\{ \prod_{\beta \geq 0} F_\beta^{(n_\beta)} \cdot \prod_{i=1}^{n} \begin{pmatrix} K_{\alpha_i};0 \\ t_i \end{pmatrix} K_{\alpha_i}^{\mathrm{Ent}(t_i/2)} \cdot \prod_{\alpha \geq 0} E_\alpha^{(m_\alpha)} : \\ \forall\ n_\beta, t_i, m_\alpha \equiv 0 \mod (\ell) \Big\}$$

Then, by [DL94, Thm. 6.3] there is a decomposition of free R-modules $\Gamma(\mathfrak{g}) = J_\ell \otimes \Gamma_\ell$ and $\Gamma_\ell/[p_\ell(q)\Gamma_\ell] \simeq U(\mathfrak{g})_{\mathbb{Q}(\epsilon)}$. Let $Q_{I_\pm} = \bigoplus_{i\in I_\pm} \mathbb{Z}\alpha_i$ and define the following R-submodules of $\Gamma(\mathfrak{l})$:

$$W_\ell = R\Big\{ \prod_{\beta\geq 0} F_\beta^{(n_\beta)} \cdot \prod_{i=1}^{n} \left(\begin{matrix} K_{\alpha_i};0 \\ t_i \end{matrix} \right) K_{\alpha_i}^{\mathrm{Ent}(t_i/2)} \cdot \prod_{\alpha\geq 0} E_\alpha^{(m_\alpha)} :$$
$$\exists\ n_\beta, t_i, m_\alpha \not\equiv 0 \mod (\ell) \text{ with } \beta\in Q_{I_-}, \alpha \in Q_{I_+}, 1\leq i\leq n \Big\}$$

$$\Theta_\ell = R\Big\{ \prod_{\beta\geq 0} F_\beta^{(n_\beta)} \cdot \prod_{i=1}^{n} \left(\begin{matrix} K_{\alpha_i};0 \\ t_i \end{matrix} \right) K_{\alpha_i}^{\mathrm{Ent}(t_i/2)} \cdot \prod_{\alpha\geq 0} E_\alpha^{(m_\alpha)} :$$
$$\forall\ n_\beta, t_i, m_\alpha \equiv 0 \mod (\ell) \text{ with } \beta\in Q_{I_-}, \alpha \in Q_{I_+}, 1\leq i\leq n \Big\}$$

Using the decomposition of $\Gamma(\mathfrak{g})$ as free R-module we get the following.

Lemma 2.2. *There is a decomposition of free R-modules $\Gamma(\mathfrak{l}) = W_\ell \otimes \Theta_\ell$. In particular, $\Gamma(\mathfrak{l})$ is a direct summand of $\Gamma(\mathfrak{g})$.*

Proof. Clearly, $\Gamma(\mathfrak{l})$ contains the free R-module $W_\ell \otimes \Theta_\ell$. Thus, it is enough to show that $\Gamma(\mathfrak{l}) \subseteq W_\ell \otimes \Theta_\ell$, but this follows directly from the fact that $\Gamma(\mathfrak{l})$ is generated as an algebra over R by the elements in Definition 2.1 and these generators satisfy the relations given in [DL94, Sec. 3.4]. □

Let $\Gamma_\epsilon(\mathfrak{l}) := \Gamma(\mathfrak{l})/[p_\ell(q)\Gamma(\mathfrak{l})]$. Then we have the following proposition.

Proposition 2.3. (*a*) *$\Gamma_\epsilon(\mathfrak{l})$ is a Hopf subalgebra of $\Gamma_\epsilon(\mathfrak{g})$.*
(*b*) *$\Gamma_\epsilon(\mathfrak{g}) \simeq \Gamma(\mathfrak{g}) \otimes_R R/[p_\ell(q)R]$ and $\Gamma_\epsilon(\mathfrak{l}) \simeq \Gamma(\mathfrak{l}) \otimes_R R/[p_\ell(q)R]$.*

Proof. We prove only (a) since (b) is straightforward. By definition, the elements E_j are $(K_{\alpha_j}, 1)$-primitives, the F_k's are $(1, K_{\alpha_k}^{-1})$-primitives and the K_{α_i}'s are group-like. Moreover, the antipode is given by $\mathcal{S}(K_{\alpha_i}) = K_{\alpha_i}^{-1}$, $\mathcal{S}(E_j) = -K_{\alpha_j}^{-1}E_j$ and $\mathcal{S}(F_k) = -F_k K_{\alpha_k}$ with $1\leq i\leq n$, $j\in I_+$ and $k\in I_-$. Hence, the subalgebra of $\Gamma(\mathfrak{l})$ generated by these elements is a Hopf subalgebra of $\Gamma(\mathfrak{g})$ and $\Gamma(\mathfrak{l})/[p_\ell(q)\Gamma(\mathfrak{g})\cap\Gamma(\mathfrak{l})]$ is a Hopf subalgebra of $\Gamma_\epsilon(\mathfrak{g})$. But by Lemma 2.2, we know that $\Gamma(\mathfrak{g}) = \Gamma(\mathfrak{l})\oplus N$ for some R-submodule N. Then $p_\ell(q)\Gamma(\mathfrak{g})\cap\Gamma(\mathfrak{l}) = p_\ell(q)(\Gamma(\mathfrak{l})\oplus N)\cap\Gamma(\mathfrak{l}) = p_\ell(q)\Gamma(\mathfrak{l})$, which implies that $\Gamma_\epsilon(\mathfrak{l}) = \Gamma(\mathfrak{l})/[p_\ell(q)\Gamma(\mathfrak{g})\cap\Gamma(\mathfrak{l})]$. □

2.1.2. *The regular Frobenius-Lusztig kernel $\mathbf{u}_\epsilon(\mathfrak{l})$.* Let $\mathbf{u}_\epsilon(\mathfrak{l})$ be the subalgebra of $\Gamma_\epsilon(\mathfrak{l})$ generated by the elements

$$\{K_{\alpha_i}, E_j, F_k :\ 1\leq i\leq n, j\in I_+, k\in I_-\}.$$

Lemma 2.4. *$\mathbf{u}_\epsilon(\mathfrak{l})$ is a Hopf subalgebra of $\Gamma_\epsilon(\mathfrak{l})$ such that $\Gamma_\epsilon(\mathfrak{l})\cap\mathbf{u}_\epsilon(\mathfrak{g}) = \mathbf{u}_\epsilon(\mathfrak{l})$ and corresponds to the triple $(\mathbb{T}, I_+, I_-)$, see* (1).

Proof. It is clear that $\mathbf{u}_\epsilon(\mathfrak{l})$ is a Hopf subalgebra of $\Gamma_\epsilon(\mathfrak{l})$. Since the Frobenius-Lusztig kernel $\mathbf{u}_\epsilon(\mathfrak{g})$ is the subalgebra of $\Gamma_\epsilon(\mathfrak{g})$ generated by the elements $\{K_{\alpha_i}, E_i, F_i : \ 1 \leq i \leq n\}$, we have that $\mathbf{u}_\epsilon(\mathfrak{l}) \subseteq \Gamma_\epsilon(\mathfrak{l}) \cap \mathbf{u}_\epsilon(\mathfrak{g})$. But from Lemma 2.2, it follows that every element of $\Gamma_\epsilon(\mathfrak{l}) \cap \mathbf{u}_\epsilon(\mathfrak{g})$ must be contained in $\mathbf{u}_\epsilon(\mathfrak{l})$. The last assertion follows immediately from Corollary 1.13. $\square$

Recall that the quantum Frobenius map $\mathrm{Fr} : \Gamma_\epsilon(\mathfrak{g}) \to U(\mathfrak{g})_{\mathbb{Q}(\epsilon)}$ is defined on the generators of $\Gamma_\epsilon(\mathfrak{g})$ by

$$\mathrm{Fr}(E_i^{(m)}) = \begin{cases} e_i^{(m/\ell)} & \text{if } \ell | m \\ 0 & \text{otherwise,} \end{cases} \qquad \mathrm{Fr}(F_i^{(m)}) = \begin{cases} f_i^{(m/\ell)} & \text{if } \ell | m \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{Fr}(\left(\begin{smallmatrix} K_{\alpha_i};0 \\ m \end{smallmatrix}\right)) = \begin{cases} \left(\begin{smallmatrix} h_i;0 \\ m \end{smallmatrix}\right) & \text{if } \ell | m \\ 0 & \text{otherwise,} \end{cases} \qquad \mathrm{Fr}(K_{\alpha_i}^{-1}) = 1, \text{ for all } 1 \leq i \leq n,$$

and one has an exact sequence of Hopf algebras– see [L90b], [DL94, Thm. 6.3]:

$$1 \to \mathbf{u}_\epsilon(\mathfrak{g}) \to \Gamma_\epsilon(\mathfrak{g}) \xrightarrow{\mathrm{Fr}} U(\mathfrak{g})_{\mathbb{Q}(\epsilon)} \to 1.$$

If we define $U(\mathfrak{l})_{\mathbb{Q}(\epsilon)} := \mathrm{Fr}(\Gamma_\epsilon(\mathfrak{l}))$, then it follows that $U(\mathfrak{l})_{\mathbb{Q}(\epsilon)}$ is a subalgebra of $U(\mathfrak{g})_{\mathbb{Q}(\epsilon)}$ and the following diagram commutes

(5)
$$\begin{tikzcd} \mathbf{u}_\epsilon(\mathfrak{g}) \arrow[r, hook] & \Gamma_\epsilon(\mathfrak{g}) \arrow[r, two heads, "\mathrm{Fr}"] & U(\mathfrak{g})_{\mathbb{Q}(\epsilon)} \\ \mathbf{u}_\epsilon(\mathfrak{l}) \arrow[r, hook] \arrow[u, hook] & \Gamma_\epsilon(\mathfrak{l}) \arrow[r, two heads, "\overline{\mathrm{Fr}}"] \arrow[u, hook] & U(\mathfrak{l})_{\mathbb{Q}(\epsilon)}, \arrow[u, hook] \end{tikzcd}$$

where $\overline{\mathrm{Fr}}$ is the restriction of Fr to $\Gamma_\epsilon(\mathfrak{l})$.

Remarks 2.5. (a) Let $\mathfrak{l}$ be the set of primitive elements $P(U(\mathfrak{l})_{\mathbb{Q}(\epsilon)})$ of $U(\mathfrak{l})_{\mathbb{Q}(\epsilon)}$. Then $\mathfrak{l}$ is a Lie subalgebra of $\mathfrak{g}$, which is in fact regular in the sense of [D57]: it is the Lie subalgebra generated by the set $\{h_i, e_j, f_k : 1 \leq i \leq n, j \in I_+, k \in I_-\}$. This agrees with Definition 1.1.

(b) $\mathrm{Ker}\,\overline{\mathrm{Fr}}$ is the two-sided ideal $\mathcal{I}$ of $\Gamma_\epsilon(\mathfrak{l})$ generated by the set

$$\Big\{E_j^{(m)}, F_k^{(m)}, \begin{pmatrix} K_{\alpha_i};0 \\ m \end{pmatrix}, K_{\alpha_i} - 1 : 1 \leq i \leq n, j \in I_+, k \in I_-, m \geq 0, \ell \nmid m\Big\},$$

and coincides with W_ℓ. Indeed, by [DL94, Thm. 6.3] we know that $\mathrm{Ker}\,\mathrm{Fr} = J_\ell$ and coincides with the two-sided ideal generated by

$$\Big\{E_i^{(m)}, F_i^{(m)}, \begin{pmatrix} K_{\alpha_i};0 \\ m \end{pmatrix}, K_{\alpha_i} - 1 : 1 \leq i \leq n, m \geq 0, \ell \nmid m\Big\}.$$

But by Lemma 2.2, $\mathrm{Ker}\,\overline{\mathrm{Fr}} = \mathrm{Ker}\,\mathrm{Fr} \cap \Gamma_\epsilon(\mathfrak{l}) = J_\ell \cap \Gamma_\epsilon(\mathfrak{l}) = W_\ell$ and the last one coincides with the ideal $\mathcal{I}$.

(*c*) Since by [DL94, Thm. 6.3], the morphism $\Gamma_\ell/[p_\ell(q)\Gamma_\ell] \to U(\mathfrak{g})_{\mathbb{Q}(\epsilon)}$ induced by the quantum Frobenius map is bijective and by definition $\Theta_\ell \subseteq \Gamma_\ell$ and $U(\mathfrak{l})_{\mathbb{Q}(\epsilon)} = \mathrm{Fr}(U(\mathfrak{g})_{\mathbb{Q}(\epsilon)})$, it follows by Lemma 2.2 that $\Theta_\ell \cap p_\ell(q)\Gamma_\ell = p_\ell(q)\Theta_\ell$ and the morphism $\Theta_\ell/[p_\ell(q)\Theta_\ell] \to U(\mathfrak{l})_{\mathbb{Q}(\epsilon)}$ is also bijective.

The following proposition gives some properties of $\mathbf{u}_\epsilon(\mathfrak{l})$.

Proposition 2.6. (*a*) *The following sequence of Hopf algebras is exact*

$$1 \to \mathbf{u}_\epsilon(\mathfrak{l}) \xrightarrow{j} \Gamma_\epsilon(\mathfrak{l}) \xrightarrow{\overline{\mathrm{Fr}}} U(\mathfrak{l})_{\mathbb{Q}(\epsilon)} \to 1. \tag{6}$$

(*b*) *There is a surjective algebra map* $\psi : \Gamma_\epsilon(\mathfrak{l}) \to \mathbf{u}_\epsilon(\mathfrak{l})$ *such that* $\psi|_{\mathbf{u}_\epsilon(\mathfrak{l})} = \mathrm{id}$.

Proof. (*a*) We need only to prove that $\operatorname{Ker}\overline{\mathrm{Fr}} = \mathbf{u}_\epsilon(\mathfrak{l})^+\Gamma_\epsilon(\mathfrak{l})$ and ${}^{\mathrm{co}\,\overline{\mathrm{Fr}}}\Gamma_\epsilon(\mathfrak{l}) = \mathbf{u}_\epsilon(\mathfrak{l})$. The first equality follows directly from Remark 2.5 (*b*), since the two-sided ideal generated by $\mathbf{u}_\epsilon(\mathfrak{l})^+$ coincides with $\mathcal{I}$. The second equality follows from Lemma 2.4, because ${}^{\mathrm{co}\,\mathrm{Fr}}\Gamma_\epsilon(\mathfrak{g}) = \mathbf{u}_\epsilon(\mathfrak{g})$ by [A96, Lemma 3.4.1] and $\mathbf{u}_\epsilon(\mathfrak{l}) = \mathbf{u}_\epsilon(\mathfrak{g}) \cap \Gamma_\epsilon(\mathfrak{l}) = {}^{\mathrm{co}\,\overline{\mathrm{Fr}}}\Gamma_\epsilon(\mathfrak{g}) \cap \Gamma_\epsilon(\mathfrak{l}) = {}^{\mathrm{co}\,\overline{\mathrm{Fr}}}\Gamma_\epsilon(\mathfrak{l})$.

(*b*) By Lemma 1.10, there exists a surjective algebra map $\varphi : \Gamma_\epsilon(\mathfrak{g}) \to \mathbf{u}_\epsilon(\mathfrak{g})$ such that $\varphi|_{\mathbf{u}_\epsilon(\mathfrak{g})} = \mathrm{id}$. If we define $\psi := \varphi|_{\mathbf{u}_\epsilon(\mathfrak{l})} : \Gamma_\epsilon(\mathfrak{l}) \to \mathbf{u}_\epsilon(\mathfrak{g})$, then $\operatorname{Im}\psi \subseteq \mathbf{u}_\epsilon(\mathfrak{l})$ and $\varphi|_{\mathbf{u}_\epsilon(\mathfrak{l})} = \mathrm{id}$, from which follows that $\operatorname{Im}\psi = \mathbf{u}_\epsilon(\mathfrak{l})$. □

2.1.3. *The quantized coordinate algebra* $\mathcal{O}_\epsilon(L)$. The inclusion $\Gamma_\epsilon(\mathfrak{l}) \hookrightarrow \Gamma_\epsilon(\mathfrak{g})$ determines by duality a Hopf algebra map $\mathrm{Res} : \Gamma_\epsilon(\mathfrak{g})^\circ \to \Gamma_\epsilon(\mathfrak{l})^\circ$. Since by Proposition 1.7, we have that $\mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)} \subseteq \Gamma_\epsilon(\mathfrak{g})^\circ$, we may define

$$\mathcal{O}_\epsilon(L)_{\mathbb{Q}(\epsilon)} := \mathrm{Res}(\mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)}).$$

Moreover, as $\mathcal{O}(G)_{\mathbb{Q}(\epsilon)} \subseteq \mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)}$, $\mathrm{Res}(\mathcal{O}(G)_{\mathbb{Q}(\epsilon)})$ is a central Hopf subalgebra of $\mathcal{O}_\epsilon(L)_{\mathbb{Q}(\epsilon)}$ and whence there exists an algebraic subgroup L of G such that $\mathrm{Res}(\mathcal{O}(G)_{\mathbb{Q}(\epsilon)}) = \mathcal{O}(L)_{\mathbb{Q}(\epsilon)}$. Next we show that L is connected and the corresponding Lie subalgebra of $\mathfrak{g}$ is no other than the Lie algebra $\mathfrak{l}$ discussed in Remark 2.5 (a).

Recall that a Lie subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ is called *algebraic* if there exists an algebraic subgroup $K \subseteq G$ such that $\mathfrak{k} = \mathrm{Lie}(K)$. We say that $\mathfrak{k}^+$ is the *algebraic hull* of $\mathfrak{k}$ if $\mathfrak{k}^+$ is an algebraic subalgebra of $\mathfrak{g}$ such that $\mathfrak{k} \subseteq \mathfrak{k}^+$ and if $\mathfrak{a}$ is an algebraic subalgebra of $\mathfrak{g}$ that contains $\mathfrak{k}$, then $\mathfrak{k}^+ \subseteq \mathfrak{a}$.

Proposition 2.7. *The algebraic group L is connected and* $\mathrm{Lie}(L) = \mathfrak{l}$.

Proof. Since $\mathcal{O}(G)_{\mathbb{Q}(\epsilon)} \subseteq U(\mathfrak{g})^\circ_{\mathbb{Q}(\epsilon)}$, dualizing diagram (5) we have $\mathcal{O}(L)_{\mathbb{Q}(\epsilon)} = \mathrm{Res}(\mathcal{O}(G)_{\mathbb{Q}(\epsilon)}) \subseteq U(\mathfrak{l})^\circ_{\mathbb{Q}(\epsilon)}$. But by [H81, XVI.3], $U(\mathfrak{l})^\circ_{\mathbb{Q}(\epsilon)}$ and consequently $\mathcal{O}(L)_{\mathbb{Q}(\epsilon)}$ are integral domains, implying that L is irreducible and therefore connected.

To show $\mathrm{Lie}(L) = \mathfrak{l}$, we prove that $\mathrm{Lie}(L)$ is the algebraic hull of $\mathfrak{l}$ and $\mathfrak{l}$ is an algebraic Lie algebra. Since $\operatorname{Ker}\mathrm{Res}|_{\mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)}} = \{f \in \mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)} : f|_{\Gamma_\epsilon(\mathfrak{l})} = 0\}$ and the inclusion of $\mathcal{O}(G)_{\mathbb{Q}(\epsilon)}$ in $\mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)}$ is given by the transpose of the quantum Frobenius map Fr (see page 11), it follows that $\mathcal{O}(L)_{\mathbb{Q}(\epsilon)} \simeq \mathcal{O}(G)_{\mathbb{Q}(\epsilon)}/J$, where

$$\begin{aligned} J &= \{f \in \mathcal{O}(G)_{\mathbb{Q}(\epsilon)} : \ \langle f, \mathrm{Fr}(x)\rangle = 0, \forall\, x \in \Gamma_\epsilon(\mathfrak{l})\} \\ &= \{f \in \mathcal{O}(G)_{\mathbb{Q}(\epsilon)} : \ \langle f, x\rangle = 0, \forall\, x \in U(\mathfrak{l})_{\mathbb{Q}(\epsilon)}\}. \end{aligned}$$

In particular, $0 = \langle f, x\rangle = x(f)$ for all $x \in U(\mathfrak{l})_{\mathbb{Q}(\epsilon)}$. Since by [FR05, Lemma 6.9], $\mathrm{Lie}(L) = \{\tau \in \mathfrak{g} : \ \tau(f) = 0, \ \forall f \in J\}$, it is clear that $\mathfrak{l} \subseteq \mathrm{Lie}(L)$. Now let $K \subseteq G$ such that $\mathfrak{l} \subseteq \mathrm{Lie}(K) =: \mathfrak{k}$ and denote by $\mathcal{I}$ the ideal of K; then $\mathfrak{k} = \{\tau \in \mathfrak{g} : \tau(\mathcal{I}) = 0\}$. As $\mathfrak{l} \subseteq \mathfrak{k}$, $\tau(\mathcal{I}) = 0$ for all $\tau \in \mathfrak{l}$. Since the pairing $\langle , \rangle$ is multiplicative, we have that $\mathcal{I} \subseteq J$ and whence $L \subseteq K$. Thus $\mathrm{Lie}(L) \subseteq \mathfrak{k}$ for all algebraic Lie subalgebra $\mathfrak{k}$ such that $\mathfrak{l} \subseteq \mathfrak{k}$, implying that $\mathrm{Lie}(L) = \mathfrak{l}^+$.

Now we show that $\mathfrak{l}$ is algebraic, implying that $\mathfrak{l} = \mathfrak{l}^+ = \mathrm{Lie}(L)$. Consider $\mathfrak{g}$ as a G-module with the adjoint action and define $G_\mathfrak{l} = \{x \in G : \ x \cdot \mathfrak{l} = \mathfrak{l}\}$ and $\mathfrak{g}_\mathfrak{l} = \{\tau \in \mathfrak{g} : \ [\tau, \mathfrak{l}] \subseteq \mathfrak{l}\}$. Then by [FR05, Ex. 8.4.7], $\mathrm{Lie}(G_\mathfrak{l}) = \mathfrak{g}_\mathfrak{l}$. Thus, it is enough to show that $\mathfrak{l}$ equals its normalizer in $\mathfrak{g}$.

By construction, we know that $\mathfrak{l} = \mathfrak{l}_+ \oplus \mathfrak{h} \oplus \mathfrak{l}_-$, where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{l}_\pm = \bigoplus_{\alpha \in \Psi_\pm} \mathfrak{g}_\alpha$, with $\Psi_\pm = \{\alpha \in \Phi : \ \mathrm{Supp}(\alpha) \subseteq I_\pm\}$. Let $x \in \mathfrak{g}_\mathfrak{l}$, then we may write $x = \sum_{\alpha\in\Phi} c_\alpha x_\alpha + x_0$ with $x_0 \in \mathfrak{h}$. Thus, for all $H \in \mathfrak{h}$ we have that $[H, x] = \sum_{\alpha\in\Phi} c_\alpha \alpha(H) x_\alpha \in \mathfrak{l}$. This implies that for all $H \in \mathfrak{h}$, $c_\alpha \alpha(H) = 0$ for all $\alpha \notin \Psi = \Psi_+ \cup \Psi_-$. Hence $c_\alpha = 0$ for all $\alpha \notin \Psi$ and $x \in \mathfrak{l}$. $\square$

Since $\mathcal{O}(L)_{\mathbb{Q}(\epsilon)}$ is a central Hopf subalgebra of $\mathcal{O}_\epsilon(L)_{\mathbb{Q}(\epsilon)}$, the quotient

$$\overline{\mathcal{O}_\epsilon(L)}_{\mathbb{Q}(\epsilon)} := \mathcal{O}_\epsilon(L)_{\mathbb{Q}(\epsilon)}/[\mathcal{O}(L)^+_{\mathbb{Q}(\epsilon)}\mathcal{O}_\epsilon(L)_{\mathbb{Q}(\epsilon)}]$$

is a Hopf algebra which is finite-dimensional. The following proposition shows that, as expected, this algebra is isomorphic to $\mathbf{u}_\epsilon(\mathfrak{l})^*$, see 2.1.2.

Proposition 2.8. (a) *The following sequence of Hopf algebras is exact*

$$1 \to \mathcal{O}(L)_{\mathbb{Q}(\epsilon)} \xrightarrow{\iota_L} \mathcal{O}_\epsilon(L)_{\mathbb{Q}(\epsilon)} \xrightarrow{\pi_L} \overline{\mathcal{O}_\epsilon(L)}_{\mathbb{Q}(\epsilon)} \to 1. \tag{7}$$

(b) *There exists a surjective Hopf algebra map* $P : \mathbf{u}_\epsilon(\mathfrak{g})^* \to \overline{\mathcal{O}_\epsilon(L)}_{\mathbb{Q}(\epsilon)}$ *making the following diagram commutative:*

$$\begin{tikzcd} 1 \arrow[r] & \mathcal{O}(G)_{\mathbb{Q}(\epsilon)} \arrow[r, "\iota"] \arrow[d, "\mathrm{res}"] & \mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)} \arrow[r, "\pi"] \arrow[d, "\mathrm{Res}"] & \mathbf{u}_\epsilon(\mathfrak{g})^* \arrow[r] \arrow[d, "P"] & 1 \\ 1 \arrow[r] & \mathcal{O}(L)_{\mathbb{Q}(\epsilon)} \arrow[r, "\iota_L"] & \mathcal{O}_\epsilon(L)_{\mathbb{Q}(\epsilon)} \arrow[r, "\pi_L"] & \overline{\mathcal{O}_\epsilon(L)}_{\mathbb{Q}(\epsilon)} \arrow[r] & 1. \end{tikzcd} \tag{8}$$

(c) $\overline{\mathcal{O}_\epsilon(L)}_{\mathbb{Q}(\epsilon)} \simeq \mathbf{u}_\epsilon(\mathfrak{l})^*$ *as Hopf algebras.*

Proof. (a) We need only to show that $\mathcal{O}(L)_{\mathbb{Q}(\epsilon)} = {}^{\mathrm{co}\,\pi_L}\mathcal{O}_\epsilon(L)_{\mathbb{Q}(\epsilon)}$. The algebra $\mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)}$ is noetherian, by Theorem 1.8 (b). Therefore $\mathcal{O}_\epsilon(L)_{\mathbb{Q}(\epsilon)}$ is also noetherian, since it is a quotient of $\mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)}$. Then by [S93, Thm.

3.3], $\mathcal{O}_\epsilon(L)_{\mathbb{Q}(\epsilon)}$ is faithfully flat over $\mathcal{O}(L)_{\mathbb{Q}(\epsilon)}$ and by [Mo93, Prop. 3.4.3] it follows that $\mathcal{O}(L)_{\mathbb{Q}(\epsilon)} = {}^{\operatorname{co}\pi_L}\mathcal{O}_\epsilon(L)_{\mathbb{Q}(\epsilon)} = \mathcal{O}_\epsilon(L)_{\mathbb{Q}(\epsilon)}^{\operatorname{co}\pi_L}$.

(b) Since the sequence (2) is exact, we have $\operatorname{Ker}\pi = \mathcal{O}(G)^+_{\mathbb{Q}(\epsilon)}\mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)}$ and $\mathbf{u}_\epsilon(\mathfrak{g})^* \simeq \mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)}/[\mathcal{O}(G)^+_{\mathbb{Q}(\epsilon)}\mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)}]$. But then, $\pi_L \operatorname{Res}(\operatorname{Ker}\pi) = \pi_L(\mathcal{O}(L)^+_{\mathbb{Q}(\epsilon)}\mathcal{O}_\epsilon(L)_{\mathbb{Q}(\epsilon)}) = 0$ and hence there exists a Hopf algebra map $P : \mathbf{u}_\epsilon(\mathfrak{g})^* \to \overline{\mathcal{O}_\epsilon(L)}_{\mathbb{Q}(\epsilon)}$ which makes the diagram (8) commutative.

(c) Dualizing diagram (5) we obtain a commutative diagram

$$\begin{tikzcd} U(\mathfrak{g})^\circ_{\mathbb{Q}(\epsilon)} \arrow[r, hook, "{}^t\mathrm{Fr}"] \arrow[d] & \Gamma_\epsilon(\mathfrak{g})^\circ \arrow[r, "F"] \arrow[d, "\mathrm{Res}"] & \mathbf{u}_\epsilon(\mathfrak{g})^* \arrow[d, "p"] \\ U(\mathfrak{l})^\circ_{\mathbb{Q}(\epsilon)} \arrow[r, hook, "{}^t\overline{\mathrm{Fr}}"'] & \Gamma_\epsilon(\mathfrak{l})^\circ \arrow[r, "f"'] & \mathbf{u}_\epsilon(\mathfrak{l})^*. \end{tikzcd} \tag{9}$$

Since $\mathcal{O}_\epsilon(L)_{\mathbb{Q}(\epsilon)} = \operatorname{Res}(\mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)})$, $\mathcal{O}(L)_{\mathbb{Q}(\epsilon)} = \operatorname{Res}(\mathcal{O}(G)_{\mathbb{Q}(\epsilon)})$ and $\mathcal{O}(G)_{\mathbb{Q}(\epsilon)} \simeq U(\mathfrak{g})^\circ_{\mathbb{Q}(\epsilon)}$, because $\mathfrak{g}$ is simple, it follows that $\mathcal{O}(L)_{\mathbb{Q}(\epsilon)} \subseteq {}^t\overline{\mathrm{Fr}}(U(\mathfrak{l})^\circ_{\mathbb{Q}(\epsilon)})$. In particular, $\mathcal{O}(L)^+_{\mathbb{Q}(\epsilon)} \subseteq \operatorname{Ker} f$. Moreover, since $F(\mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)}) = \pi(\mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)}) = \mathbf{u}_\epsilon(\mathfrak{g})^*$ we have that $\mathbf{u}_\epsilon(\mathfrak{l})^* = f\operatorname{Res}(\mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)}) = f(\mathcal{O}_\epsilon(L)_{\mathbb{Q}(\epsilon)})$. Hence, there exists a surjective Hopf algebra map $\beta : \overline{\mathcal{O}_\epsilon(L)}_{\mathbb{Q}(\epsilon)} \to \mathbf{u}_\epsilon(\mathfrak{l})^*$; and $\dim \overline{\mathcal{O}_\epsilon(L)}_{\mathbb{Q}(\epsilon)} \geq \dim \mathbf{u}_\epsilon(\mathfrak{l})^*$.

We show next that there exists a surjective morphism $\mathbf{u}_\epsilon(\mathfrak{l})^* \to \overline{\mathcal{O}_\epsilon(L)}_{\mathbb{Q}(\epsilon)}$ implying that β is an isomorphism. Consider the map $p : \mathbf{u}_\epsilon(\mathfrak{g})^* \to \mathbf{u}_\epsilon(\mathfrak{l})^*$ as in (9) and let $a \in \operatorname{Ker} p$. Since $\mathbf{u}_\epsilon(\mathfrak{g})$ is finite-dimensional, the coordinate functions of the regular representation of $\mathbf{u}_\epsilon(\mathfrak{g})$ span linearly $\mathbf{u}_\epsilon(\mathfrak{g})^*$ and we may assume that a is a coordinate function of a finite-dimensional representation M of $\mathbf{u}_\epsilon(\mathfrak{g})$. As p is just the map given by the restriction, we have that a must be trivial on every basis of $\mathbf{u}_\epsilon(\mathfrak{l})$, in particular the following:

$$\Big\{ \prod_{\beta\geq 0} F_\beta^{n_\beta} \cdot \prod_{i=1}^{n} K_{\alpha_i}^{t_i} \cdot \prod_{\alpha \geq 0} E_\alpha^{m_\alpha} : 0 \leq n_\beta, t_i, m_\alpha < \ell, \\ \beta \in Q_{I_-}, 1 \leq i \leq n, \alpha \in Q_{I_+} \Big\}.$$

On the other hand, we know by Lemma 1.10 that there exists a surjective algebra map $\varphi : \Gamma_\epsilon(\mathfrak{g}) \to \mathbf{u}_\epsilon(\mathfrak{g})$ such that $\varphi|_{\mathbf{u}_\epsilon(\mathfrak{g})} = \mathrm{id}$. Hence, the $\mathbf{u}_\epsilon(\mathfrak{g})$-module M admits a $\Gamma_\epsilon(\mathfrak{g})$-module structure via φ. Since M is finite-dimensional and $K_{\alpha_i}^\ell$ acts as the identity for every $1 \leq i \leq n$, it follows that each operator K_{α_i} is diagonalizable with eigenvalues ϵ_i^m for some $m \in \mathbb{N}$. This implies by definition that the coordinate function $\varphi^*(a)$ of the $\Gamma_\epsilon(\mathfrak{g})$-module M must be contained in $\mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)}$. Thus, using the definition of φ

we have that $\operatorname{Res}\varphi^*(a)$ must annihilate the set

$$W_\ell = \mathbb{Q}(\epsilon)\Big\{ \prod_{\beta\geq 0} F_\beta^{(n_\beta)} \cdot \prod_{i=1}^{n} \left(\begin{array}{c} K_{\alpha_i};0 \\ t_i \end{array} \right) K_{\alpha_i}^{\mathrm{Ent}(t_i/2)} \cdot \prod_{\alpha\geq 0} E_\alpha^{(m_\alpha)} : \\ \exists\, n_\beta, t_i, m_\alpha \not\equiv 0 \mod (\ell) \text{ with } \beta \in Q_{I_-}, 1\leq i\leq n, \alpha\in Q_{I_+}\Big\}.$$

Since by Lemma 2.2, $\Gamma(\mathfrak{l}) = W_\ell \otimes \Theta_\ell$ as free R-modules and by Remark 2.5, $\operatorname{Ker}\overline{\mathrm{Fr}} = W_\ell$ and the map $\Theta_l/[p_\ell(q)\Theta_\ell] \to U(\mathfrak{l})_{\mathbb{Q}(\epsilon)}$ induced by the restriction of the quantum Frobenius map $\overline{\mathrm{Fr}}$ is bijective. Then there exists $b \in U(\mathfrak{l})^\circ_{\mathbb{Q}(\epsilon)}$ such that ${}^t\overline{\mathrm{Fr}}(b) = \operatorname{Res}(\varphi^*(a))$. Hence,

$$P(a) = P(\pi(\varphi^*(a))) = \pi_L(\operatorname{Res}(\varphi^*(a))) = \pi_L({}^t\overline{\mathrm{Fr}}(b)) = \varepsilon(b) = \varepsilon(a) = 0,$$

and $a \in \operatorname{Ker} P$. Thus $\operatorname{Ker} p \subseteq \operatorname{Ker} P$ and there exists a surjective map $\mathbf{u}_\epsilon(\mathfrak{l})^* \to \overline{\mathcal{O}_\epsilon(L)}_{\mathbb{Q}(\epsilon)}$. □

Remark 2.9. By the Proposition above, we have the following commutative diagram of exact sequences of Hopf algebras

$$\begin{tikzcd} 1 \arrow[r] & \mathcal{O}(G)_{\mathbb{Q}(\epsilon)} \arrow[r, "\iota"] \arrow[d, "\mathrm{res}"] & \mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)} \arrow[r, "\pi"] \arrow[d, "\mathrm{Res}"] & \mathbf{u}_\epsilon(\mathfrak{g})^* \arrow[r] \arrow[d, "p"] & 1 \\ 1 \arrow[r] & \mathcal{O}(L)_{\mathbb{Q}(\epsilon)} \arrow[r, "\iota_L"] & \mathcal{O}_\epsilon(L)_{\mathbb{Q}(\epsilon)} \arrow[r, "\pi_L"] & \mathbf{u}_\epsilon(\mathfrak{l})^* \arrow[r] & 1 \end{tikzcd} \tag{10}$$

2.2. **Second Step.** We consider now the complex form of the algebras defined above. Denote the $\mathbb{C}$-form of the Frobenius-Lusztig kernels just by $\mathbf{u}_\epsilon(\mathfrak{g})$ and $\mathbf{u}_\epsilon(\mathfrak{l})$.

The following proposition tell us how to construct Hopf algebras from a central exact sequence and a surjective Hopf algebra map. We perform it in a general setting and then we apply it to our situation. The characterization of these algebras as pushouts will be crucial.

Proposition 2.10. *Let A and K be Hopf algebras, B a central Hopf subalgebra of A such that A is left or right faithfully flat over B and $p: B \to K$ a surjective Hopf algebra map. Then $H = A/AB^+$ is a Hopf algebra and A fits into the exact sequence $1 \to B \xrightarrow{\iota} A \xrightarrow{\pi} H \to 1$. If we set $\mathcal{J} = \operatorname{Ker} p \subseteq B$, then $(\mathcal{J}) = A\mathcal{J}$ is a Hopf ideal of A and $A/(\mathcal{J})$ is the pushout given by the following diagram:*

$$\begin{tikzcd} B \arrow[r, hook, "\iota"] \arrow[d, "p"'] & A \arrow[d, "q"] \\ K \arrow[r, hook, "j"'] & A/(\mathcal{J}). \end{tikzcd}$$

Moreover, K can be identified with a central Hopf subalgebra of $A/(\mathcal{J})$ and $A/(\mathcal{J})$ fits into the exact sequence

$$1 \to K \to A/(\mathcal{J}) \to H \to 1. \tag{11}$$

Proof. The first assertion follows directly from [Mo93, Prop. 3.4.3]. Since B is central in A, $(\mathcal{J})$ is a two-sided ideal of A. Moreover, from the fact that ε and Δ are algebra maps and $\mathcal{S}(\mathcal{J}) \subseteq \mathcal{J}$, it follows that $(\mathcal{J})$ is indeed a Hopf ideal. Identify K with $B/\mathcal{J}$. Then the map $j : K \to A/(\mathcal{J})$ given by $j(b + \mathcal{J}) = \iota(b) + (\mathcal{J})$ defines a morphism of Hopf algebras because ι is a Hopf algebra map. Since A is faithfully flat over B, by [S92, Cor. 1.8], B is a direct summand in A as a B-module, say $A = B \oplus M$. Then $(\mathcal{J}) \cap B = \mathcal{J}A \cap B = (\mathcal{J}B \oplus \mathcal{J}M) \cap B = (\mathcal{J} \oplus \mathcal{J}M) \cap B = \mathcal{J}$. Thus, if $j(b + \mathcal{J}) = 0$ then $\iota(b) \in (\mathcal{J})$ and this implies that $b \in (\mathcal{J}) \cap B = \mathcal{J}$ by the equality above. Hence, j is injective.

Let us see now that $A/(\mathcal{J})$ is a pushout: let C be a Hopf algebra and suppose that there exist Hopf algebra maps $\varphi_1 : K \to C$ and $\varphi_2 : A \to C$ such that $\varphi_1 p = \varphi_2 \iota$. We have to show that there exists a unique Hopf algebra map $\phi : A/(\mathcal{J}) \to C$ such that $\phi q = \varphi_2$ and $\phi j = \varphi_1$.

$$\begin{tikzcd}
B \arrow[r, "\iota"] \arrow[d, "p"'] & A \arrow[d, "q"] \arrow[ddr, bend left, "\varphi_2"] & \\
K \arrow[r, "j"'] \arrow[drr, bend right, "\varphi_1"'] & A/(\mathcal{J}) \arrow[dr, dashed, "\exists! \phi"] & \\
 & & C
\end{tikzcd}$$

Since $\varphi_2((\mathcal{J})) = \varphi_2(A\mathcal{J}) = \varphi_2(A)\varphi_2(\iota(\mathcal{J})) = \varphi_2(A)\varphi_1(p(\mathcal{J})) = 0$, there exists a unique Hopf algebra map $\phi : A/(\mathcal{J}) \to C$ such that $\phi q = \varphi_2$. Moreover, let $x \in K$ and $b \in B$ such that $p(b) = x$. Then $\phi j(x) = \phi j p(b) = \phi q \iota(b) = \varphi_2 \iota(b) = \varphi_1 p(b) = \varphi_1(x)$, from which follows that $\phi j = \varphi_1$.

Denote also by K the image of K under j. To see that K is central in $A/(\mathcal{J})$ we have to verify that $j(c)\bar{a} = \bar{a}j(c)$ for all $\bar{a} \in A/(\mathcal{J})$, $c \in K$. Since p is surjective, for all $c \in K$ there exists $b \in B$ such that $p(b) = c$ and since q is an algebra map, it follows that $\bar{a}j(c) = q(a)j(p(b)) = q(a)q(\iota(b)) = q(a\iota(b)) = q(\iota(b)a) = q(\iota(b))q(a) = j(c)\bar{a}$, because B is central in A. In particular, the quotient $\widetilde{H} = [A/(\mathcal{J})]/[K^+(A/(\mathcal{J}))]$ is a Hopf algebra. To see that $A/(\mathcal{J})$ is a central extension of K by $\widetilde{H}$, by [Mo93, Prop. 3.4.3] it is enough to show that $A/(\mathcal{J})$ is flat over K and K is a direct summand of $A/(\mathcal{J})$ as K-modules, since by [S92, Cor. 1.8] this implies that $A/(\mathcal{J})$ is faithfully flat over K.

First we show that $A/(\mathcal{J})$ is flat over K. Let M_1 and M_2 be two right K-modules and let $f : M_1 \to M_2$ be an injective homomorphism. In particular, they admit a B-module structure via the map $p : B \to K$, which we denote by $\overline{M}_i$ for $i = 1, 2$; thus f is an injective homomorphism of B-modules. Since

A is faithfully flat over B, the homomorphism of A-modules $f \otimes \mathrm{id} : \overline{M}_1 \otimes_B A \to \overline{M}_2 \otimes_B A$ is also injective. As $\mathcal{J}$ is central in A, we have for $i = 1, 2$ that $(\overline{M}_i \otimes_B A)(\mathcal{J}) = 0$. Then the A-modules are also $A/(\mathcal{J})$-modules and $\overline{M}_i \otimes_B A \simeq M_i \otimes_K A/(\mathcal{J})$ as $A/(\mathcal{J})$-modules by the construction of $\overline{M}_i$. Hence the homomorphism of $A/(\mathcal{J})$-modules $f \otimes \mathrm{id} : M_1 \otimes_K A/(\mathcal{J}) \to M_2 \otimes_K A/(\mathcal{J})$ is injective and $A/(\mathcal{J})$ is flat over K.

As $A = B \oplus M$ as B-modules, we have that $(\mathcal{J}) = A\mathcal{J} = \mathcal{J} \oplus M\mathcal{J}$, where $M\mathcal{J}$ is a B-submodule of M and $\mathcal{J} = B \cap (\mathcal{J} \oplus M\mathcal{J})$. Hence $A/(\mathcal{J}) = (B \oplus M)/(\mathcal{J} \oplus M\mathcal{J}) = K \oplus (M/M\mathcal{J})$ as K-modules, which implies that K is a direct summand of $A/(\mathcal{J})$.

In conclusion, $A/(\mathcal{J})$ fits into an exact sequence of Hopf algebras

$$1 \to K \xrightarrow{j} A/(\mathcal{J}) \xrightarrow{r} \widetilde{H} \to 1.$$

Since the map $\Psi : K^+(A/(\mathcal{J})) \to (B^+A)/(\mathcal{J})$ defined by $\Psi(\overline{b}\overline{a}) = \overline{ba}$ is a k-linear isomorphism, it follows that $\widetilde{H} = (A/(\mathcal{J}))/[K^+(A/(\mathcal{J}))] \simeq (A/(\mathcal{J}))/[(B^+A)/(\mathcal{J})] \simeq A/B^+A = H$ and therefore $A/(\mathcal{J})$ fits into an exact sequence (11). $\square$

Let Γ be an algebraic group and let $\sigma : \Gamma \to G$ an injective homomorphism of algebraic groups such that $\sigma(\Gamma) \subseteq L$. Then we have a surjective Hopf algebra map ${}^t\sigma : \mathcal{O}(L) \to \mathcal{O}(\Gamma)$. Applying the pushout construction given in Proposition 2.10, we obtain a Hopf algebra $A_{\mathfrak{l},\sigma}$ which is part of an exact sequence of Hopf algebras and fits into the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}(G) \arrow[r, "\iota"] \arrow[d, "\mathrm{res}"'] & \mathcal{O}_\epsilon(G) \arrow[r, "\pi"] \arrow[d, "\mathrm{Res}"] & \mathbf{u}_\epsilon(\mathfrak{g})^* \arrow[r] \arrow[d, "p"] & 1 \\
1 \arrow[r] & \mathcal{O}(L) \arrow[r, "\iota_L"] \arrow[d, "{}^t\sigma"'] & \mathcal{O}_\epsilon(L) \arrow[r, "\pi_L"] \arrow[d, "\nu"] & \mathbf{u}_\epsilon(\mathfrak{l})^* \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathcal{O}(\Gamma) \arrow[r, "j"] & A_{\mathfrak{l},\sigma} \arrow[r, "\bar{\pi}"] & \mathbf{u}_\epsilon(\mathfrak{l})^* \arrow[r] & 1.
\end{tikzcd} \tag{12}$$

Remark 2.11. Let $1 \to K \to A \to H \to 1$ be an exact sequence of Hopf algebras. If $\beta : A \otimes_K A \to A \otimes H$, $\beta(x, y) = xy_{(0)} \otimes y_{(1)}$ denotes the Galois map, then β is surjective, since $H \simeq A/K^+A$. If moreover H is finite-dimensional, A is a finitely generated projective K-module, by [KT81, Thm. 1.7]. In particular, if $\dim K$ is finite, then $\dim A = \dim K \dim H$ is also finite. In our case, if Γ is finite we obtain that $\dim A_{\mathfrak{l},\sigma} = |\Gamma| \dim \mathbf{u}_\epsilon(\mathfrak{l})$.

2.3. **Third Step.** In this subsection we make the third and last step of the construction. It consists essentially on taking a quotient by a Hopf ideal generated by differences of central group-like elements of $A_{\mathfrak{l},\sigma}$. The crucial point here is the description of H as a quotient of $\mathbf{u}_\epsilon(\mathfrak{l})^*$ and the existence of a coalgebra morphism $\psi^* : \mathbf{u}_\epsilon(\mathfrak{l})^* \to \mathcal{O}_\epsilon(L)$.

Recall that from the beginning of this section we fixed a surjective Hopf algebra map $r : \mathbf{u}_\epsilon(\mathfrak{g})^* \to H$ and H^* is determined by the triple (Σ, I_+, I_-). Since the Hopf subalgebra $\mathbf{u}_\epsilon(\mathfrak{l})$ is determined by the triple $(\mathbb{T}, I_+, I_-)$ with $\mathbb{T} \supseteq \Sigma$, we have that $H^* \subseteq \mathbf{u}_\epsilon(\mathfrak{l}) \subseteq \mathbf{u}_\epsilon(\mathfrak{g})$. Denote by $v : \mathbf{u}_\epsilon(\mathfrak{l})^* \to H$ the surjective Hopf algebra map induced by this inclusion. Then H is a quotient of $\mathbf{u}_\epsilon(\mathfrak{l})^*$ which fits into the following commutative diagram

$$\begin{tikzcd} \mathbf{u}_\epsilon(\mathfrak{g})^* \arrow[r, two heads, "p"] \arrow[dr, "r"'] & \mathbf{u}_\epsilon(\mathfrak{l})^* \arrow[d, two heads, "v"] \\ & H. \end{tikzcd}$$

Remark 2.12. Let $I = I_+ \cup -I_-$, $I^c = \Pi - I$ and $\mathbb{T}_I = \{K_{\alpha_i} :\ i \in I\}$. Let $s = |I^c|$. By Corollary 1.13, we know that $\mathbb{T}_I \subseteq \Sigma \subseteq \mathbb{T} = \mathbb{T}_I \times \mathbb{T}_{I^c}$. If we set $\Omega = \Sigma \cap \mathbb{T}_{I^c}$, it follows clearly that $\Sigma \simeq \mathbb{T}_I \times \Omega$.

Thus, giving a subgroup Σ such that $\mathbb{T}_I \subseteq \Sigma \subseteq \mathbb{T}$ is the same as giving a subgroup $\Omega \subseteq \mathbb{T}_{I^c}$, and this is the same as giving a subgroup $N \subseteq \widehat{\mathbb{T}_{I^c}}$. Namely, N is the kernel of the group homomorphism $\rho : \widehat{\mathbb{T}_{I^c}} \to \widehat{\Omega}$ induced by the inclusion. In particular, we have that $|\Sigma| = |\mathbb{T}_I||\Omega| = \ell^{n-s}|\Omega| = \frac{\ell^n}{|N|}$.

Definition 2.13. For all $1 \leq i \leq n$ such that $\alpha_i \notin I_+$ or $\alpha_i \notin I_-$ we define $D_i \in G(\mathbf{u}_\epsilon(\mathfrak{l})^*) = \mathrm{Alg}(\mathbf{u}_\epsilon(\mathfrak{l}), \mathbb{C})$ on the generators of $\mathbf{u}_\epsilon(\mathfrak{l})$ by

$$\begin{aligned} D_i(E_j) &= 0 \qquad \forall\, j :\ \alpha_j \in I_+, \qquad & D_i(F_k) &= 0 \qquad \forall\, k :\ \alpha_k \in I_-, \\ D_i(K_{\alpha_t}) &= 1 \qquad \forall\, t \neq i,\ 1 \leq t \leq n, \qquad & D_i(K_{\alpha_i}) &= \epsilon_i, \end{aligned}$$

where ϵ_i is a primitive ℓ-th root of 1. If $\alpha_i \notin I_+$ or $\alpha_i \notin I_-$, then E_i or F_i is not a generator of $\mathbf{u}_\epsilon(\mathfrak{l})$, respectively. Hence, D_i is a well-defined algebra map, since it verifies all the defining relations of $\Gamma_\epsilon(\mathfrak{g})$ [DL94, Sec. 3.4], see [G07, 5.2.12] for details.

Let $I^c = \{\alpha_{i_1}, \ldots, \alpha_{i_s}\}$ and let $N \subseteq \widehat{\mathbb{T}_{I^c}}$, correspond to Σ as in Remark 2.12. We define for all $z = (z_1, \ldots, z_s) \in \widehat{\mathbb{T}_{I^c}}$ the following group-like element

$$D^z := D_{i_1}^{z_1} \cdots D_{i_s}^{z_s}.$$

Recall that (M) denotes the two-sided ideal generated by a subset M of an algebra R.

Lemma 2.14. *(a) If $\alpha_i \in I^c$ then D_i is central in $\mathbf{u}_\epsilon(\mathfrak{l})^*$. In particular D^z is central for all $z \in \widehat{\mathbb{T}_{I^c}}$.*
(b) $H \simeq \mathbf{u}_\epsilon(\mathfrak{l})^*/(D^z - 1 | z \in N)$.

Proof. (a) We have to show that $D_i f = f D_i$ for all $f \in \mathbf{u}_\epsilon(\mathfrak{l})^*$. First observe that D_i coincide with the counit of $\mathbf{u}_\epsilon(\mathfrak{l})$ in all elements of the basis which do not contain some positive power of K_{α_i}. By Lemma 2.2 we know that

$\mathbf{u}_\epsilon(\mathfrak{l})$ has a basis of the form

$$\Big\{ \prod_{\beta\geq 0} F_\beta^{n_\beta} \cdot \prod_{i=1}^{n} K_{\alpha_i}^{t_i} \cdot \prod_{\alpha\geq 0} E_\alpha^{m_\alpha} : \quad 0 \leq n_\beta, t_i, m_\alpha < \ell,$$
$$\text{with } \beta \in Q_{I_-}, \alpha \in Q_{I_+}, 1 \leq i \leq n \Big\}.$$

Thus, using the defining relations of $\Gamma_\epsilon(\mathfrak{g})$ [DL94, Sec. 3.4], we may assume that this basis is of the form $K_{\alpha_i}^{t_i} M$ with $0 \leq t_i < \ell$ and M does not contain any power of K_{α_i}. Then for every element of this basis we have

$$\begin{aligned} D_i f(K_{\alpha_i}^{t_i} M) &= D_i(K_{\alpha_i}^{t_i} M_{(1)}) f(K_{\alpha_i}^{t_i} M_{(2)}) = D_i(K_{\alpha_i}^{t_i}) D_i(M_{(1)}) f(K_{\alpha_i}^{t_i} M_{(2)}) \\ &= \epsilon_i^{t_i} \varepsilon(M_{(1)}) f(K_{\alpha_i}^{t_i} M_{(2)}) = \epsilon_i^{t_i} f(K_{\alpha_i}^{t_i} M) \\ &= f D_i(K_{\alpha_i}^{t_i} M), \end{aligned}$$

(b) By (a) we know that D^z is a central group-like element of $\mathbf{u}_\epsilon(\mathfrak{l})^*$ for all $z \in N$. Hence the quotient $\mathbf{u}_\epsilon(\mathfrak{l})^*/(D^z - 1 | z \in N)$ is a Hopf algebra.

On the other hand, following Corollary 1.13 we know that H^* is determined by the triple (Σ, I_+, I_-) and consequently H^* is included in $\mathbf{u}_\epsilon(\mathfrak{l})$. If we denote $v : \mathbf{u}_\epsilon(\mathfrak{l})^* \to H$ the surjective map induced by this inclusion, we have that $\operatorname{Ker} v = \{f \in \mathbf{u}_\epsilon(\mathfrak{l})^* : \ f(h) = 0, \ \forall\ h \in H^*\}$. But $D^z - 1 \in \operatorname{Ker} v$ for all $z \in N$, since $D^z(\omega) = \rho(z)(\omega) = 1$ for all $\omega \in \Omega$. Hence there exists a surjective Hopf algebra map

$$\gamma : \mathbf{u}_\epsilon(\mathfrak{l})^*/(D^z - 1 |\ z \in N) \twoheadrightarrow H.$$

Combining Corollary 1.13 with the PBW-basis of H and $\mathbf{u}_\epsilon(\mathfrak{l})$ we have that

$$\begin{aligned} \dim H &= \ell^{|I_+|+|I_-|} |\Sigma| = \ell^{|I_+|+|I_-|} \ell^{n-s} |\Omega| = \ell^{|I_+|+|I_-|} \ell^{n-s} |\widehat{\Omega}| = \ell^{|I_+|+|I_-|} \frac{\ell^n}{|N|} \\ &= \dim(\mathbf{u}_\epsilon(\mathfrak{l})^*/(D^z - 1 |\ z \in N)), \end{aligned}$$

which implies that γ is an isomorphism. $\square$

Remark 2.15. The lemma above is very similar to a result used by E. Müller in the case of type A_n [M00, Sec. 4] for the classification of the finite-dimensional quotients of $\mathcal{O}_\epsilon(SL_N)$. The new point of view here consists in regarding H as a quotient of the dual of $\mathbf{u}_\epsilon(\mathfrak{l})$.

Before going on with the construction we need the following technical lemma. Let $\mathbf{X} = \{D^z |\ z \in \widehat{\mathbb{T}_{I^c}}\}$ be the set of central group-like elements of $\mathbf{u}_\epsilon(\mathfrak{l})^*$ given by Lemma 2.14.

Lemma 2.16. *There exists a subgroup* $\mathbf{Z} := \{\partial^z |\ z \in \widehat{\mathbb{T}_{I^c}}\}$ *of* $G(A_{\mathfrak{l},\sigma})$ *isomorphic to* $\mathbf{X}$ *consisting of central elements.*

Proof. By Proposition 2.6 (b), we know that there exists an algebra map $\psi : \Gamma_\epsilon(\mathfrak{l}) \to \mathbf{u}_\epsilon(\mathfrak{l})$; it induces a coalgebra map $\psi^* : \mathbf{u}_\epsilon(\mathfrak{l})^* \to \Gamma_\epsilon(\mathfrak{l})^\circ$ such that the following diagram commutes

$$\begin{tikzcd}
\Gamma_\epsilon(\mathfrak{g})^\circ \arrow[d, "\mathrm{Res}"'] & \mathbf{u}_\epsilon(\mathfrak{g})^* \arrow[l, "\varphi^*"'] \arrow[d, "p"] \\
\Gamma_\epsilon(\mathfrak{l})^\circ & \mathbf{u}_\epsilon(\mathfrak{l})^*. \arrow[l, "\psi^*"]
\end{tikzcd}$$

Here, φ^* is the coalgebra map induced by the algebra map $\varphi : \Gamma_\epsilon(\mathfrak{g}) \to \mathbf{u}_\epsilon(\mathfrak{l})$ given by Lemma 1.10, whose restriction to $\Gamma_\epsilon(\mathfrak{l})$ defines ψ. Furthermore, by the proof of Proposition 2.6 (c), $\operatorname{Im}\varphi^* \subseteq \mathcal{O}_\epsilon(G)$; since $\operatorname{Res}(\mathcal{O}_\epsilon(G)) = \mathcal{O}_\epsilon(L)$, it follows that $\operatorname{Im}\psi^* \subseteq \mathcal{O}_\epsilon(L)$. Consequently, we obtain a group of group-like elements $\mathbf{Y} = \{d^z = \psi^*(D^z) | \ z \in \widehat{\mathbb{T}_{I^c}}\}$ in $\mathcal{O}_\epsilon(L)$. Moreover, by Lemma 2.2 and the definitions of ψ and the elements D_i, the elements of $\mathbf{Y}$ are central.

Since the map $\nu : \mathcal{O}_\epsilon(L) \to A_{\mathfrak{l},\sigma}$ given by the pushout construction is surjective, the image of $\mathbf{Y}$ defines a group of central group-like elements in $A_{\mathfrak{l},\sigma}$:

$$\mathbf{Z} = \{\partial^z = \nu(d^z) | \ z \in \widehat{\mathbb{T}_{I^c}}\}.$$

Besides, $|\mathbf{Z}| = |\mathbf{Y}| = |\mathbf{X}| = \ell^s$. Indeed, $\bar{\pi}(\mathbf{Z}) = \bar{\pi}\nu(\mathbf{Y}) = \pi_L(\mathbf{Y}) = \pi_L\psi^*(\mathbf{X}) = \mathbf{X}$ since the diagram (12) is commutative and $\pi_L\psi^* = \mathrm{id}$. Hence $|\bar{\pi}(\mathbf{Z})| = |\mathbf{X}|$, from which the assertion follows. □

We are now ready for our first main result.

Theorem 2.17. *Let $\mathcal{D} = (I_+, I_-, N, \Gamma, \sigma, \delta)$ be a subgroup datum. Then there exists a Hopf algebra $A_{\mathcal{D}}$ which is a quotient of $\mathcal{O}_\epsilon(G)$ and fits into the exact sequence*

$$1 \to \mathcal{O}(\Gamma) \xrightarrow{\hat{\iota}} A_{\mathcal{D}} \xrightarrow{\hat{\pi}} H \to 1.$$

Concretely, $A_{\mathcal{D}}$ is given by the quotient $A_{\mathfrak{l},\sigma}/J_\delta$ where J_δ is the two-sided ideal generated by the set $\{\partial^z - \delta(z) | z \in N\}$ and the following diagram of exact sequences of Hopf algebras is commutative

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}(G) \arrow[r, "\iota"] \arrow[d, "\mathrm{res}"'] & \mathcal{O}_\epsilon(G) \arrow[r, "\pi"] \arrow[d, "\mathrm{Res}"] & \mathbf{u}_\epsilon(\mathfrak{g})^* \arrow[r] \arrow[d, "p"] & 1 \\
1 \arrow[r] & \mathcal{O}(L) \arrow[r, "\iota_L"] \arrow[d, "t_\sigma"'] & \mathcal{O}_\epsilon(L) \arrow[r, "\pi_L"] \arrow[d, "\nu"] & \mathbf{u}_\epsilon(\mathfrak{l})^* \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathcal{O}(\Gamma) \arrow[r, "j"] \arrow[d, equal] & A_{\mathfrak{l},\sigma} \arrow[r, "\bar{\pi}"] \arrow[d, "t"] & \mathbf{u}_\epsilon(\mathfrak{l})^* \arrow[r] \arrow[d, "v"] & 1 \\
1 \arrow[r] & \mathcal{O}(\Gamma) \arrow[r, "\hat{\iota}"] & A_{\mathcal{D}} \arrow[r, "\hat{\pi}"] & H \arrow[r] & 1.
\end{tikzcd} \tag{13}$$

Proof. By Remark 2.12, N determines a subgroup Σ of $\mathbb{T}$ and the triple (Σ, I_+, I_-) give rise to a surjective Hopf algebra map $r : \mathbf{u}_\epsilon(\mathfrak{g})^* \to H$. Since $\sigma : \Gamma \to L \subseteq G$ is injective, by the first two steps developed before one can construct a Hopf algebra $A_{\mathfrak{l},\sigma}$ which is a quotient of $\mathcal{O}_\epsilon(G)$ and an extension of $\mathcal{O}(\Gamma)$ by $\mathbf{u}_\epsilon(\mathfrak{l})^*$, where $\mathbf{u}_\epsilon(\mathfrak{l})$ is the Hopf subalgebra of $\mathbf{u}_\epsilon(\mathfrak{g})$ associated to the triple $(\mathbb{T}, I_+, I_-)$. Moreover, by Lemma 2.14 (b), H is the quotient of $\mathbf{u}_\epsilon(\mathfrak{l})^*$ by the two-sided ideal $(D^z - 1|\ z \in N)$. If $\delta : N \to \widehat{\Gamma}$ is a group map, then the elements $\delta(z)$ are central group-like elements in $A_{\mathfrak{l},\sigma}$ for all $z \in N$, and the two-sided ideal J_δ of $A_{\mathfrak{l},\sigma}$ generated by the set $\{\partial^z - \delta(z) | z \in N\}$ is a Hopf ideal. Hence, by [M00, Prop. 3.4 (c)] the following sequence is exact

$$1 \to \mathcal{O}(\Gamma)/\mathfrak{J} \to A_{\mathfrak{l},\sigma}/J_\delta \to \mathbf{u}_\epsilon(\mathfrak{l})^*/\bar{\pi}(\mathfrak{J}_\delta) \to 1,$$

where $\mathfrak{J} = J_\delta \cap \mathcal{O}(\Gamma)$. Since $\bar{\pi}(\partial^z) = D^z$ and $\bar{\pi}(\delta(z)) = 1$ for all $z \in N$, we have that $\bar{\pi}(J_\delta)$ is the two-sided ideal of $\mathbf{u}_\epsilon(\mathfrak{l})^*$ given by $(D^z - 1|\ z \in N)$, which implies by Lemma 2.14 (b) that $\mathbf{u}_\epsilon(\mathfrak{l})^*/\bar{\pi}(\mathfrak{J}_\delta) = H$. Hence, if we denote $A_{\mathcal{D}} := A_{\mathfrak{l},\sigma}/J_\delta$, we can re-write the exact sequence of above as

$$1 \to \mathcal{O}(\Gamma)/\mathfrak{J} \to A_{\mathcal{D}} \to H \to 1. \tag{14}$$

To end the proof it is enough to see that $\mathfrak{J} = J_\delta \cap \mathcal{O}(\Gamma) = 0$. Clearly, J_δ coincides with the two-sided ideal $(\partial^z\delta(z)^{-1} - 1|\ z \in N)$ of $A_{\mathfrak{l},\sigma}$. Moreover, $\Upsilon := \{\partial^z\delta(z)^{-1}|\ z \in N\}$ is a subgroup of central group-like elements of $G(A_{\mathfrak{l},\sigma})$ and $J_\delta = (g-1|\ g \in \Upsilon) = A_{\mathfrak{l},\sigma}\mathbb{C}[\Upsilon]^+$. Let $\partial N = \{\partial^z|\ z \in N\}$. Then clearly the subalgebra $B := \mathcal{O}(\Gamma)\mathbb{C}[\partial N]$ is a central Hopf subalgebra of A_σ which contains $\mathbb{C}[\Upsilon]$. Further, $B \simeq \mathcal{O}(\tilde{\Gamma})$ for some algebraic group $\tilde{\Gamma}$ and one has the following exact sequence of Hopf algebras

$$1 \to \mathcal{O}(\Gamma) \to \mathcal{O}(\tilde{\Gamma}) \to R \to 1,$$

where $R = \mathcal{O}(\tilde{\Gamma})/\mathcal{O}(\tilde{\Gamma})\mathcal{O}(\Gamma)^+$. But $R \simeq \bar{\pi}(\mathcal{O}(\tilde{\Gamma})) = \mathbb{C}[N]$, since

$$\begin{aligned}\bar{\pi}(\mathcal{O}(\tilde{\Gamma})) &= [\mathcal{O}(\tilde{\Gamma}) + \mathcal{O}(\Gamma)^+A_{\mathfrak{l},\sigma}]/[\mathcal{O}(\Gamma)^+A_{\mathfrak{l},\sigma}] \simeq \mathcal{O}(\tilde{\Gamma})/[\mathcal{O}(\tilde{\Gamma}) \cap (\mathcal{O}(\Gamma)^+A_{\mathfrak{l},\sigma})] \\ &\simeq \mathcal{O}(\tilde{\Gamma})/\mathcal{O}(\tilde{\Gamma})\mathcal{O}(\Gamma)^+.\end{aligned}$$

The last isomorphism follows from the fact that $\mathcal{O}(\tilde{\Gamma}) \cap (\mathcal{O}(\Gamma)^+A_{\mathfrak{l},\sigma}) = \mathcal{O}(\tilde{\Gamma})\mathcal{O}(\Gamma)^+$. Indeed, since $\mathcal{O}(\tilde{\Gamma})$ is a central Hopf subalgebra of the noetherian algebra $A_{\mathfrak{l},\sigma}$, by [S92, Thm. 3.3], $\mathcal{O}(\tilde{\Gamma})$ is a direct summand of $A_{\mathfrak{l},\sigma}$ as $\mathcal{O}(\tilde{\Gamma})$-module, say $A_{\mathfrak{l},\sigma} = \mathcal{O}(\tilde{\Gamma}) \oplus M$. Then $\mathcal{O}(\Gamma)^+A_{\mathfrak{l},\sigma} = \mathcal{O}(\Gamma)^+\mathcal{O}(\tilde{\Gamma}) \oplus \mathcal{O}(\Gamma)^+M$ and the claim follows since $\mathcal{O}(\tilde{\Gamma}) \cap \mathcal{O}(\Gamma)^+M = 0$. Hence we have an exact sequence

$$1 \to \mathcal{O}(\Gamma) \to \mathcal{O}(\tilde{\Gamma}) \xrightarrow{\bar{\pi}} \mathbb{C}[N] \to 1,$$

which is cleft by the proof of Lemma 2.16, since $\bar{\pi}$ admits a coalgebra section. Moreover, this section on $\mathbb{C}[N]$ is by definition a bialgebra section, implying that $\mathcal{O}(\tilde{\Gamma}) \simeq \mathcal{O}(\Gamma) \otimes \mathbb{C}[\partial N]$.

Let $\Lambda = \frac{1}{|\Upsilon|}\sum_{z\in N}\delta(z)\partial^{-z}$ be the integral of $\mathbb{C}[\Upsilon]$ and denote by L_Λ the endomorphism of $\mathcal{O}(\tilde{\Gamma})$ given by left multiplication of Λ. Since $\mathcal{O}(\tilde{\Gamma}) \simeq \mathcal{O}(\Gamma)\otimes\mathbb{C}[\partial N] \simeq \mathcal{O}(\Gamma)\otimes\mathbb{C}[\Upsilon]$, it follows that $\operatorname{Ker} L_\Lambda = \mathcal{O}(\Gamma)(\mathbb{C}[\Upsilon])^+$. But since $A_{\mathfrak{l},\sigma} = \mathcal{O}(\tilde{\Gamma})\oplus M$ as $\mathcal{O}(\tilde{\Gamma})$-modules, we have that $J_\delta\cap\mathcal{O}(\tilde{\Gamma}) = A_{\mathfrak{l},\sigma}(\mathbb{C}[\Upsilon])^+\cap\mathcal{O}(\tilde{\Gamma}) = \mathcal{O}(\tilde{\Gamma})(\mathbb{C}[\Upsilon])^+ = \mathcal{O}(\Gamma)(\mathbb{C}[\Upsilon])^+ = \operatorname{Ker} L_\Lambda$. Hence $J_\delta\cap\mathcal{O}(\Gamma) = \operatorname{Ker} L_\Lambda\cap\mathcal{O}(\Gamma) = 0$ for if $x\in\operatorname{Ker} L_\Lambda\cap\mathcal{O}(\Gamma)$, then

$$0 = \Lambda x = \frac{1}{|\Upsilon|}\sum_{z\in N}(\delta(z)\otimes\partial^{-z})(x\otimes 1) = \frac{1}{|\Upsilon|}\sum_{z\in N}\delta(z)x\otimes\partial^{-z},$$

which implies that $\delta(z)x = 0$ for all $z\in N$, because the elements ∂^z are linearly independent. Thus $x=0$ since $\delta(z)$ is invertible for all $z\in N$. $\square$

Remark 2.18. (a) If Γ is finite-dimensional, then $\mathcal{O}(\Gamma) = \mathbb{C}^\Gamma$ and by Remark 2.11, $\dim A_{\mathcal{D}} = |\Gamma|\dim H$. In this case, $\mathcal{D}$ is a finite subgroup datum and the last step of the proof of the theorem above follows easily by dimension arguments. Indeed, by [M00, Lemma 4.8], we have that $\dim A_{\mathcal{D}} = \dim A_{\mathfrak{l},\sigma}/|\Upsilon|$. Since $A_{\mathfrak{l},\sigma}$ and $A_{\mathcal{D}}$ are extensions, it follows that

(15)
$$\dim\mathbb{C}^\Gamma\frac{\dim\mathbf{u}_\epsilon(\mathfrak{l})}{|\Upsilon|} = \dim A_{\mathcal{D}} = \dim(\mathbb{C}^\Gamma/\mathfrak{J})\dim H = \dim(\mathbb{C}^\Gamma/\mathfrak{J})\frac{\dim\mathbf{u}_\epsilon(\mathfrak{l})}{|N|}.$$

Since $\bar{\pi}(\Upsilon) = \{D^z|\ z\in N\}$ and $\bar{\pi}(\partial^z\delta(z)^{-1}) = D^z = 1$ if and only if $z=0$, we have that $|\Upsilon| = |N|$. Thus, from the equality (15) it follows that $\mathbb{C}^\Gamma = \mathbb{C}^\Gamma/\mathfrak{J}$.

(b) All exact sequences in the rows of diagram (13) are of the type $\mathcal{B}\hookrightarrow\mathcal{A}\twoheadrightarrow\mathcal{H}$, where $\mathcal{B}$ is central in $\mathcal{A}$ and $\mathcal{H}$ is finite-dimensional. Thus, by [KT81, Thm. 1.7], $\mathcal{B}\subset\mathcal{A}$ is an $\mathcal{H}$-Galois extension and $\mathcal{A}$ is a finitely-generated projective $\mathcal{B}$-module. Moreover, using Lemma 1.10 and Proposition 2.6 (b), one can see that the first three exact sequences are cleft.

2.4. **Relations between quantum subgroups.** Let U be any Hopf algebra and consider the category $\mathcal{QUOT}(U)$, whose objects are surjective Hopf algebra maps $q: U\to A$. If $q: U\to A$ and $q': U\to A'$ are such maps, then an arrow $q\xrightarrow{\alpha} q'$ in $\mathcal{QUOT}(U)$ is a Hopf algebra map $\alpha: A\to A'$ such that $\alpha q = q'$. In this language, a *quotient* of U is just an isomorphism class of objects in $\mathcal{QUOT}(U)$; let $[q]$ denote the class of the map q. There is a partial order in the set of quotients of U, given by $[q]\leq[q']$ iff there exists an arrow $q\xrightarrow{\alpha} q'$ in $\mathcal{QUOT}(U)$. Notice that $[q]\leq[q']$ and $[q']\leq[q]$ implies $[q]=[q']$.

Our aim is to describe the partial order in the set $[q_{\mathcal{D}}]$, $\mathcal{D}$ a subgroup datum, of quotients $q_{\mathcal{D}}: \mathcal{O}_\epsilon(G)\twoheadrightarrow A_{\mathcal{D}}$ given by Theorem 2.17. Eventually, this will be the partial order in the set of all quotients of $\mathcal{O}_\epsilon(G)$. We begin by the following definition. By an abuse of notation we write $[A_{\mathcal{D}}] = [q_{\mathcal{D}}]$.

Definition 2.19. Let $\mathcal{D} = (I_+, I_-, N, \Gamma, \sigma, \delta)$ and $\mathcal{D}' = (I'_+, I'_-, N', \Gamma', \sigma', \delta')$ be subgroup data. We say that $\mathcal{D}\leq\mathcal{D}'$ iff

- $I'_+ \subseteq I_+$ and $I'_- \subseteq I_-$.
  In particular, this condition implies that $I' \subseteq I$, $\mathbb{T}_{I'} \subseteq \mathbb{T}_I$ and $\mathbb{T}_{I^c} \subseteq \mathbb{T}_{I'^c}$. Since $\Sigma = \mathbb{T}_I \times \Omega$ and $\Sigma' = \mathbb{T}_{I'} \times \Omega'$, we have that $\Omega' \subseteq \Omega \subseteq \mathbb{T}_{I^c} \subseteq \mathbb{T}_{I'^c}$. As $\mathbb{T}_{I'^c} = \mathbb{T}_{I^c} \times \mathbb{T}_{I'^c - I^c}$, the restriction map $\widehat{\mathbb{T}_{I'^c}} \twoheadrightarrow \widehat{\mathbb{T}_{I^c}}$ admits a canonical section η and $\eta(N) \subseteq N'$.
- There exists a morphism of algebraic groups $\tau : \Gamma' \to \Gamma$ such that $\sigma\tau = \sigma'$.
- $\delta'\eta = {}^t\tau\delta$.

Furthermore, we say that $\mathcal{D} \simeq \mathcal{D}'$ iff $\mathcal{D} \leq \mathcal{D}'$ and $\mathcal{D}' \leq \mathcal{D}$. This means that

- $I_+ = I'_+$ and $I_- = I'_-$.
- There exists an isomorphism of algebraic groups $\tau : \Gamma' \to \Gamma$ such that $\sigma\tau = \sigma'$.
- $N = N'$ and $\delta' = {}^t\tau\delta$.

Theorem 2.20. *Let $\mathcal{D}$ and $\mathcal{D}'$ be subgroup data. Then*
(a) $[A_{\mathcal{D}}] \leq [A_{\mathcal{D}'}]$ iff $\mathcal{D} \leq \mathcal{D}'$.
(b) $[A_{\mathcal{D}}] = [A_{\mathcal{D}'}]$ iff $\mathcal{D} \simeq \mathcal{D}'$.

Proof. Let $q = q_{\mathcal{D}}$ and $q' = q_{\mathcal{D}'}$. Suppose that $[A_{\mathcal{D}}] \leq [A_{\mathcal{D}'}]$, that is, there exists a surjective Hopf algebra map $\alpha : A_{\mathcal{D}} \to A_{\mathcal{D}'}$ such that $\alpha q = q'$. Since by Theorem 2.17, $\hat{\imath}\ {}^t\sigma = q\iota$ and $\hat{\imath}'\ {}^t\sigma' = q'\iota$, we have that $\alpha\hat{\imath}\ {}^t\sigma = \alpha q\iota = q'\iota = \hat{\imath}'\ {}^t\sigma'$. Thus, the Hopf algebra map $\beta := \alpha\hat{\imath} : \mathcal{O}(\Gamma) \to \mathcal{O}(\Gamma')$ is surjective with $\operatorname{Im}\beta \subseteq \operatorname{Im}\ {}^t\sigma$ and its transpose defines an injective map of algebraic groups $\tau : \Gamma' \to \Gamma$ such that $\sigma\tau = \sigma'$.

Again by Theorem 2.17, we know that both $A_{\mathcal{D}}$ and $A_{\mathcal{D}'}$ are central extensions by $H \simeq A_{\mathcal{D}}/A_{\mathcal{D}}\mathcal{O}(\Gamma)^+$ and $H' \simeq A_{\mathcal{D}'}/A_{\mathcal{D}'}\mathcal{O}(\Gamma')^+$, respectively. Since $\hat{\pi}'\alpha(A_{\mathcal{D}}\mathcal{O}(\Gamma)^+) = \hat{\pi}'(A_{\mathcal{D}'}\mathcal{O}(\Gamma')^+) = 0$, there exists a surjective Hopf algebra map $\gamma : H \to H'$ such that the following diagram commutes

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}(G) \arrow[r, "\iota"] \arrow[d, "{}^t\sigma"'] \arrow[dd, bend right, dotted, "{}^t\sigma'"'] & \mathcal{O}_\epsilon(G) \arrow[r, "\pi"] \arrow[d, "q"] \arrow[dd, bend right, dotted, "q'"'] & \mathbf{u}_\epsilon(\mathfrak{g})^* \arrow[r] \arrow[d, "r"'] \arrow[dd, bend left, dotted, "r'"] & 1 \\
1 \arrow[r] & \mathcal{O}(\Gamma) \arrow[r, "\hat{\imath}"] \arrow[d, "\beta"] & A_{\mathcal{D}} \arrow[r, "\hat{\pi}"] \arrow[d, "\alpha"] & H \arrow[r] \arrow[d, "\gamma"'] & 1 \\
1 \arrow[r] & \mathcal{O}(\Gamma') \arrow[r, "\hat{\imath}'"] & A_{\mathcal{D}'} \arrow[r, "\hat{\pi}'"] & H' \arrow[r] & 1.
\end{tikzcd}$$

Since ${}^tr : H^* \hookrightarrow \mathbf{u}_\epsilon(\mathfrak{g})$ and ${}^tr' : (H')^* \hookrightarrow \mathbf{u}_\epsilon(\mathfrak{g})$ are just the inclusions, it follows that ${}^t\gamma : (H')^* \hookrightarrow H^*$ is the same inclusion. If H^* and $(H')^*$ are determined by the triples (Σ, I_+, I_-) and (Σ', I'_+, I'_-), it follows that $\Sigma' \subseteq \Sigma$, $I'_+ \subseteq I_+$, $I'_- \subseteq I_-$, whence $\eta(N) \subseteq N'$. Thus, $\mathbf{u}_\epsilon(\mathfrak{l}') \subseteq \mathbf{u}_\epsilon(\mathfrak{l})$ by Lemma 2.4.

Now by Theorem 2.17, $\delta(z) = t(\partial^z)$ in $A_{\mathcal{D}}$ and $\delta'(z') = t'(\partial^{z'})$ in $A_{\mathcal{D}'}$, for all $z \in N$ and $z' \in N'$. Thus, for all $z \in N$ we have

$${}^t\tau\delta(z) = \alpha\delta(z) = \alpha t(\partial^z) = \alpha t\nu(\psi^*(D^z)) = t'\nu'((\psi')^*\eta(D^z)) = \delta'(\eta(z)),$$

where the fourth equality follows from the construction of the quotients $A_{\mathcal{D}}$, $A_{\mathcal{D}'}$ and $\alpha q = q'$. All this implies that $\mathcal{D} \leq \mathcal{D}'$.

Suppose now that $\mathcal{D} \leq \mathcal{D}'$. This implies that $\mathbf{u}_\epsilon(\mathfrak{l}') \subseteq \mathbf{u}_\epsilon(\mathfrak{l})$ and by construction, there exists a Hopf algebra map $\kappa : \mathcal{O}_\epsilon(L) \to \mathcal{O}_\epsilon(L')$ such that

$$\begin{tikzcd}
\mathcal{O}_\epsilon(G) \arrow[r, two heads, "\mathrm{Res}"] \arrow[dr, two heads, "\mathrm{Res}'"'] & \mathcal{O}_\epsilon(L) \arrow[d, two heads, "\kappa"] \\
 & \mathcal{O}_\epsilon(L')
\end{tikzcd}$$

commutes. Since ${}^t\tau\ {}^t\sigma = {}^t\sigma'$, there exists a commutative diagram

$$\begin{tikzcd}
\mathcal{O}(L) \arrow[rr, "\iota_L"] \arrow[d, "{}^t\sigma"'] & & \mathcal{O}_\epsilon(L) \arrow[d, "\nu"'] \arrow[ddr, bend left, "t'\nu'\kappa"] & \\
\mathcal{O}(\Gamma) \arrow[rr, "\bar{\iota}"'] \arrow[dr, bend right, "{}^t\tau"'] & & A_{\mathfrak{l},\sigma} & \\
 & \mathcal{O}(\Gamma') \arrow[rr, "\bar{\iota}'"'] & & A_{\mathcal{D}'}.
\end{tikzcd}$$

As $A_{\mathfrak{l},\sigma}$ is a pushout, there exists a surjective Hopf algebra map $\tilde{\alpha} : A_{\mathfrak{l},\sigma} \to A_{\mathcal{D}'}$ such that $\tilde{\alpha}\nu = t'\nu'\kappa$. Since $A_{\mathcal{D}} = A_{\mathfrak{l},\sigma}/J_\delta$, to show the existence of a surjective map $\alpha : A_{\mathcal{D}} \to A_{\mathcal{D}'}$ such that $\alpha q = q'$, it is enough to prove that $\tilde{\alpha}(J_\delta) = 0$. But J_δ is the two-sided ideal of $A_{\mathfrak{l},\sigma}$ generated by $\delta(z) - \partial^z$ with $z \in N$; now

$$\begin{aligned}
\tilde{\alpha}(\delta(z) - \partial^z) &= {}^t\tau\delta(z) - \tilde{\alpha}(\nu\psi^*(D^z)) = {}^t\tau\delta(z) - t'\nu'\eta(z) \\
&= {}^t\tau\delta(z) - \delta'\eta(z) = 0,
\end{aligned}$$

by assumption. Hence, $\tilde{\alpha}(J_\delta) = 0$. This finishes the proof of (a). Now (b) follows immediately. $\square$

3. Determining quantum subgroups

Let $q : \mathcal{O}_\epsilon(G) \to A$ be a surjective Hopf algebra map. We prove now that it is isomorphic to $q_{\mathcal{D}} : \mathcal{O}_\epsilon(G) \to A_{\mathcal{D}}$ for some subgroup datum $\mathcal{D}$. This concludes the proof of Theorem 1.

The Hopf subalgebra $K = q(\mathcal{O}(G))$ is central in A and whence A is an H-extension of K, where H is the Hopf algebra $H = A/AK^+$. Indeed, it follows directly from [Mo93, Prop. 3.4.3], because A is faithfully flat over K by [S92, Thm. 3.3]. Since K is a quotient of $\mathcal{O}(G)$, there exists an algebraic group Γ and an injective map of algebraic groups $\sigma : \Gamma \to G$ such that $K \simeq \mathcal{O}(\Gamma)$. Moreover, since $q(\mathcal{O}_\epsilon(G)\mathcal{O}(G)^+) = AK^+$, we have that $\mathcal{O}_\epsilon(G)\mathcal{O}(G)^+ \subseteq \operatorname{Ker} \hat{\pi} q$, where $\hat{\pi} : A \to H$ is the canonical projection. Since $\mathbf{u}_\epsilon(\mathfrak{g})^* \simeq \mathcal{O}_\epsilon(G)/[\mathcal{O}_\epsilon(G)\mathcal{O}(G)^+]$, there exists a surjective map $r : \mathbf{u}_\epsilon(\mathfrak{g})^* \to$

H and by Proposition 1.12, H^* is determined by a triple (Σ, I_+, I_-). In particular, we have the following commutative diagram

(16)
$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}(G) \arrow[r, "\iota"] \arrow[d, "{}^t\sigma"'] & \mathcal{O}_\epsilon(G) \arrow[r, "\pi"] \arrow[d, "q"] & \mathbf{u}_\epsilon(\mathfrak{g})^* \arrow[r] \arrow[d, "r"] & 1 \\
1 \arrow[r] & \mathcal{O}(\Gamma) \arrow[r, "\hat{\iota}"] & A \arrow[r, "\hat{\pi}"] & H \arrow[r] & 1.
\end{tikzcd}$$

Let N correspond to Σ as in Remark 2.12. Our aim is to show that there exists δ such that $A \simeq A_{\mathcal{D}}$ for the subgroup datum $\mathcal{D} = (I_+, I_-, N, \Gamma, \sigma, \delta)$. Recall the Lie algebra $\mathfrak{l}$ from Definition 1.1 and the Hopf algebra $\mathbf{u}_\epsilon(\mathfrak{l}) \supseteq H^*$ from 2.1.2. Denote by $v : \mathbf{u}_\epsilon(\mathfrak{l})^* \to H$ the surjective Hopf algebra map induced by this inclusion.

Lemma 3.1. *The diagram* (16) *factorizes through the exact sequence*

$$1 \longrightarrow \mathcal{O}(L) \xrightarrow{\iota_L} \mathcal{O}_\epsilon(L) \xrightarrow{\pi_L} \mathbf{u}_\epsilon(\mathfrak{l})^* \longrightarrow 1,$$

that is, there exist Hopf algebra maps u, w such that the following diagram with exact rows commutes:

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}(G) \arrow[r, "\iota"] \arrow[d, "\mathrm{res}"'] \arrow[dd, bend right=60, dotted, "{}^t\sigma"'] & \mathcal{O}_\epsilon(G) \arrow[r, "\pi"] \arrow[d, "\mathrm{Res}"] \arrow[dd, bend right=60, dotted, "q"'] & \mathbf{u}_\epsilon(\mathfrak{g})^* \arrow[r] \arrow[d, "p"'] \arrow[dd, bend left=60, dotted, "r"] & 1 \\
1 \arrow[r] & \mathcal{O}(L) \arrow[r, "\iota_L"] \arrow[d, "u"] & \mathcal{O}_\epsilon(L) \arrow[r, "\pi_L"] \arrow[d, "w"] & \mathbf{u}_\epsilon(\mathfrak{l})^* \arrow[r] \arrow[d, "v"'] & 1 \\
1 \arrow[r] & \mathcal{O}(\Gamma) \arrow[r, "\hat{\iota}"] & A \arrow[r, "\hat{\pi}"] & H \arrow[r] & 1.
\end{tikzcd}$$

Proof. To show the existence of the maps u and w it is enough to show that $\operatorname{Ker}\operatorname{Res} \subseteq \operatorname{Ker} q$, since u is simply $w\iota_L$. This clearly implies that $v\pi_L = \hat{\pi}w$.

Let $\check{U}_\epsilon(\mathfrak{b}_+)$ and $\check{U}_\epsilon(\mathfrak{b}_-)$ be the Borel subalgebras of $\check{U}_\epsilon(\mathfrak{g})$ (see [DL94] and [J96, Cap. 4]), and let $\mathbb{A}_\epsilon$ be the subalgebra of $\check{U}_\epsilon(\mathfrak{b}_+) \otimes \check{U}_\epsilon(\mathfrak{b}_-)$ generated by the elements

$$\{1 \otimes e_j, f_j \otimes 1, K_{-\lambda} \otimes K_\lambda : \ 1 \leq j \leq n, \lambda \in P\},$$

where P is the weight lattice. By [DL94, Sec. 4.3], this algebra has a basis given by the set $\{fK_{-\lambda} \otimes K_\lambda e\}$, where $\lambda \in P$ and e, f are monomials in e_α and f_β respectively, α, $\beta \in Q_+$. Moreover, $\mathbb{A}_\epsilon$ is a (Q_-, P, Q_+)-graded algebra whose gradation is given by

$$\deg(f_j \otimes 1) = (-\alpha_j, 0, 0), \qquad \deg(1 \otimes e_j) = (0, 0, \alpha_j),$$
$$\deg(K_{-\lambda} \otimes K_\lambda) = (0, \lambda, 0),$$

for all $1 \leq j \leq n$, $\lambda \in P$. By [DL94, 4.3 and 6.5], there exists an injective algebra map $\mu_\epsilon : \mathcal{O}_\epsilon(G) \to \mathbb{A}_\epsilon$ such that $\mu_\epsilon(\mathcal{O}(G)) \subseteq \mathbb{A}_0$, where $\mathbb{A}_0$ is the subalgebra of $\mathbb{A}_\epsilon$ generated by the elements

$$\{1 \otimes e_j^\ell, f_j^\ell \otimes 1, K_{-\ell\lambda} \otimes K_{\ell\lambda} : \ 1 \leq j \leq n, \lambda \in P\}.$$

Hence, it is enough to show that $\mu_\epsilon(\operatorname{Ker}\operatorname{Res}) \subseteq \mu_\epsilon(\operatorname{Ker} q)$.

Claim: $\mu_\epsilon(\operatorname{Ker}\operatorname{Res})$ is the two-sided ideal $\mathcal{I}$ generated by the elements

$$\{1 \otimes e_k, f_j \otimes 1 : \ \alpha_k \notin I_-, \alpha_j \notin I_+\}.$$

Indeed, let $\lambda \in P_+$ and let $\psi_\lambda \in \Gamma_\epsilon(\mathfrak{g})^\circ$ such that

$$\psi_\lambda(FME) = \delta_{1,E}\delta_{1,F}M(\lambda), \qquad \psi_{-\lambda}(EMF) = \delta_{1,E}\delta_{1,F}M(-\lambda),$$

for all elements FME of the PBW basis of $\Gamma_\epsilon(\mathfrak{g})$, where $M \in Q$ and the form $M(\lambda)$ is simply the linear extension of the bilinear form $< \alpha_j, \lambda >= \epsilon^{d_i(\alpha_i,\lambda)}$ for all $\lambda \in P$, $1 \leq i \leq n$. By [DL94, Sec. 4.4], there exist matrix coefficients $\psi_{\pm\lambda}^{\pm\alpha}$, and $\alpha \in Q_+$ such that

$$\psi_{-\lambda}^{\alpha}(EMF) = \psi_{-\lambda}(EMFE_\alpha), \qquad \psi_{-\lambda}^{-\alpha}(EMF) = \psi_{-\lambda}(F_\alpha EMF),$$

for all elements EMF of the PBW basis of $\Gamma_\epsilon(\mathfrak{g})$. Moreover, one has that

$$\mu_\epsilon(\psi_{-\varpi_i}) = K_{-\varpi_i}\otimes K_{\varpi_i}, \qquad \mu_\epsilon(\psi_{-\varpi_i}^{\alpha_k}) = K_{-\varpi_i} \otimes K_{\varpi_i}e_k,$$
$$\mu_\epsilon(\psi_{-\varpi_i}^{-\alpha_j}) = f_j K_{-\varpi_i} \otimes K_{\varpi_i},$$

for all $1 \leq i,j \leq n$. Through a direct computation one can see that $\psi_{-\varpi_i}^{\alpha_k}$, $\psi_{-\varpi_i}^{-\alpha_j} \in \operatorname{Ker}\operatorname{Res}$ and

$$\mu_\epsilon(\psi_{\varpi_i}\psi_{-\varpi_i}^{\alpha_k}) = 1 \otimes e_k \qquad \mu_\epsilon(\psi_{-\varpi_i}^{-\alpha_j}\psi_{\varpi_i}) = f_j \otimes 1.$$

for all $\alpha_k \notin I_-$, $\alpha_j \notin I_+$. Hence, the generators of $\mathcal{I}$ are in $\mu_\epsilon(\operatorname{Ker}\operatorname{Res})$.

Conversely, if $h \in \operatorname{Ker}\operatorname{Res}$, then $h|_{\Gamma_\epsilon(\mathfrak{l})} = 0$ and by definition we have that

$$< \mu_\epsilon(h), EM \otimes NF >=< h, EMNF >= 0,$$

for all elements $EMNF$ of the PBW basis of $\Gamma_\epsilon(\mathfrak{l})$. Thus, using the existence of perfect pairings (see [DL94, Sec. 3.2]) and evaluating in adequate elements, it follows that each term of the basis $\{fK_{-\lambda} \otimes K_\lambda e\}$ that appears in $\mu_\epsilon(h)$ must lie in $\mathcal{I}$.

Since $0 = \pi_L \operatorname{Res}(h) = r\pi(h) = \hat{\pi}q(h)$, we have that $q(h) \in \operatorname{Ker}\hat{\pi} = \mathcal{O}(\Gamma)^+A = q(\mathcal{O}(G)^+\mathcal{O}_\epsilon(G))$. Then there exist $a \in \mathcal{O}(G)^+\mathcal{O}_\epsilon(G)$ and $c \in \operatorname{Ker} q$ such that $h = a + c$; in particular, for all generators t of $\mathcal{I}$ we have that $t = \mu_\epsilon(a) + \mu_\epsilon(c)$, where $\mu_\epsilon(a)$ is contained in $\mathbb{A}_0$. Comparing degrees in both sides of the equality we have that $\mu_\epsilon(a) = 0$, which implies that each generator of $\mathcal{I}$ must lie in $\mu_\epsilon(\operatorname{Ker} q)$. □

The following lemma shows the convenience of characterizing the quotients $A_{\mathfrak{l},\sigma}$ of $\mathcal{O}_\epsilon(G)$ as pushouts.

Lemma 3.2. *$\sigma(\Gamma) \subseteq L$ and therefore A is a quotient of $A_{\mathfrak{l},\sigma}$ given by the pushout. Moreover, the following diagram commutes*

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}(G) \arrow[r, "\iota"] \arrow[d, "\mathrm{res}"'] & \mathcal{O}_\epsilon(G) \arrow[r, "\pi"] \arrow[d, "\mathrm{Res}"] & \mathbf{u}_\epsilon(\mathfrak{g})^* \arrow[r] \arrow[d, "p"] & 1 \\
1 \arrow[r] & \mathcal{O}(L) \arrow[r, "\iota_L"] \arrow[d, "u"'] & \mathcal{O}_\epsilon(L) \arrow[r, "\pi_L"] \arrow[d, "\nu"] & \mathbf{u}_\epsilon(\mathfrak{l})^* \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathcal{O}(\Gamma) \arrow[r, "j"] \arrow[d, equal] & A_{\mathfrak{l},\sigma} \arrow[r, "\bar{\pi}"] \arrow[d, "t"] & \mathbf{u}_\epsilon(\mathfrak{l})^* \arrow[r] \arrow[d, "v"] & 1 \\
1 \arrow[r] & \mathcal{O}(\Gamma) \arrow[r, "\hat{\iota}"] & A \arrow[r, "\hat{\pi}"] & H \arrow[r] & 1.
\end{tikzcd} \tag{17}$$

Proof. Recall the maps u, w defined in the lemma above; we have that $w\iota_L = \hat{\iota}u$, that is, the following diagram commutes

$$\begin{tikzcd}
\mathcal{O}(L) \arrow[r, "\iota_L"] \arrow[d, "u"'] & \mathcal{O}_\epsilon(L) \arrow[d, "\nu"] \arrow[ddr, bend left, "w"] & \\
\mathcal{O}(\Gamma) \arrow[r, "j"'] \arrow[drr, bend right, "\hat{\iota}"'] & A_{\mathfrak{l},\sigma} & \\
 & & A.
\end{tikzcd}$$

Since $A_{\mathfrak{l},\sigma}$ is a pushout, there exists a unique Hopf algebra map $t : A_{\mathfrak{l},\sigma} \to A$ such that $ts = w$ and $tj = \hat{\iota}$. This implies that $\operatorname{Ker} \bar{\pi} = j(\mathcal{O}(\Gamma))^+ A_{\mathfrak{l},\sigma} \subseteq \operatorname{Ker} \hat{\pi}t$ and therefore the diagram (17) is commutative. □

Let (Σ, I_+, I_-) be the triple that determines H. Recall that by Remark 2.12, giving a group Σ such that $\mathbb{T}_I \subseteq \Sigma \subseteq \mathbb{T}$ is the same as giving a subgroup $N \subseteq \widehat{\mathbb{T}_{I^c}}$. In fact, by Lemma 2.16, we know that the Hopf algebra $A_{\mathfrak{l},\sigma}$ contains a set of central group-like elements $\mathbf{Z} = \{\partial^z |\ z \in \widehat{\mathbb{T}_{I^c}}\}$ such that $\bar{\pi}(\partial^z) = D^z$ for all $z \in \widehat{\mathbb{T}_{I^c}}$ and $H = \mathbf{u}_\epsilon(\mathfrak{l})^*/(D^z - 1|\ z \in N)$. To see that $A = A_{\mathcal{D}}$ for a subgroup datum $\mathcal{D} = (I_+, I_-, N, \Gamma, \sigma, \delta)$ it remains to find a group map $\delta : N \to \widehat{\Gamma}$ such that $A \simeq A_{\mathfrak{l},\sigma}/J_\delta$. This is given by the last lemma of the paper.

Lemma 3.3. *There exists a group homomorphism $\delta : N \to \widehat{\Gamma}$ such that $J_\delta = (\partial^z - \delta(z)|\ z \in N)$ is a Hopf ideal of $A_{\mathfrak{l},\sigma}$ and $A \simeq A_{\mathcal{D}} = A_{\mathfrak{l},\sigma}/J_\delta$.*

Proof. Let $\partial^z \in \mathbf{Z}$. Then $\hat{\pi}t(\partial^z) = v\bar{\pi}(\partial^z) = 1$ for all $z \in N$, by Lemma 2.14 (b). Since $t(\partial^z)$ is a group-like element, this implies that $t(\partial^z) \in A^{\mathrm{co}\,\hat{\pi}} = \mathcal{O}(\Gamma)$. As $G(\mathcal{O}(\Gamma)) = \widehat{\Gamma}$, we have a group homomorphism δ given by

$$\delta : N \to \widehat{\Gamma}, \qquad \delta(z) = t(\partial^z) \quad \forall\ z \in N.$$

The two-sided ideal of $A_{\mathrm{I},\sigma}$ given by $J_\delta = (\partial^z - \delta(z)|\ z \in N)$ is clearly a Hopf ideal and $t(J_\delta) = 0$. Consequently we have a surjective Hopf algebra map $\theta : A_{\mathcal{D}} \twoheadrightarrow A$, which makes the following diagram commutative

$$\begin{tikzcd} 1 \arrow[r] & \mathcal{O}(\Gamma) \arrow[r, "\tilde{\iota}"] \arrow[d, equal] & A_{\mathcal{D}} \arrow[r, "\tilde{\pi}"] \arrow[d, two heads, "\theta"] & H \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathcal{O}(\Gamma) \arrow[r, "\hat{\iota}"] & A \arrow[r, "\hat{\pi}"] & H \arrow[r] & 1. \end{tikzcd} \tag{18}$$

Then θ is an isomorphism by Corollary 1.15. $\square$

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