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On Nichols Algebras With Standard Braiding

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REPÚBLICA ARGENTINA

ON NICHOLS ALGEBRAS WITH STANDARD BRAIDING

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Abstract. The class of standard braided vector spaces, introduced by Andruskiewitsch and the author in `arXiv:math/0703924v2` to understand the proof of a theorem of Heckenberger [H2], is slightly more general than the class of braided vector spaces of Cartan type. In the present paper, we classify standard braided vector spaces with finite-dimensional Nichols algebra. For any such braided vector space, we give a PBW-basis, a closed formula of the dimension and a presentation by generators and relations of the associated Nichols algebra.

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Introduction

A breakthrough in the development of the theory of Hopf algebras was the discovery of quantized enveloping algebra by Drinfeld and Jimbo [Dr, Ji]. This special class of Hopf algebras was intensively studied by many authors and from many points of view. In particular, finite-dimensional analogues of quantized enveloping algebras were introduced and investigated by Lusztig [L1, L2].

About ten years ago, a classification program of pointed Hopf algebras was launched by Andruskiewitsch and Schneider [AS1], see also [AS5]. The success of this program depends on finding solutions to several questions, among them:

> [A, Question 5.9] Given a braided vector space of diagonal type V, such that the entries of its matrix are roots of 1, compute the dimension of the associated Nichols algebra $\mathfrak{B}(V)$. If it is finite, give a nice presentation of $\mathfrak{B}(V)$.

Partial answers to this question were given in [AS2, H2] for the class of braided vector spaces of Cartan type. These answers were already crucial to prove a classification theorem for finite-dimensional Hopf algebras whose group is abelian with prime divisors of the order great than 7 [AS6]. Later, a complete answer to the first part of [A, Question 5.9] was given in [H3].

The notion of standard braided vector space, a special kind of diagonal braided vector space, was introduced in [AA], see Definition 3.5 below. This class includes properly the class of braided vector spaces of Cartan type.

The purpose of this paper is to develop from scratch the theory of standard braided vector spaces. Here are our main contributions:

- We give a complete classification of standard braided vector spaces with finite-dimensional Nichols algebras. As usual, we may assume the connectedness of the corresponding braiding. It turns out that standard braided vector spaces are of Cartan type when the associated Cartan matrix is of type C, D, E or F, see Proposition 3.8. For types A, B, G there are standard braided vector spaces not of Cartan type; these are listed in Propositions 3.9, 3.10 and 3.11. Those of type A_2 and B_2 appeared already in [Gr]. Our classification does not rely on [H3], but we can identify our examples in the tables of [H3].
- We describe a concrete PBW-basis of the Nichols algebra of a standard braided vector space as in the previous point; this follows from the general theory of Kharchenko [Kh] together with Theorem 1 of

[H2]. As an application, we give closed formulas for the dimension of these Nichols algebras.

- We present a concrete set of defining relations of the Nichols algebras of standard braided vector spaces as in the previous points. This is an answer to the second part of [A, Question 5.9] in the standard case. We note that this seems to be new even for Cartan type, for some values of the roots of 1 appearing in the picture. Essentially, these relations are either quantum Serre relations or powers of root vectors; but in some cases, there are some substitutes of the quantum Serre relations due to the smallness of the intervening root vectors. Some of these substitutes can be recognized already in the relations in [AD].

Here is the plan of this article. In section 1, we collect different tools that will be used in the following sections. Namely, we recall the definition of Lyndon words and give some properties about them, such as the Shirshov decomposition, in subsection 1.1. In 1.2, we discuss the notions of hyperletter and hyperword following [Kh] (they are called superletter and superword in loc. cit.); these are certain specific iterations of braided commutators applied to Lyndon words. Next, in subsection 1.3, a PBW basis is given for any quotient of the tensor algebra of a diagonal braided vector space V by a Hopf ideal using these hyperwords. This applies in particular to Nichols algebras.

In section 2, after some technical preparations, we present a transformation of a braided graded Hopf algebra into another, with different space of degree one. This generalizes an analogous transformation for Nichols algebras given in [H2, Prop. 1] – see Subsection 2.3.

In section 3 we classify standard braided vector spaces with finite dimensional Nichols algebra. In subsection 3.1, we prove that if the set of PBW generators is finite, then the associated generalized Cartan matrix is of finite type. So in subsection 3.2 we obtain all the standard braidings associated to Nichols algebras of finite dimension.

Section 4 is devoted to PBW-bases of Nichols algebras of standard braided vector spaces with finite Cartan matrix. In subsection 4.1 we prove that there is exactly one PBW generator whose degree corresponds with each positive root associated to the finite Cartan matrix. We give a set of PBW-generators in subsection 4.2, following a nice presentation from [LR]. As a consequence, we compute the dimension in Subsection 4.3.

The main result of this paper is the explicit presentation by generators and relations of Nichols algebras of standard braided vector spaces with finite Cartan matrix, given in section 5. This result relies on the explicit PBW-basis and the transformation described in Subsection 2.3. In subsection 5.1, we state some relations for Nichols algebras of standard braidings, and prove some facts about the coproduct. Subsections 5.2, 5.3 and 5.4 contain the explicit presentation for types A_θ, B_θ and G_2, respectively. For this, we establish relations among the elements of the PBW-basis, inspired in [AD] and [Gr]. We finally prove the presentation in the case of Cartan type in 5.5. To our knowledge, this is the first self-contained exposition of Nichols algebras of braided vector spaces of Cartan type.

Notation. We fix an algebraically closed field k of characteristic 0; all vector spaces, Hopf algebras and tensor products are considered over k.

Given $n \in \mathbb{N}$ and $q \in \mathsf{k}$, $q \notin \cup_{0\leq j\leq n}\mathbb{G}_j$, we denote

$$\binom{n}{j}_q = \frac{(n)q!}{(k)_q!(n-k_q}!, \quad \text{where } (n)_q! = \prod_{j=1}^{n}(k)_q, \quad \text{and } (k)_q = \sum_{j=0}^{k-1} q^j.$$

For each $n = (n_1, \dots, n_\theta) \in \mathbb{Z}^\theta$, we set $x^n = x_1^{n_1} \cdots x_\theta^{n_\theta} \in \mathsf{k}[[x_1^{\pm 1}, \dots, x_\theta^{\pm 1}]]$. Also we denote

$$\mathsf{q}_h(\mathrm{t}) := \frac{\mathrm{t}^h - 1}{\mathrm{t} - 1} \in \mathsf{k}[\mathrm{t}], \quad h \in \mathbb{N}; \quad \mathsf{q}_\infty(\mathrm{t}) := \frac{1}{1-\mathrm{t}} = \sum_{s=0}^{\infty} \mathrm{t}^s \in \mathsf{k}[[\mathrm{t}]].$$

For each $N \in \mathbb{N}$, $\mathbb{G}_N$ denotes the set of primitive N-th roots of 1 in k.

For each $\theta \in \mathbb{N}$ and each $\mathbb{Z}^\theta$-graded vector spaces $\mathfrak{B}$, we denote by $\mathcal{H}_{\mathfrak{B}} = \sum_{n\in\ \mathbb{Z}^\theta} \dim \mathfrak{B}^n$ the Hilbert series associated to $\mathfrak{B}$.

Let $C = \oplus_{n\in\mathbb{N}_0} C_{i+j}$ be a $\mathbb{N}_0$-graded coalgebra, with projections $\pi_n : C \to C_n$. Given $i, j \geq 0$, we denote by

$$\Delta_{i,j} := (\pi_i \otimes \pi_j) \circ \Delta : C_{i+j} \to C_i \otimes C_j,$$

the (i,j)-th component of the comultiplication.

1. PBW-basis

Let A be an algebra, $P, S \subset A$ and $h : S \mapsto \mathbb{N} \cup \{\infty\}$. Let also $<$ be a linear order on S. Let us denote by $B(P, S, <, h)$ the set

$$\left\{p\, s_1^{e_1} \dots s_t^{e_t} : t \in \mathbb{N}_0, \quad s_1 > \cdots > s_t, \quad s_i \in S, \quad 0 < e_i < h(s_i), \quad p \in P\right\}.$$

If $B(P, S, <, h)$ is a linear basis of A, then we say that $(P, S, <, h)$ is a set of *PBW generators* with height h, and that $B(P, S, <, h)$ is a *PBW-basis* of A. Occasionally, we shall simply say that S is a PBW-basis of A.

In this Section, we describe– following [Kh]– an appropriate PBW-basis of a braided graded Hopf algebra $\mathfrak{B} = \oplus_{n\in\mathbb{N}}\mathfrak{B}^n$ such that $\mathfrak{B}^1 \cong V$, where V is a braided vector space of diagonal type. This applies in particular, to the Nichols algebra $\mathfrak{B}(V)$. In Subsection 1.1 we recall the classical construction of Lyndon words. Let V be a vector space V together with a fixed basis. Then there is a basis of the tensor algebra $T(V)$ by certain words satisfying a special condition, called Lyndon words. Each Lyndon word has a canonical decomposition as a product of a pair of smaller Lyndon words, called the Shirshov decomposition.

We briefly remind the notions of braided vector space (V, c) of diagonal type and Nichols algebra in Subsection 1.2. Then we recall– in Subsection

1.3– the definition of the hyperletter $[l]_c$, for any Lyndon word l; this is the braided commutator of the hyperletters corresponding to the words in the Shirshov decomposition. The hyperletters are a set of generators for a PBW-basis of $T(V)$ and their classes form a PBW-basis of $\mathfrak{B}$.

1.1. **Lyndon words.**

Let $\theta \in \mathbb{N}$. Let X be a set with θ elements and fix a numeration $x_1, \dots, x_\theta$ of X; this induces a total order on X. Let $\mathbb{X}$ be the corresponding vocabulary (the set of words with letters in X) and consider the lexicographical order on $\mathbb{X}$.

Definition 1.1. An element $u \in \mathbb{X}$, $u \neq 1$, is called a *Lyndon word* if u is smaller than any of its proper ends; that is, if $u = vw$, $v, w \in \mathbb{X} - \{1\}$, then $u < w$. The set of Lyndon words is denoted by L.

We shall need the following properties of Lyndon words.

(1) Let $u \in \mathbb{X} - X$. Then u is Lyndon if and only if for any representation $u = u_1u_2$, with $u_1, u_2 \in \mathbb{X}$ not empty, one has $u_1u_2 = u < u_2u_1$.
(2) Any Lyndon word begins by its smallest letter.
(3) If $u_1, u_2 \in L, u_1 < u_2$, then $u_1u_2 \in L$.

The basic Theorem about Lyndon words, due to Lyndon, says that any word $u \in \mathbb{X}$ has a unique decomposition

$$u = l_1 l_2 \dots l_r, \tag{1.1}$$

with $l_i \in L$, $l_r \leq \dots \leq l_1$, as a product of non increasing Lyndon words. This is called the *Lyndon decomposition* of $u \in \mathbb{X}$; the $l_i \in L$ appearing in the decomposition (1.1) are called the *Lyndon letters* of u.

The lexicographical order of $\mathbb{X}$ turns out to be the same as the lexicographical order in the Lyndon letters. Namely, if $v = l_1 \dots l_r$ is the Lyndon decomposition of v, then $u < v$ if and only if:

(i) the Lyndon decomposition of u is $u = l_1 \dots l_i$, for some $1 \leq i < r$, or
(ii) the Lyndon decomposition of u is $u = l_1 \dots l_{i-1} l l'_{i+1} \dots l'_s$, for some $1 \leq i < r$, $s \in \mathbb{N}$ and $l, l'_{i+1}, \dots, l'_s$ in L, with $l < l_i$.

Here is another useful characterization of Lyndon words.

Lemma 1.2. *Let $u \in \mathbb{X} - X$. Then $u \in L$ if and only if there exist $u_1, u_2 \in L$ with $u_1 < u_2$ such that $u = u_1u_2$.*

Proof. See [Kh, p.6, Shirshov Th.]. □

Definition 1.3. Let $u \in L - X$. A decomposition $u = u_1u_2$, with $u_1, u_2 \in L$ such that u_2 is the smallest end among those proper non-empty ends of u is called the *Shirshov decomposition* of u.

Let $u, v, w \in L$ be such that $u = vw$. Then $u = vw$ is the Shirshov decomposition of u if and only if either $v \in X$, or else if $v = v_1v_2$ is the Shirshov decomposition of v, then $w \leq v_2$.

1.2. **Braided vector spaces of diagonal type and Nichols algebras.** A braided vector space is a pair (V, c), where V is a vector space and $c \in \mathrm{Aut}(V \otimes V)$ is a solution of the braid equation:

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

We extend the braiding to $c : T(V) \otimes T(V) \to T(V) \otimes T(V)$ in the usual way. If $x, y \in T(V)$, then the braided commutator is

$$[x, y]_c := \text{multiplication } \circ (\mathrm{id} - c)\,(x \otimes y)\,. \tag{1.2}$$

Assume that $\dim V < \infty$ and pick a basis $X = \{x_1, \dots, x_\theta\}$ of V; we may then identify $\Bbbk\mathbb{X}$ with $T(V)$. We consider the following gradings of the algebra $T(V)$:

(i) The usual $\mathbb{N}_0$-grading $T(V) = \oplus_{n \geq 0} T^n(V)$. If ℓ denotes the length of a word in $\mathbb{X}$, then $T^n(V) = \oplus_{x \in \mathbb{X},\, \ell(x) = n} \Bbbk x$.

(ii) Let $\mathbf{e}_1, \dots, \mathbf{e}_\theta$ be the canonical basis of $\mathbb{Z}^\theta$. Then $T(V)$ is also $\mathbb{Z}^\theta$-graded, where the degree is determined by $\deg x_i = \mathbf{e}_i$, $1 \leq i \leq \theta$.

A braided vector space (V, c) is of *diagonal type* with respect to the basis $x_1, \dots x_\theta$ if there exist $q_{ij} \in \Bbbk^\times$ such that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$, $1 \leq i, j \leq \theta$. Let $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \Bbbk^\times$ be the bilinear form determined by $\chi(\mathbf{e}_i, \mathbf{e}_j) = q_{ij}$, $1 \leq i, j \leq \theta$. Then

$$c(u \otimes v) = \chi(\deg u, \deg v) v \otimes u \tag{1.3}$$

for any $u, v \in \mathbb{X}$, where $q_{u,v} = \chi(\deg u, \deg v) \in \Bbbk^\times$. In this case, the braided commutator satisfies a "braided" Jacobi identity as well as braided derivation properties, namely

$$[[u, v]_c\,, w]_c = [u, [v, w]_c]_c - \chi(\alpha, \beta) v\ [u, w]_c + \chi(\beta, \gamma)\, [u, w]_c\ v, \tag{1.4}$$

$$[u, v\ w]_c = [u, v]_c\, w + \chi(\alpha, \beta) v\ [u, w]_c\,, \tag{1.5}$$

$$[u\ v, w]_c = \chi(\beta, \gamma)\, [u, w]_c\ v + u\ [v, w]_c\,, \tag{1.6}$$

for any homogeneous $u, v, w \in T(V)$, of degrees $\alpha, \beta, \gamma \in \mathbb{N}^\theta$, respectively.

We denote by ${}^H_H\mathcal{YD}$ the category of Yetter-Drinfeld module over H, where H is a Hopf algebra with bijective antipode. Any $V \in {}^H_H\mathcal{YD}$ becomes a braided vector space [M]. If H is the group algebra of a finite abelian group, then any $V \in {}^H_H\mathcal{YD}$ is a braided vector space of diagonal type. Indeed, $V = \oplus_{g \in \Gamma, \chi \in \widehat{\Gamma}} V_g^\chi$, where $V_g^\chi = V^\chi \cap V_g$, $V_g = \{v \in V \mid \delta(v) = g \otimes v\}$, $V^\chi = \{v \in V \mid g \cdot v = \chi(g) v \text{ for all } g \in \Gamma\}$. The braiding is given by $c(x \otimes y) = \chi(g) y \otimes x$, for all $x \in V_g$, $g \in \Gamma$, $y \in V^\chi$, $\chi \in \widehat{\Gamma}$.

Reciprocally, any braided vector space of diagonal type can be realized as a Yetter-Drinfeld module over the group algebra of an abelian group.

If $V \in {}^H_H\mathcal{YD}$, then the tensor algebra $T(V)$ admits a unique structure of graded braided Hopf algebra in ${}^H_H\mathcal{YD}$ such that $V \subseteq \mathcal{P}(V)$. Following [AS5], we consider the class $\mathfrak{S}$ of all the homogeneous two-sided ideals $I \subseteq T(V)$ such that

- I is generated by homogeneous elements of degree ≥ 2,
- I is a Yetter-Drinfeld submodule of $T(V)$,
- I is a Hopf ideal: $\Delta(I) \subset I \otimes T(V) + T(V) \otimes I$.

The Nichols algebra $\mathfrak{B}(V)$ associated to V is the quotient of $T(V)$ by the maximal element $I(V)$ of $\mathfrak{S}$.

Let (V, c) be a braided vector space of diagonal type, and assume that $q_{ij} = q_{ji}$ for all i, j. Let Γ be the free abelian group of rank θ, with basis $g_1, \dots, g_\theta$, and define the characters $\chi_1, \dots, \chi_\theta$ of Γ by

$$\chi_j(g_i) = q_{ij}, \quad 1 \leq i, j \leq \theta.$$

Consider V as a Yetter-Drinfeld module over $\mathsf{k}\Gamma$ by defining $x_i \in V_{g_i}^{\chi_i}$.

We shall need the following proposition.

Proposition 1.4. [L3, Prop. 1.2.3], [AS5, Prop. 2.10]. *Let $a_1, \dots, a_\theta \in \mathsf{k}^\times$. There is a unique bilinear form $(|) : T(V) \times T(V) \to \mathsf{k}$ such that $(1|1) = 1$, and:*

$$(x_i | x_j) = \delta_{ij} a_i, \quad \text{for all } i, j; \tag{1.7}$$

$$(x | yy') = (x_{(1)} | y)(x_{(2)} | y'), \quad \text{for all } x, y, y' \in T(V); \tag{1.8}$$

$$(xx' | y) = (x | y_{(1)})(x' | y_{(2)}), \quad \text{for all } x, x', y \in T(V). \tag{1.9}$$

This form is symmetric and also satisfies

$$(x|y) = 0, \quad \text{for all } x \in T(V)_g,\ y \in T(V)_h,\ g, h \in \Gamma,\ g \neq h. \tag{1.10}$$

The quotient $T(V)/I(V)$, where

$$I(V) := \{x \in T(V) : (x|y) = 0,\ \forall y \in T(V)\}$$

is the radical of the form, is canonically isomorphic to the Nichols algebra of V. Thus, $(|)$ induces a non degenerate bilinear form on $\mathfrak{B}(V)$ denoted by the same name. □

If (V, c) is of diagonal type, then the ideal $I(V)$ is $\mathbb{Z}^\theta$-homogeneous hence $\mathfrak{B}(V)$ is $\mathbb{Z}^\theta$-graded. See [AS4] for details. The following statement, that we include for later reference, is well-known.

Lemma 1.5. *Let V a braided vector space of diagonal type, and consider its Nichols algebra $\mathfrak{B}(V)$.*

(a) *If q_{ii} is a root of unit of order $N > 1$, then $x_i^N = 0$.*

(b) *If $i \neq j$, then $(ad_c x_i)^r(x_j) = 0$ if and only if*

$$(r)!_{q_{ii}} \prod_{0 \leq k \leq r-1} (1 - q_{ii}^k q_{ij} q_{ji}) = 0.$$

(c) *If $i \neq j$ and $q_{ij} q_{ji} = q_{ii}^r$, for some r such that $0 \leq -r < \operatorname{ord}(q_{ii})$, then $(ad_c x_i)^{1-r}(x_j) = 0$.* □

1.3. **PBW basis of a quotient of the tensor algebra by a Hopf ideal.** Let (V,c) be a braided vector space with a basis $X=\{x_1,\dots,x_\theta\}$; identify $T(V)$ with $\mathsf{k}\mathbb{X}$. An important graded endomorphism $[-]_c$ of $\mathsf{k}\mathbb{X}$ is given by

$$[u]_c := \begin{cases} u, & \text{if } u=1 \text{ or } u\in X;\\ [[v]_c,[w]_c]_c, & \text{if } u\in L,\ \ell(u)>1 \text{ and } u=vw\\ & \qquad \text{is the Shirshov decomposition;}\\ [u_1]_c\dots[u_t]_c, & \text{if } u\in \mathbb{X}-L\\ & \qquad \text{with Lyndon decomposition } u=u_1\dots u_t; \end{cases}$$

Let us now assume that (V,c) is of diagonal type with respect to the basis $x_1,\dots,x_\theta$, with matrix (q_{ij}).

Definition 1.6. The *hyperletter* corresponding to $l\in L$ is the element $[l]_c$. A *hyperword* is a word in hyperletters, and a *monotone hyperword* is a hyperword of the form $W=[u_1]_c^{k_1}\dots[u_m]_c^{k_m}$, where $u_1>\dots>u_m$.

Remark 1.7. If $u\in L$, then $[u]_c$ is a homogeneous polynomial with coefficients in $\mathbb{Z}[q_{ij}]$ and $[u]_c\in u+\mathsf{k}\mathbb{X}^{\ell(u)}_{>u}$.

The hyperletters inherit the order from the Lyndon words; this induces in turn an ordering in the hyperwords (the lexicographical order on the hyperletters). Now, given monotone hyperwords W,V, it can be shown that

$$W=[w_1]_c\dots[w_m]_c>V=[v_1]_c\dots[v_t]_c,$$

where $w_1\geq\dots\geq w_r, v_1\geq\dots\geq v_s$, if and only if

$$w=w_1\dots w_m>v=v_i\dots v_t.$$

Furthermore, the principal word of the polynomial W, when decomposed as sum of monomials, is w with coefficient 1.

Theorem 1.8. *(Rosso, see* [R2]*). Let $m,n\in L$, with $m<n$. Then the braided commutator $[[m]_c,[n]_c]_c$ is a $\mathbb{Z}[q_{ij}]$-linear combination of monotone hyperwords $[l_1]_c\dots[l_r]_c, l_i\in L$, such that*

- *the hyperletters of those hyperwords satisfy $n>l_i\geq mn$,*
- *$[mn]_c$ appears in the expansion with non-zero coefficient,*
- *any hyperword appearing in this decomposition satisfies*

$$\deg(l_1\dots l_r)=\deg(mn).$$

□

A crucial result of Rosso describes the behavior of the coproduct of $T(V)$ in the basis of hyperwords.

Lemma 1.9. [R2]. *Let $u \in \mathbb{X}$, and $u = u_1 \dots u_r v^m$, $v, u_i \in L, v < u_r \leq \dots \leq u_1$ the Lyndon decomposition of u. Then*

$$\begin{aligned} \Delta\left([u]_c\right) &= 1 \otimes [u]_c + \sum_{i=0}^{m} \binom{n}{i}_{q_{v,v}} [u_1]_c \dots [u_r]_c [v]_c^i \otimes [v]_c^{n-i} \\ &+ \sum_{\substack{l_1 \geq \dots \geq l_p > l, l_i \in L \\ 0 \leq j \leq m}} x^{(j)}_{l_1,\dots,l_p} \otimes [l_1]_c \dots [l_p]_c [v]_c^j ; \end{aligned}$$

here each $x^{(j)}_{l_1,\dots,l_p}$ is $\mathbb{Z}^\theta$-homogeneous, and

$$\deg(x^{(j)}_{l_1,\dots,l_p}) + \deg(l_1 \dots l_p v^j) = \deg(u).$$

□

As in [U], we consider another order in $\mathbb{X}$; it is implicit in [Kh].

Definition 1.10. Let $u, v \in \mathbb{X}$. We say that $u \succ v$ if and only if either $\ell(u) < \ell(v)$, or else $\ell(u) = \ell(v)$ and $u > v$ (lexicographical order). This $\succ$ is a total order, called the *deg-lex order*.

Note that the empty word 1 is the maximal element for $\succ$. Also, this order is invariant by right and left multiplication.

Let now I be a proper ideal of $T(V)$, and set $R = T(V)/I$. Let $\pi : T(V) \to R$ be the canonical projection. Let us consider the subset of $\mathbb{X}$:

$$G_I := \{u \in \mathbb{X} : u \notin \mathsf{k}\mathbb{X}_{\succ u} + I\}.$$

Notice that

(a) If $u \in G_I$ and $u = vw$, then $v, w \in G_I$.
(b) Any word $u \in G_I$ factorizes uniquely as a non-increasing product of Lyndon words in G_I.

Proposition 1.11. [Kh], *see also* [R2]. *The set $\pi(G_I)$ is a basis of R.* □

In what follows, I is a Hopf ideal. We seek to find a PBW-basis by hyperwords of the quotient R of $T(V)$. For this, we look at the set

$$S_I := G_I \cap L. \tag{1.11}$$

We then define the function $h_I : S_I \to \{2, 3, \dots\} \cup \{\infty\}$ by

$$h_I(u) := \min\left\{t \in \mathbb{N} : u^t \in \mathsf{k}\mathbb{X}_{\succ u^t} + I\right\}. \tag{1.12}$$

The next result plays a fundamental role in this paper.

Theorem 1.12. [Kh]. *Keep the notation above. Then*

$$B'_I := B\left(\{1 + I\}, [S_I]_c + I, <, h_I\right)$$

is a PBW-basis of $H = T(V)/I$. □

See [Kh] for proofs of the next consequences of the Theorem 1.12.

Corollary 1.13. *A word u belongs to G_I if and only if the corresponding hyperletter $[u]_c$ is not a linear combination, module I, of hyperwords $[w]_c$, $w \succ u$, where all the hyperwords belong to B_I.* $\square$

Proposition 1.14. *In the conditions of the Theorem 1.12, if $v \in S_I$ is such that $h_I(v) < \infty$, then $q_{v,v}$ is a root of unit. In this case, if t is the order of $q_{v,v}$, then $h_I(v) = t$.* $\square$

Corollary 1.15. *If $h_I(v) := h < \infty$, then $[v]^h$ is a linear combination of hyperwords $[w]_c$, $w \succ u^t$.* $\square$

2. Transformations of braided graded Hopf algebras

In Subsection 2.3, we shall introduce a transformation over certain graded braided Hopf algebras, generalizing [H2, Prop. 1]. It is instrumental step in the proof of Theorem 5.25, one of the main results of this article.

2.1. Preliminaries on braided graded Hopf algebras.

Let H be the group algebra of an abelian group Γ. Let $V \in {}^H_H\mathcal{YD}$ with a basis $X = \{x_1, \dots, x_\theta\}$ such that $x_i \in V_{g_i}^{\chi_i}$, $1 \leq i \leq \theta$. Let $q_{ij} = \chi_j(g_i)$, so that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$, $1 \leq i, j \leq \theta$.

We fix an ideal I in the class $\mathfrak{S}$; *we assume that I is $\mathbb{Z}^\theta$-homogeneous.* Let $\mathfrak{B} := T(V)/I$: this is a braided graded Hopf algebra, $\mathfrak{B}^0 = \mathsf{k}1$ and $\mathfrak{B}^1 = V$. By definition of $I(V)$, there exists a canonical epimorphism of braided graded Hopf algebras $\pi : \mathfrak{B} \to \mathfrak{B}(V)$. Let $\sigma_i : \mathfrak{B} \to \mathfrak{B}$ be the algebra automorphism given by the action of g_i.

Proposition 2.1. *(See for example* [AS5, 2.8]*).*

1. *For each $1 \leq i \leq \theta$, there exists a uniquely determined (id, σ_i)-derivation $D_i : \mathfrak{B} \to \mathfrak{B}$ with $D_i(x_j) = \delta_{i,j}$ for all j.*
2. *$I = I(V)$ if and only if $\cap_{i=1}^\theta \ker D_i = \mathsf{k}1$.* $\square$

These operators are defined for each $x \in \mathfrak{B}^k, k \geq 1$ by the formula

$$\Delta_{n-1,1}(x) = \sum_{i=1}^{\theta} D_i(x) \otimes x_i.$$

Analogously, we can define operators $F_i : \mathfrak{B} \to \mathfrak{B}$ by $F_i(1) = 0$,

$$\Delta_{1,n-1}(x) = \sum_{i=1}^{\theta} x_i \otimes F_i(x), \qquad x \in \oplus_{k>0} \mathfrak{B}^k.$$

Let χ be as in 1.4. Consider the action $\triangleright$ of $\mathsf{k}\mathbb{Z}^\theta$ on $\mathfrak{B}$ given by

$$\mathbf{e}_i \triangleright b = \chi(\mathbf{u}, \mathbf{e}_i) b, \qquad b \text{ homogeneous of degree } \mathbf{u} \in \mathbb{Z}^\theta. \tag{2.1}$$

Then, such operators F_i satisfy $F_i(x_j) = \delta_{i,j}$ for all j, and

$$F_i(b_1 b_2) = F_i(b_1) b_2 + (\mathbf{e}_i \triangleright b_1) F_i(b_2), \quad b_1, b_2 \in \mathfrak{B}.$$

Let $z_r^{(ij)} := (ad_c x_i)^r (x_j)$, $i, j \in \{1, \dots, \theta\}, i \neq j$ and $r \in \mathbb{N}$.

Remark 2.2. The operators D_i, F_i satisfy

$$D_i^L(x_i^n) = (n)_{q_{ii}} x_i^{n-1}, \tag{2.2}$$

$$D_i\left((ad_c x_i)^r (x_{j_1} \dots x_{j_s})\right) = 0, \ \forall r, s \geq 1, j_k \neq i, \tag{2.3}$$

$$D_j\left(z_r^{(ij)}\right) = \prod_{k=0}^{r-1}\left(1 - q_{ii}^k q_{ij} q_{ji}\right) x_i^r, \forall r \geq 0, \tag{2.4}$$

$$F_i\left(z_m^{(ij)}\right) = (m)_{q_{ii}}(1 - q_{ii}^{m-1} q_{ij} q_{ji}) z_{m-1}^{(ij)}, \tag{2.5}$$

$$F_j\left(z_m^{(ij)}\right) = 0, \quad m \geq 1. \tag{2.6}$$

The proof of the first three identities is as in [AS4, Lemma 3.7]; the proof of the last two is by induction on m.

For each pair $1 \leq i, j \leq \theta, i \neq j$, we define

$$M_{i,j}(\mathfrak{B}) \ := \ \{(ad_c x_i)^m (x_j) : m \in \mathbb{N}_0\}; \tag{2.7}$$

$$m_{ij} \ := \ \min\{m \in \mathbb{N} : (m+1)_{q_{ii}}(1 - q_{ii}^m q_{ij} q_{ji}) = 0\}. \tag{2.8}$$

Then either $q_{ii}^{m_{ij}} q_{ij} q_{ji} = 1$, or $q_{ii}^{m_{ij}+1} = 1$, if $q_{ii}^m q_{ij} q_{ji} \neq 1$ for all $m = 0, 1, \dots, m_{ij}$.

If $\mathfrak{B} = \mathfrak{B}(V)$, then we simply denote $M_{i,j} = M_{i,j}(\mathfrak{B}(V))$. Note that $(\operatorname{ad}_c x_i)^{m_{ij}+1} x_j = 0$ and $(\operatorname{ad}_c x_i)^{m_{ij}} x_j \neq 0$, by Lemma 1.5, so

$$|M_{i,j}| = m_{ij} + 1.$$

By Theorem 1.12, the braided graded Hopf algebra $\mathfrak{B}$ has a PBW-basis consisting of homogeneous elements (with respect to the $\mathbb{Z}^\theta$-grading). As in [H2], we can even assume that

- ⊛ The height of a PBW-generator $[u], \deg(u) = d$, is finite if and only if $2 \leq \operatorname{ord}(q_{u,u}) < \infty$, and in such case, $h_{I(V)}(u) = \operatorname{ord}(q_{u,u})$.

This is possible because if the height of $[u], \deg(u) = d$, is finite, then $2 \leq ord(q_{u,u}) = m < \infty$, by Proposition 1.14. And if $2 \leq \operatorname{ord}(q_{u,u}) = m < \infty$, but $h_{I(V)}(u)$ is infinite, we can add $[u]^m$ to the PBW basis: in this case, $h_{I(V)}(u) = \operatorname{ord}(q_{u,u})$, and $q_{u^m,u^m} = q_{u,u}^{m^2} = 1$.

Let $\Delta^+(\mathfrak{B}) \subseteq \mathbb{N}^n$ be the set of degrees of the generators of the PBW-basis, counted with their multiplicities and let also $\Delta(\mathfrak{B}) = \Delta^+(\mathfrak{B}) \cup (-\Delta^+(\mathfrak{B}))$: $\Delta^+(\mathfrak{B})$ is independent of the choice of the PBW-basis with the property ⊛ (see [AA, Lemma 2.18] for a proof of this statement).

2.2. **Auxiliary results.**

Let I be $\mathbb{Z}^\theta$-homogeneous ideal in $\mathfrak{S}$ and $\mathfrak{B} = T(V)/I$ as in Subsection 2.1. We shall use repeatedly the following fact.

Remark 2.3. If $x_i^N = 0$ in $\mathfrak{B}$ with N minimal (this is called the order of nilpotency of x_i), then q_{ii} is a root of 1 of order N. Hence $(ad_c x_i)^N x_j = 0$.

The following result extends (18) in the proof of [H2, Proposition 1].

Lemma 2.4. *For each $i \in \{1, \dots, \theta\}$, let $\mathcal{K}_i$ be the subalgebra generated by $\cup_{j \neq i} M_{i,j}(\mathfrak{B})$ and denote by n_i the order of q_{ii}. Then there are isomorphisms of graded vector spaces*

- $\ker(D_i) \cong \mathcal{K}_i$, *if* $\operatorname{ord} q_{ii}$ *is the order of nilpotency of* x_i, *or*
- $\ker(D_i) \cong \mathcal{K}_i \otimes \mathsf{k}\left[x_i^{n_i}\right]$, *if* $\operatorname{ord} q_{ii} < \infty$ *but* x_i *is not nilpotent.*

Moreover,

$$\mathfrak{B} \cong \mathcal{K}_i \otimes \mathsf{k}\left[x_i\right]. \tag{2.9}$$

Proof. We assume for simplicity $i = 1$ and consider the PBW basis obtained in the Theorem 1.12. Now, $x_1 \in S_I$, and it is the least element of S_I, so each element of B'_I is of the form $[u_1]^{s_1} \dots [u_k]^{s_k} x_1^s$, with $u_k < \dots < u_1, u_i \in S_I \setminus \{x_1\}, 0 < s_i < h_I(u_i), 0 \leq s < h_I(x_1)$. Call $S' = S_I \setminus \{x_1\}$, and

$$B_2 := B\left(1 + I, \left[S'\right]_c + I, <, h_I|_{S'}\right),$$

that is, the PBW set generated by $[S']_c + I$, whose height is the restriction of the height of the PBW basis corresponding to S'. We have

$$\mathfrak{B} \cong \mathsf{k}B_2 \otimes \mathsf{k}\left[x_1\right].$$

By (2.3), any $(ad_c x_1)^r(x_j) \in \ker(D_1)$; as D_1 is a skew-derivation, we have $\mathcal{K}_1 \subseteq \ker(D_1)$.

Now, if $v \in S', v = x_{j_1} \dots x_{j_s}, \quad j_1, \dots, j_s \geq 2$, then $[v]_c \in \mathcal{K}_1$, because it is a homogeneous polynomial in $x_{j_1}, \dots, x_{j_s}$, and each $x_{j_p} \in \mathcal{K}_1$.

Let $v \in L$ be a word in letters $x_2, \dots, x_\theta$, of degree $\mathbf{v} \in N^\theta$. Then $x_1 v \in L$, and

$$[x_1 v]_c = x_1 [v]_c - \chi(\mathbf{e}_1, \mathbf{v}) [v]_c x_1 = \sum_{u \geq v, \deg(u) = \mathbf{v}} \alpha_u (x_1 u - \chi(\mathbf{e}_1, \mathbf{v}) x_1),$$

where $\alpha_u \in \mathsf{k}$. If $u = x_{j_1} \dots x_{j_s}, \quad j_1, \dots, j_s \in \{2, \dots, \theta\}$, we have

$$\begin{aligned} x_1 u - q_{x_1,u} u x_1 &= x_1 x_{j_1} \dots x_{j_s} - q_{1j_1} \dots q_{1j_s} x_{j_1} \dots x_{j_s} x_1 \\ &= ad_c(x_1)(x_{j_1}) x_{j_2} \dots x_{j_s} + q_{1j_1} x_{j_1} (ad_c x_1)(x_{j_2}) x_{j_3} \dots x_{j_s} \\ &\quad + \dots + q_{1j_1} \dots q_{1j_{s-1}} x_{j_1} \dots x_{j_{s-1}} (ad_c x_1)(x_{j_s}). \end{aligned}$$

Then $x_1 u - q_{x_1,u} u x_1 \in \mathcal{K}_1$, so $[x_1 v]_c \in \mathcal{K}_1$.

Now let $v \notin L$ be a word in letters $x_2, \dots, x_\theta$; consider $v = u_1 \dots u_p$ its Lyndon decomposition, where $u_p \leq \dots \leq u_1$, $u_i \in L$, $p \geq 2$. The Shirshov decomposition of $x_1 v$ is $(x_1 u_1 \dots u_{p-1}, u_p)$, so

$$[x_1 v]_c = [x_1 u_1 \dots u_{p-1}]_c [u_p]_c - q_{x_1 u_1 \dots u_{p-1}, u_p} [u_p]_c [x_1 u_1 \dots u_{p-1}]_c,$$

and by induction on p we can prove that $[x_1 v]_c \in \mathcal{K}_1$, because each $[u_p] \in \mathcal{K}_1$, and we proved already the case $p = 1$.

We next prove, by induction on t, that $\left[x_1^t u\right]_c \in \mathcal{K}_1, \ \forall t \in \mathbb{N}$, where u is a word in letters $x_2, \dots, x_\theta$: the case $t = 1$ is the previous one. Then we consider $t \geq 2$ and $\left[x_1^{t-1} u\right]_c \in \mathcal{K}_1$. The Shirshov decomposition of $x_1^t u$ is $(x_1, x_1^{t-1} u)$, so $\left[x_1^t u\right] = x_1 \left[x_1^{t-1} u\right] - q_{x_1, x_1^{t-1} u} \left[x_1^{t-1} u\right] x_1$.

By induction hypothesis, $\left[x_1^{t-1}u\right] = \sum \alpha_i B_1^{(i)} \dots B_{n_i}^{(i)}$, for some $\alpha_i \in \mathsf{k}$, and $B_p^{(i)} \in \cup_{j=2}^{\theta} M_{1,j}$. using that $(ad_c x_1)$ is an skew derivation,

$$\begin{aligned} x_1 B_1^{(i)} \cdots B_{n_i}^{(i)} &- \chi(\mathbf{e}_1, (t-1)\mathbf{e}_1 + \mathbf{u}) B_1^{(i)} \cdots B_{n_i}^{(i)} x_1 \\ =&(ad_c x_1)(B_1^{(i)}) B_2^{(i)} \cdots B_{n_i}^{(i)} \\ &+ \chi(\mathbf{e}_1, \deg B_1^{(i)}) B_1^{(i)} (ad_c x_1)(B_2^{(i)}) B_3^{(i)} \cdots B_{n_i}^{(i)} + \dots \\ &+ \chi(\mathbf{e}_1, \sum_{j=1}^{n_i} \deg B_j^{(i)}) B_1^{(i)} \cdots B_{n_i-1}^{(i)} (ad_c x_1)(B_{n_i}^{(i)}). \end{aligned}$$

Note that if $B_p^{(i)} \in M_{1,j_l}$, then $(ad_c x_1)(B_p^{(i)}) \in M_{1,j_l}$, so $\left[x_1^t u\right] \in \mathcal{K}_1$.

For the last case, let $u \in L \setminus \{x_1\}$ be a word that begins with the letter x_1 (it is the least letter); there exist $s \geq 1, t_1, \dots, t_s \geq 1$ and non empty words $u_1, \dots, u_s$ in letters $x_2, \dots, x_\theta$ such that

$$u = x_1^{t_1} u_1 \dots x_1^{t_s} u_s.$$

We prove that $[u]_c \in \mathcal{K}_1$ by induction on s, where the case $s = 1$ is as before. So for $s > 1$, if $u = vw$, where (v, w) is the Shirshov decomposition of u, w must begin with the letter x_1, because $s > 1$ and w is the least proper end of u. Then there exists $k \in \mathbb{N}, 1 \leq k < s$ such that

$$v = x_1^{t_1} u_1 \dots x_1^{t_k} u_k, \quad w = x_1^{t_{k+1}} u_{k+1} \dots x_1^{t_s} u_s.$$

By inductive hypothesis, $[v]_c, [w]_c \in \mathcal{K}_1$, and finally

$$[u]_c = [v]_c [w]_c - \chi(\deg v, \deg w) [w]_c [v]_c \in \mathcal{K}_1.$$

Then we prove that $L \setminus \{x_1\} \subseteq \mathcal{K}_1$, and B_2 is generated by $L \setminus \{x_1\}$; that is, $\mathsf{k}B_2 \subseteq \mathcal{K}_1$, and $D_1(B_2) = 0$.

If $u \in \ker(D_1)$, we can write $[u]_c = \sum_{w \in B'_I} \alpha_w [w]_c$. If w does not end with x_1, then $w \in B_2$, and $D_1([w]_c) = 0$. But if $w = u_w x_1^{t_w}$, $\;[u_w]_c \in B_2, 0 < t_w < h_I(x_1)$, we have

$$D_1\left([w]_c\right) = (t_w)_{q_{11}^{-1}} [u_w]_c x_1^{t_w - 1},$$

where $(t_w)_{q_{11}^{-1}} \neq 0$ if n_i does not divide t_w. Then

$$0 = D_1([u]_c) = \sum_{w \in B'_I / t_w > 0} \alpha_w (t_w)_{q_{11}^{-1}} [u_w]_c x_1^{t_w - 1},$$

But $[u_w]_c x_1^{t_w-1} \in B_2$, and B_2 is a basis, so $\alpha_w = 0$ for each w such that n_i does not divide t_w. This concludes the proof. □

2.3. **Transformations of certain braided graded Hopf algebras.**

Let I be $\mathbb{Z}^\theta$-homogeneous ideal in $\mathfrak{S}$ and $\mathfrak{B} = T(V)/I$ as in the previous Subsections. *We fix $i \in \{1, \dots, \theta\}$.*

Remark 2.5. $\mathrm{ord} q_{ii} = \min\{k \in \mathbb{N} : F_i^k = 0\}$.

Proof. Note that, if $k \in \mathbb{N}$, then $F_i(x_i^k) = (k)_{q_{ii}} x_i^{k-1}$, and for all $k \in \mathbb{N}$,

$$F_i^k(x_i^k) = (k)_{q_{ii}^{-1}}!.$$

That is, if $F_i^k = 0$, then $(k)_{q_{ii}^{-1}}! = 0$. Hence $\operatorname{ord} q_{ii} \leq \min\{k \in \mathbb{N} : F_i^k = 0\}$. Reciprocally, if q_{ii} is a root of 1 of order k, then $F_i^k(x_i^t) = 0$ for all $t \geq k$ by the previous claim, and $F_i^k(x_i^t) = 0$ for all $t < k$ by degree arguments. Since $F_i(x_j) = 0$ for $j \neq i$, $F_i^k = 0$. $\square$

We now extend some considerations in [H2, p. 180]. We consider the Hopf algebra

$$H_i := \begin{cases} \mathsf{k}\langle y, e_i, e_i^{-1} | e_i y - q_{ii}^{-1} y e_i, y^{N_i} \rangle & \text{where } N_i \text{ is the order of nilpotency,} \\ & \text{of } x_i \text{ in } \mathfrak{B}, \text{ if } x_i \text{ is nilpotent;} \\ \mathsf{k}\langle y, e_i, e_i^{-1} | e_i y - q_{ii}^{-1} y e_i \rangle & \text{if } x_i \text{ is not nilpotent;} \end{cases}$$

$$\Delta(e_i) = e_i \otimes e_i, \quad \Delta(y) = e_i \otimes y + y \otimes 1.$$

Notice that Δ is well-defined by Remark 2.3. We also consider the action $\triangleright$ of H_i on $\mathfrak{B}$ given by

$$e_i \triangleright b = \chi(\mathbf{u}, \mathbf{e}_i) b, \qquad y \triangleright b = F_i(b),$$

if b is homogeneous of degree $\mathbf{u} \in \mathbb{N}^\theta$, extending the previous one defined in (2.1). The action is well-defined by Remark 2.3 and because

$$(e_i y) \triangleright b = e_i \triangleright (F_i(b)) = q_{ii}^{-1} F_i(e_i \triangleright b) = \left(q_{ii}^{-1} y e_i\right) \triangleright b, \ \forall b \in \mathfrak{B}.$$

It is easy to see that $\mathfrak{B}$ is an H_i-module algebra; hence we can form

$$\mathcal{A}_i := \mathfrak{B} \# H_i.$$

Also, if we denote explicitly by $\cdot$ the multiplication in $\mathcal{A}_i$, we have

$$(1\#y) \cdot (b\#1) = (e_i \triangleright b\#1) \cdot (1\#y) + F_i(b)\#1, \quad \forall b \in \mathfrak{B}. \tag{2.10}$$

As in [H2], $\mathcal{A}_i$ is a left Yetter-Drinfeld module over $\mathsf{k}\Gamma$, where the action and the coaction are given by

$$\begin{aligned} g_k \cdot x_j\#1 &= q_{kj} x_j\#1, & \delta(x_j\#1) &= g_j \otimes x_j\#1, \\ g_k \cdot 1\#y &= q_{ki}^{-1} 1\#y, & \delta(1\#y) &= g_i^{-1} \otimes 1\#y, \\ g_k \cdot 1\#e_i &= 1\#e_i, & \delta(1\#e_i) &= 1 \otimes 1\#e_i, \end{aligned}$$

for each pair $k, j \in \{1, \dots, \theta\}$. Also, $\mathcal{A}_i$ is a $\mathsf{k}\Gamma$-module algebra.

We now prove a generalization of [H2, Proposition 1] in the more general context of our braided Hopf algebras $\mathfrak{B}$. Although the general strategy of the proof is similar as in *loc. cit.*, many points need slightly different argumentations here.

Theorem 2.6. *Keep the notation above. Assume that $M_{i,j}(\mathfrak{B})$ is finite and*

$$|M_{i,j}(\mathfrak{B})| = m_{ij} + 1, \quad j \in \{1, \dots, \theta\}, j \neq i. \tag{2.11}$$

(i) *Let V_i be the vector subspace of $\mathcal{A}_i$ generated by*

$$\{(ad_c x_i)^{m_{ij}}(x_j)\#1 : j \neq i\} \cup \{1\#y\}.$$

The subalgebra $s_i(\mathfrak{B})$ of $\mathcal{A}_i$ generated by V_i is a graded algebra such that $s_i(\mathfrak{B})^1 \cong V_i$. There exist skew derivations $Y_i : s_i(\mathfrak{B}) \to s_i(\mathfrak{B})$ such that, for all $b_1, b_2 \in s_i(\mathfrak{B})$, and $l, j \in \{1, \dots, \theta\}, j \neq i$,

$$Y_j(b_1 b_2) = b_1 Y_j(b_2) + Y_j(b_2)\left(g_i^{-m_{ij}} g_j^{-1} \cdot b_2\right), \tag{2.12}$$

$$Y_i(b_1 b_2) = b_1 Y_i(b_2) + Y_i(b_1)\left(g_i^{-1} \cdot b_1\right), \tag{2.13}$$

$$Y_l((ad_c x_i)^{m_{ij}}(x_j)\#1) = \delta_{lj}, \qquad Y_l(1\#y) = \delta_{li}. \tag{2.14}$$

(ii) *The Hilbert series of $s_i(\mathfrak{B})$ satisfies*

$$\mathcal{H}_{s_i(\mathfrak{B})} = \left(\prod_{\alpha \in \Delta^+(\mathfrak{B}) \setminus \{\mathbf{e}_i\}} \mathfrak{q}_{h_\alpha}(X^{s_i(\alpha)}) \right) \mathfrak{q}_{h_i}(x_i). \tag{2.15}$$

Therefore, if $s_i(\mathfrak{B})$ is a graded braided Hopf algebra,

$$\Delta^+(s_i(\mathfrak{B})) = \left\{ s_i\left(\Delta^+(\mathfrak{B})\right) \setminus \{-\mathbf{e}_i\} \right\} \cup \{\mathbf{e}_i\}.$$

(iii) *If $\mathfrak{B} = \mathfrak{B}(V)$, then the algebra $s_i(\mathfrak{B})$ is isomorphic to the Nichols algebra $\mathfrak{B}(V_i)$.*

Proof. We prove (i). Note that V_i is a Yetter-Drinfeld submodule over $\mathsf{k}\Gamma$ of $\mathcal{A}_i$. Now, $\mathcal{A}_i \cong \mathfrak{B} \otimes H_i$ as graded vector spaces. Let $\mathcal{K}_i$ be the subalgebra generated by $\cup_{j \neq i} M_{i,j}(\mathfrak{B})$, as in Lemma 2.4. Then $s_i(\mathfrak{B}) \subseteq \mathcal{K}_i \otimes \mathsf{k}[y]$, since F_i is a skew-derivation and $F_i\left(z_k^{(ij)}\right) = (k)_{q_{ii}}(1 - q_{ii}^{k-1} q_{ij} q_{ji}) z_{k-1}^{(ij)}$, by (2.5). From (2.10),

$$(1\#y) \cdot \left(z_{m_{ij}}^{(ij)}\#1\right) = \left(z_{m_{ij}}^{(ij)}\#1\right) \cdot (1\#y) + F_i\left(z_{m_{ij}}^{(ij)}\right)\#1.$$

Also, as $m_{ij} + 1 = |M_{i,j}(\mathfrak{B})|$, we have $(m_{ij})_{q_{ii}}(1 - q_{ii}^{m_{ij}-1} q_{ij} q_{ji}) \neq 0$, so $z_{m_{ij}-1}^{(ij)}\#1 \in s_i(\mathfrak{B})$, and by induction each $z_k^{(ij)}\#1, k = 0, \dots, m_{ij} - 1$ is an element of $s_i(\mathfrak{B})$. Then $\mathcal{K}_i \otimes \mathsf{k}[y] \subseteq s_i(\mathfrak{B})$, and therefore

$$s_i(\mathfrak{B}) = \mathcal{K}_i \otimes \mathsf{k}[y]. \tag{2.16}$$

Thus, $s_i(\mathfrak{B})$ is a graded algebra in ${}^{\mathsf{k}\Gamma}_{\mathsf{k}\Gamma}\mathcal{YD}$ with $s_i(\mathfrak{B})^1 = V_i$. We have to find the skew derivations $Y_l \in End(s_i(\mathfrak{B}))$, $l = 1, \dots, \theta$. Set $Y_i := g_i^{-1} \circ \mathrm{ad}(x_i\#1)|_{s_i(\mathfrak{B})}$. Then, for each $b \in \mathcal{K}_i$ and each $j \neq i$

$$\mathrm{ad}(x_i\#1)(b\#1) = (\mathrm{ad}_c\, x_i)(b)\#1,$$
$$\mathrm{ad}(x_i\#1)\left((\mathrm{ad}_c\, x_i)^{m_{ij}}(x_j)\#1\right) = (\mathrm{ad}_c\, x_i)^{m_{ij}+1}(x_j)\#1 = 0.$$

Also,

$$\begin{aligned} Y_i(1\#y) &= g_i^{-1}\cdot((x_i\#1)\cdot(1\#y)-(g_i\cdot(1\#y))\cdot(x_i\#1)) \\ &= g_i^{-1}\cdot\left(x_i\#y+1-q_{ii}\left(q_{ii}^{-1}x_i\#y\right)\right)=1. \end{aligned}$$

Thus $Y_i \in End(s_i(\mathfrak{B}))$ satisfies (2.14).

Therefore, $\operatorname{ad}(x_i\#1)(b_1b_2)=\operatorname{ad}(x_i\#1)(b_1)b_2+(g_i\cdot b_1)\operatorname{ad}(x_i\#1)(b_2)$, for each pair $b_1,b_2\in s_i(\mathfrak{B})$, so we conclude that $\operatorname{ad}(x_i\#1)(s_i(\mathfrak{B}))\subseteq s_i(\mathfrak{B})$, and $Y_i\in End(s_i(\mathfrak{B}))$ satisfies (2.13).

Before proving that Y_i satisfies (2.12), we need to establish some preliminary facts. Let us fix $j\neq i$, and let $z_k^{(ij)}=(ad_c x_i)^k(x_j)$ as before. We define inductively

$$\hat{z}_0^{(ij)}:=D_j,\quad \hat{z}_{k+1}^{(ij)}:=D_i\hat{z}_k^{(ij)}-q_{ii}^kq_{ij}\hat{z}_{k+1}^{(ij)}D_i\in End(\mathfrak{B}).$$

We calculate

$$\begin{aligned} \lambda_{ij} &:= \hat{z}_{m_{ij}}^{(ij)}\left(z_{m_{ij}}^{(ij)}\right)=\sum_{s=0}^{m_{ij}}a_sD_i^{m_{ij}-s}D_jD_i^s\left(z_{m_{ij}}^{(ij)}\right) \\ &= (D_i)^{m_{ij}}(D_j)\left(z_{m_{ij}}^{(ij)}\right)=\alpha_{m_{ij}}\,(m_{ij})_{q_{ii}}!\in \mathsf{k}^\times, \end{aligned}$$

where $a_s=(-1)^k\binom{m}{k}_{q_{ii}}q_{ii}^{k(k-1)/2}q_{ij}^k$.

Note that $(D_i)^{m_{ij}+1}D_j(b)=0,\forall b\in M_{i,k},k\neq i,j$, and

$$(D_i)^{m_{ij}+1}D_j(z_r^{(ij)})=(D_i)^{m_{ij}+1}\left(q_{ji}^{-r}\alpha_rx_i^r\right)=0,\quad \forall r\leq m_{ij},$$

so $(D_i)^{m_{ij}+1}D_j(\mathcal{K}_i)=0$. This implies that, for each $b\in\mathcal{K}_i$, $\hat{z}_{m_{ij}}^{(ij)}(b)\in\mathcal{K}_i$. Then, we define $Y_j\in End(s_i(\mathfrak{B}))$ by

$$Y_j\left(b\#y^m\right):=q_{ii}^{mm_{ij}}q_{ji}^m\lambda_{ij}^{-1}\hat{z}_{m_{ij}}^{(ij)}(b)\#y^m,\qquad b\in\mathcal{K}_i,m\in\mathbb{N}.$$

We have $Y_j(1\#y)=0$, and if $l\neq i,j$, $Y_j((\operatorname{ad}_c x_i)^{m_{il}}(x_l)\#1)=0$. By the choice of λ_{ij}, $Y_j((\operatorname{ad}_c x_i)^{m_{ij}}(x_j)\#1)=1$.

Now, using that $D_k(g_l\cdot b)=q_{kl}g_l\cdot(D_k(b))$, for each $b\in\mathfrak{B}$, and $k,l\in\{1,\dots,\theta\}$, we prove inductively that for $b_1,b_2\in\mathcal{K}_i$,

$$\hat{z}_k^{(ij)}(b_1b_2)=b_1\hat{z}_k^{(ij)}(b_2)+\hat{z}_k^{(ij)}(b_1)(g_i^kg_j\cdot b_2).$$

Then,

$$\begin{aligned} Y_j\left(b_1\#1\cdot b_2\#1\right) &= Y_j\left(b_1b_2\#1\right)=\lambda_{ij}^{-1}\hat{z}_{m_{ij}}(b_1b_2)\#1 \\ &= b_2\#1\cdot Y_j(b_2\#1)+Y_j(b_1\#1)\cdot\left(g_i^{m_{ij}}g_j\cdot(b_2\#1)\right). \end{aligned}$$

By induction on the degree we prove that F_i commute with D_i,D_j, so

$$\hat{z}_{m_{ij}}^{(ij)}(F_i(b))=F_i\left(\hat{z}_{m_{ij}}^{(ij)}(b)\right),\quad \forall b\in\mathfrak{B}.$$

Consider $b \in \mathcal{K}_i \subseteq \ker(D_i)$,

$$\begin{aligned} Y_j\left(b\#1 \cdot 1\#y\right) &= Y_j\left(b\#y\right) = q_{ii}^{m_{ij}} q_{ji} \hat{z}^{(ij)}_{m_{ij}}(b)\#y \\ &= b\#1 \cdot Y_j\left(1\#y\right) + Y_j\left(b\#1\right) \cdot \left(g_i^{m_{ij}} g_j \cdot (1\#y)\right), \end{aligned}$$

where we use that $Y_j(1\#y) = 0$. Then as

$$b_1\#1 \cdot b_2\#y^t = b_1\#1 \cdot b_2\#1 \cdot (1\#y)^t,$$

(2.12) is valid for products of this form. To prove it in the general case, note that

$$(b_1\#y^t) \cdot (b_2\#y^s) = (b_1\#1) \cdot (1\#y)^t \cdot (b_2\#y^s)$$

At this point, we have to prove (2.12) for $b \in \mathcal{K}_i \ker(D_i)$, $s \in \mathbb{N}$:

$$\begin{aligned} Y_j\left(1\#y \cdot b\#y^s\right) &= Y_j\left(F_i(b)\#y^s + (e_i \triangleright b\#y) \cdot 1\#y\right) \\ &= q_{ii}^{m_{ij}s} q_{ji}^s \lambda_{ij}^{-1} \hat{z}^{(ij)}_{m_{ij}}\left(F_i(b)\right)\#y^s \\ &\quad + q_{ii}^{m_{ij}(s+1)} q_{ji}^{s+1} \lambda_{ij}^{-1} \cdot \hat{z}^{(ij)}_{m_{ij}}\left(e_i \triangleright b\right)\#y^{s+1} \\ &= F_i\left(q_{ii}^{m_{ij}(s+1)} q_{ji}^{s+1} \lambda_{ij}^{-1} \hat{z}^{(ij)}_{m_{ij}}(b)\right)\#y^s \\ &\quad + q_{ii}^{m_{ij}} q_{ji}\left(e_i \triangleright \left(q_{ii}^{m_{ij}s} q_{ji}^s \lambda_{ij}^{-1} \hat{z}^{(ij)}_{m_{ij}}(b)\right)\#y^s\right) \\ &= (1\#y) \cdot Y_j\left(b\#y^s\right) \\ &= 1\#y \cdot Y_j\left(b\#y^s\right) + Y_j\left(1\#y\right) \cdot \left(g_i^{m_{ij}} g_j \cdot b\#y^s\right), \end{aligned}$$

where we use that $\hat{z}^{(ij)}_{m_{ij}}(e_i \triangleright b) = q_{ii}^{m_{ij}} q_{ji} e_i \triangleright (\hat{z}^{(ij)}_{m_{ij}}(b))$.

To prove (ii), note that the algebra H_i is $\mathbb{Z}^\theta$-graded, with

$$\deg y = -\mathbf{e}_i, \deg e_i^{\pm 1} = 0.$$

Hence, the algebra $\mathcal{A}_i$ is $\mathbb{Z}^\theta$-graded, because $\mathfrak{B}$ and H_i are graded, and (2.10) holds.

Hence, consider the abstract basis $\{u_j\}_{j\in\{1,\dots,\theta\}}$ of V_i, with the grading $\deg u_j = \mathbf{e}_j$, $\mathfrak{B}(V_i)$ is $\mathbb{Z}^\theta$-graded. Consider also the algebra homomorphism $\Omega : T(V_i) \to s_i(\mathfrak{B})$ given by

$$\Omega(u_j) := \begin{cases} (ad_c x_i)^{m_{ij}}(x_j) & j \neq i \\ y & j = i. \end{cases}$$

By the first part of the Theorem, Ω is an epimorphism, so it induces an isomorphism between $s_i(\mathfrak{B})' := T(V_i)/\ker\Omega$ and $s_i(\mathfrak{B})$, that we also denote Ω. Note:

- $\deg \Omega(u_j) = \deg\left((ad_c x_i)^{m_{ij}}(x_j)\right) = \mathbf{e}_j + m_{ij}\mathbf{e}_i = s_i(\deg \mathbf{u}_j)$, if $j \neq i$;
- $\deg \Omega(u_i) = \deg(y) = -\mathbf{e}_i = s_i(\deg \mathbf{u}_i)$.

As Ω is an algebra homomorphism, $\deg(\Omega(\mathbf{u})) = s_i(\deg(\mathbf{u}))$, for all $\mathbf{u} \in s_i(\mathfrak{B})'$. As $s_i^2 = \mathrm{id}$, $s_i(\deg(\Omega(\mathbf{u}))) = \deg(\mathbf{u})$, for all $\mathbf{u} \in s_i(\mathfrak{B})'$, and $\mathfrak{H}_{s_i(\mathfrak{B})'} = s_i(\mathfrak{H}_{s_i(\mathfrak{B})})$.

From this point, the proof goes exactly as in [AA, Theorem 3.2].

The statement in (iii) is exactly [H2, Prop. 1]. □

By Theorem 2.6, the initial braided vector space with matrix $(q_{kj})_{1\leq k,j\leq\theta}$ is transformed into another braided vector space of diagonal type V_i, with matrix $(\overline{q}_{kj})_{1\leq k,j\leq\theta}$, where $\overline{q}_{jk}=q_{ii}^{m_{ij}m_{ik}}q_{ik}^{m_{ij}}q_{ji}^{m_{ik}}q_{jk},\ j,k\in\{1,\dots,\theta\}$.

If $j\neq i$, then $\overline{m_{ij}}=\min\left\{m\in\mathbb{N}:(m+1)_{\overline{q}_{ii}}\left(\overline{q}_{ii}^{m}\overline{q}_{ij}\overline{q}_{ji}=0\right)\right\}=m_{ij}$.

For later use the previous Theorem in Section 5, we recall a result from [AHS], adapted to diagonal braided vector spaces.

Lemma 2.7. *Let V a diagonal braided vector space, and I an ideal of $T(V)$. Call $\mathfrak{B}:=T(V)/I$, and assume that there exist (id,σ_i)-derivations $D_i:\mathfrak{B}\to\mathfrak{B}$ with $D_i(x_j)=\delta_{i,j}$ for all j. Then, $I\subseteq I(V)$.*

That is, the canonical surjective algebra morphisms from $T(V)$ onto $\mathfrak{B}$, $\mathfrak{B}(V)$ induce a surjective algebra morphism $\mathfrak{B}\to\mathfrak{B}(V)$.

Proof. See [AHS, Lemma 2.8(ii)] □

3. Standard braidings

In [H3], Heckenberger classifies diagonal braidings whose set of PBW generators is finite. Standard braidings form an special subclass, which includes properly braidings of Cartan type.

we first recall the definition of standard braiding from [AA], and the notion of Weyl groupoid, introduced in [H2]. Then we present the classification of standard braidings, and compare them with [H3].

As in Heckenberger's works, we use the notion of *generalized Dynkin diagram associated* to a braided vector space of diagonal type, with matrix $(q_{ij})_{1\leq i,j\leq\theta}$: this is a graph with θ vertices, each of them labeled with the corresponding q_{ii}, and an edge between two vertices i,j labeled with $q_{ij}q_{ji}$ if this scalar is different from 1. So two braided vector spaces of diagonal type have the same generalized Dynkin diagram if and only if they are twist equivalent. We shall assume that the generalized Dynkin diagram is connected, by [AS2, Lemma 4.2].

Summarizing, the main result of this Section says:

Theorem 3.1. *Any standard braiding is twist equivalent with some of the following*

- *a braiding of Cartan type,*
- *a braiding of type A_θ listed in Proposition 3.9,*
- *a braiding of type B_θ listed in Proposition 3.10,*
- *a braiding of type G_2 listed in Proposition 3.11.*

The generalized Dynkin diagrams appearing in Propositions 3.9 and 3.10 correspond to the rows 1,2,3,4,5,6 in [H3, Table C]. Also, the generalized Dynkin diagrams in Proposition 3.11 are (T8) in [H1, Section 3]. However, our classification does not rely on [H3].

3.1. **Definitions of Weyl groupoid and standard braidings.**

Let $E = (\mathbf{e}_1, \dots, \mathbf{e}_\theta)$ be the canonical basis of $\mathbb{Z}^\theta$. Consider an arbitrary matrix $(q_{ij})_{1\le i,j\le\theta} \in (\mathsf{k}^\times)^{\theta\times\theta}$, and fix once and for all the bilinear form $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \mathsf{k}^\times$ determined by

$$\chi(\mathbf{e}_i, \mathbf{e}_j) = q_{ij}, \qquad 1 \le i, j \le \theta. \tag{3.1}$$

If $F = (\mathbf{f}_1, \dots, \mathbf{f}_\theta)$ is another ordered basis of $\mathbb{Z}^\theta$, then we set $\widetilde{q}_{ij} = \chi(\mathbf{f}_i, \mathbf{f}_j)$, $1 \le i, j \le \theta$. We call $(\widetilde{q}_{ij})$ the *braiding matrix with respect to the basis* F. Fix $i \in \{1, \dots, \theta\}$. If $1 \le i, j \le \theta$, then we consider the set

$$\widetilde{M}_{ij} := \{m \in \mathbb{N}_0 : (m+1)_{\widetilde{q}_{ii}} (\widetilde{q}_{ii}^m \widetilde{q}_{ij} \widetilde{q}_{ji} - 1) = 0\}.$$

If this set is nonempty, then its minimal element is denoted $\widetilde{m}_{ij}$ (which of course depends on the basis F). Define also $\widetilde{m}_{ii} = 2$. Let $s_{i,F} \in GL(\mathbb{Z}^\theta)$ be the pseudo-reflection given by $s_{i,F}(\mathbf{f}_j) := \mathbf{f}_j + \widetilde{m}_{ij}\mathbf{f}_i, \quad j \in \{1, \dots, \theta\}$.

Let G be a group acting on a set X. We define the *transformation groupoid* as $\mathfrak{G} = G \times X$, with the structure of groupoid given by the operation $(g, x)(h, y) = (gh, y)$ if $x = h(y)$, but undefined otherwise.

Definition 3.2. Consider $\mathfrak{X}$ the set of all ordered bases of $\mathbb{Z}^\theta$, and the canonical action of $GL(\mathbb{Z}^\theta)$ over $\mathfrak{X}$. The smallest subgroupoid of the transformation groupoid $GL(\mathbb{Z}^\theta) \times \mathfrak{X}$ that satisfies the following properties:

- $(\mathrm{id}, E) \in W(\chi)$,
- if $(\mathrm{id}, F) \in W(\chi)$ and $s_{i,F}$ is defined, then $(s_{i,F}, F) \in W(\chi)$,

is called the *Weyl groupoid* $W(\chi)$ of the bilinear form χ.

Let $\mathfrak{P}(\chi) = \{F : (\mathrm{id}, F) \in W(\chi)\}$ be the set of points of the groupoid $W(\chi)$. The set

$$\Delta(\chi) = \bigcup_{F\in\mathfrak{P}(\chi)} F. \tag{3.2}$$

is called the *generalized root system* associated to χ.

We record for later use the following evident facts.

Remark 3.3. Let $i \in \{1, \dots, \theta\}$ such that $s_{i,E}$ is defined. Let $F = s_{i,E}(E)$ and $(\widetilde{q}_{ij})$ the braiding matrix with respect to the basis F. Assume that

- $q_{ii} = -1$ (and then, $m_{ik} = 0$ if $q_{ik}q_{ki} = 1$ or $m_{ik} = 1$, for each $k \ne i$);
- there exists $j \ne i$ such that $q_{jj}q_{ji}q_{ij} = 1$ (that is, $m_{ij} = m_{ji} = 1$).

Then, $\widetilde{q}_{jj} = -1$.

Proof. Simply, $\widetilde{q}_{jj} = q_{ii}q_{ij}q_{ji}q_{jj} = q_{ii} = -1$. □

Remark 3.4. If the m_{ij} satisfies $q_{ii}^{m_{ij}} q_{ij}q_{ji} = 1$ for all $j \ne i$, then the braiding of V_i is twist equivalent with the corresponding to V.

Let $\alpha : W(\chi) \to GL(\theta, \mathbb{Z})$, $\alpha(s, F) = s$ if $(s, F) \in W(\chi)$, and denote by $W_0(\chi)$ the subgroup generated by the image of α.

Definition 3.5. [AA] We say that χ is *standard* if for any $F \in \mathfrak{P}(\chi)$, the integers m_{rj} are defined, for all $1 \leq r, j \leq \theta$, and the integers m_{rj} for the bases $s_{i,F}(F)$ coincide with those for F for all i, r, j. Clearly it is enough to assume this for the canonical basis E.

We assume now that χ is standard. We set $C := (a_{ij}) \in \mathbb{Z}^{\theta \times \theta}$, where $a_{ij} = -m_{ij}$: it is a generalized Cartan matrix.

Proposition 3.6. [AA] *$W_0(\chi) = \langle s_{i,E} : 1 \leq i \leq \theta \rangle$. Furthermore $W_0(\chi)$ acts freely and transitively on $\mathfrak{P}(\chi)$.* □

Hence, $W_0(\chi)$ is a Coxeter group, and $W_0(\chi)$ and $\mathfrak{P}(\chi)$ have the same cardinal.

Lemma 3.7. [AA] *The following are equivalent:*

(1) *The groupoid $W(\chi)$ is finite.*
(2) *The set $\mathfrak{P}(\chi)$ is finite.*
(3) *The generalized root system $\Delta(\chi)$ is finite.*
(4) *The group $W_0(\chi)$ is finite.*

If C is symmetrizable, (1)-(4) are equivalent to

(5) *The Cartan matrix C is of finite type.* □

We shall prove in Theorem 4.1, that if $\Delta(\chi)$ is finite, then the matrix C is symmetrizable, hence of finite type. Then, $\mathfrak{B}(V)$ is of finite dimension if and only if the Cartan matrix C is of finite type.

3.2. **Classification of standard braidings.** We now classify standard braidings such that the Cartan matrix is of finite type. We begin by types $C_\theta, D_\theta, E_l \quad (l = 6, 7, 8)$ and F_4: these standard braidings are necessarily of Cartan type.

Proposition 3.8. *Let V be a braided vector space of standard type, $\theta = \dim V$, and $C = (a_{ij})_{i,j \in \{1,\dots,\theta\}}$ the corresponding Cartan matrix, of type $C_\theta, D_\theta, E_l \quad (l = 6, 7, 8)$ or F_4. Then V is of Cartan type (associated to the corresponding matrix of finite type).*

Proof. *Let V be standard of type C_θ, $\theta \geq 3$.*

$$\circ^{1} \text{———} \circ^{2} \text{———} \circ^{3} \quad \dots \qquad \circ^{\theta-2} \text{———} \circ^{\theta-1} \Longleftarrow \circ^{\theta} \tag{3.3}$$

Note that, if we suppose $q_{\theta-1,\theta-1} = -1$, as $m_{\theta-1,\theta} = 2$ and $q_{\theta-1,\theta-1}^3 \neq 1$, we have

$$1 = q_{\theta-1,\theta-1}^2 q_{\theta-1,\theta} q_{\theta,\theta-1} = q_{\theta-1,\theta} q_{\theta,\theta-1},$$

so $m_{\theta,\theta-1} = m_{\theta-1,\theta} = 0$, but this is a contradiction. Then $q_{\theta-1,\theta-1} \neq -1$, and $m_{\theta-1,\theta-2} = 1$, so $q_{\theta-1,\theta-1} q_{\theta-1,\theta-2} q_{\theta-2,\theta-1} = 1$. Using Remark 3.3 when $i = \theta - 2, j = \theta - 1$, as $\widetilde{q}_{\theta-1,\theta-1} \neq -1$ when we transform by $s_{\theta-2}$ (since the new braided vector space is also standard), we have $q_{\theta-2,\theta-2} \neq -1$, so

$$q_{\theta-2,\theta-2} q_{\theta-2,\theta-1} q_{\theta-1,\theta-2} = q_{\theta-2,\theta-2} q_{\theta-2,\theta-3} q_{\theta-3,\theta-2} = 1,$$

and $q_{\theta-1,\theta-1} = q_{\theta-2,\theta-2}$. Inductively,

$$q_{kk}q_{k,k-1}q_{k-1,k} = q_{kk}q_{k,k+1}q_{k+1,k} = q_{11}q_{12}q_{21} = 1, \quad k = 2, ..., \theta - 1$$

and $q_{11} = q_{22} = \ldots = q_{\theta-1,\theta-1}$. So we look at $q_{\theta\theta}$: as $m_{\theta,\theta-1} = 1$, we have $q_{\theta\theta} = -1$ or $q_{\theta\theta}q_{\theta,\theta-1}q_{\theta-1,\theta} = 1$. If $q_{\theta\theta} = -1$, transforming by s_θ, we have

$$\widetilde{q}_{\theta-1,\theta-1} = -q^{-1}, \quad \widetilde{q}_{\theta-1,\theta}\widetilde{q}_{\theta,\theta-1} = q^2,$$

and as $m_{\theta-1,\theta-2} = 1$, we have $q^2 = -1$. Then

$$q_{\theta\theta}q_{\theta,\theta-1}q_{\theta-1,\theta} = 1, \quad q_{\theta\theta} = q^2,$$

and the braiding is of Cartan type in both cases.

Let V be standard of type D_θ, $\theta \geq 4$.

We prove the statement by induction on θ. Let V be of standard D_4 type, and suppose that $q_{22} = -1$. Let $(\widetilde{q}_{ij})$ the braiding matrix with respect to $F = s_{2,E}(E)$. We calculate for each pair $j \neq k \in \{1, 3, 4\}$:

$$\widetilde{q}_{jk}\widetilde{q}_{kj} = ((-1)q_{2k}q_{j2}q_{jk})((-1)q_{2j}q_{k2}q_{kj}) = (q_{2k}q_{k2})(q_{2j}q_{j2}),$$

where we use that $q_{jk}q_{kj} = 1$. As also $\widetilde{q}_{jk}\widetilde{q}_{kj} = 1$, we have $q_{2k}q_{k2} = (q_{2j}q_{j2})^{-1}, \quad j \neq k$, so $q_{2k}q_{k2} = -1$, $k = 1, 3, 4$, since $q_{2k}q_{k2} \neq 1$. In this case, the braiding is of Cartan type, with $q = -1$. Suppose then $q_{22} \neq -1$. From the fact that $m_{2j} = 1$, we have

$$q_{22}q_{2j}q_{j2} = 1, \quad j = 1, 3, 4.$$

For each j, applying Remark 3.3, as $\overline{q}_{22} \neq -1$, we have $q_{jj} \neq -1$, so $q_{jj}q_{2j}q_{j2} = 1, \quad j = 1, 3, 4$, and the braiding is of Cartan type.

$$\text{(3.4)} \qquad \begin{array}{ccccccc} \circ^{1} & \text{———} & \circ^{2} & \text{———} & \circ^{3}\ldots & \circ^{\theta-2} & \text{———} & \circ^{\theta} \\ & & & & & | & & \\ & & & & & \circ_{\theta-1} & & \end{array}$$

We now suppose the statement valid for θ. Let V be a standard braided vector space of type $D_{\theta+1}$. The subspace generated by $x_2, \ldots, x_{\theta+1}$ is a standard braided vector space associated to the matrix $(q_{ij})_{i,j=2,\ldots,\theta+1}$, of type D_θ, so it is of Cartan type. To finish, apply Remark 3.3 when $i = 1, j = 2$, so we obtain that V is of Cartan type with $q = -1$, or if $q_{22} \neq -1$, we have $q_{11} \neq -1$, and $q_{11}q_{12}q_{21} = 1$, and in this case it is of Cartan type too (because also $q_{1k}q_{k1} = 1$ when $k > 2$).

Let V be standard of type E_6. Note that $1, 2, 3, 4, 5$ determine a braided vector subspace, which is standard of type D_5, so it is of Cartan type. Then to prove that $q_{66}q_{65}q_{56} = 1$, we use Remark 3.3 as above.

$$\begin{array}{ccccccccc} \circ^{1} & \text{———} & \circ^{2} & \text{———} & \circ^{3} & \text{———} & \circ^{5} & \text{———} & \circ^{6} \\ & & & & | & & & & \\ & & & & \circ_{4} & & & & \end{array} \tag{3.5}$$

If V is standard of type E_7 or E_8, we proceed similarly by reduction to E_6, respectively E_7.

$$\begin{array}{ccccccccccc} \circ^{1} & \text{———} & \circ^{2} & \text{———} & \circ^{3} & \text{———} & \circ^{4} & \text{———} & \circ^{6} & \text{———} & \circ^{7} \\ & & & & & & | & & & & \\ & & & & & & \circ_{5} & & & & \end{array} \tag{3.6}$$

$$\begin{array}{ccccccccccccc} \circ^{1} & \text{———} & \circ^{2} & \text{———} & \circ^{3} & \text{———} & \circ^{4} & \text{———} & \circ^{5} & \text{———} & \circ^{7} & \text{———} & \circ^{8} \\ & & & & & & & & | & & & & \\ & & & & & & & & \circ_{6} & & & & \end{array} \tag{3.7}$$

Let V be standard of type F_4. The vertices $2, 3, 4$ determine a braided subspace, which is standard of type C_3, so the q_{ij} satisfy the corresponding relations. Let $(\widetilde{q}_{ij})$ the braiding matrix with respect to $F = s_{2,E}(E)$. As $\widetilde{q}_{13}\widetilde{q}_{31} = 1$ and $q_{22}q_{23}q_{32} = 1$, we have $q_{22}q_{12}q_{21} = 1$.

$$\circ^{1} \text{———} \circ^{2} \Longrightarrow \circ^{3} \text{———} \circ^{4} \tag{3.8}$$

Now, if we suppose $q_{11} = -1$, applying Remark 3.3 we have $q_{22} = -1 = q_{21}q_{12}$, and then it is corresponding vector space of Cartan F_4 type associated to $q \in \mathbb{G}_4$. If $q_{11} \neq -1$, then $q_{11}q_{12}q_{21} = 1$, and also it is of Cartan type. $\square$

To finish the classification of standard braidings, we describe the standard braidings that are not of Cartan type. They are associated to Cartan matrices of type A_θ, B_θ or G_2.

We use the same notation as in [H3]; $C(\theta, q; i_1, \dots, i_j)$ corresponds to the generalized Dynkin diagram

$$\circ^{1} \text{———} \circ^{2} \text{———} \circ^{3} \ \dots \qquad \circ^{\theta-1} \text{———} \circ^{\theta} \tag{3.9}$$

where

- $q = q_{\theta-1,\theta}q_{\theta,\theta-1}q_{\theta\theta}^2$ holds, $1 \leq i_1 < \ldots < i_j \leq \theta$;
- equation $q_{i-1,i}q_{i,i-1} = q$, where $1 \leq i \leq \theta$, is valid if and only if $i \in \{i_1, i_2, \dots, i_j\}$, so each $q_{i_t,i_t} = -1, \quad t = 1, \dots, j$;
- $q_{ii} = q^{\pm 1}$ if $i \neq i_1, \dots, i_j$.

Then, the labels of vertices between i_t and i_{t+1} are all equal, and they are labeled with the inverse of the scalar associated to the vertices between i_{t+1} and i_{t+2}; the same is valid for the scalars that appear in the edges.

Proposition 3.9. *Let V be a braided vector space of diagonal type. Then V is standard of A_θ type if and only if its generalized Dynkin diagram is of the form:*

$$C(\theta, q; i_1, \dots, i_j). \tag{3.10}$$

Note that the previous braiding is of Cartan type if and only if $j = 0$, or $j = n$ with $q = -1$.

Proof. Let V be a braided vector space of standard A_θ type. For each vertex $1 < i < \theta$ we have $q_{ii} = -1$ or $q_{ii}q_{i,i-1}q_{i-1,i} = q_{ii}q_{i,i+1}q_{i+1,i} = 1$, and the corresponding formulas for $i = 1, \theta$. So suppose that $1 < i < \theta$ and $q_{ii} = -1$. We transform by s_i and obtain

$$\widetilde{q}_{i-1,i+1} = -q_{i,i+1}q_{i-1,i}q_{i-1,i+1}, \quad \widetilde{q}_{i+1,i-1} = -q_{i,i-1}q_{i+1,i}q_{i+1,i-1},$$

and using that $m_{i-1,i+1} = \widetilde{m}_{i-1,i+1} = 0$, we have $q_{i-1,i+1}q_{i+1,i-1} = 1$ and $\widetilde{q}_{i-1,i+1}\widetilde{q}_{i+1,i-1} = 1$, so we deduce that $q_{i,i+1}q_{i+1,i} = (q_{i,i-1}q_{i-1,i})^{-1}$. Then the corresponding matrix (q_{ij}) is of the form (3.10).

Now, consider V of the form 3.10. Assume $q_{ii} = q^{\pm 1}$; if we transform by s_i, then the braided vector space V_i is twist equivalent with V by Remark 3.4. Thus, $\overline{m}_{ij} = m_{ij}$.

Assume $q_{ii} = -1$. We transform by s_i and calculate

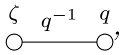
$$\begin{aligned} \widetilde{q}_{jj} &= (-1)^{m_{ij}^2}(q_{ij}q_{ji})^{m_{ij}}q_{jj} \\ &= \begin{cases} q_{jj}, & |j-i| > 1; \\ (-1)q^{\mp 1}q^{\pm 1} = -1, & j = i \pm 1, \ q_{jj} = q^{\pm 1}; \\ (-1)q^{\pm 1}(-1) = q^{\pm 1}, & j = i \pm 1, \ q_{jj} = -1. \end{cases} \end{aligned}$$

Also, $\widetilde{q}_{ij}\widetilde{q}_{ji} = q_{ij}^{-1}q_{ji}^{-1}$ if $|j-i| > 1$, and

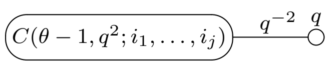
$$\widetilde{q}_{kj}\widetilde{q}_{jk} = (q_{ik}q_{ki})^{m_{ij}}(q_{ij}q_{ji})^{m_{ik}}q_{kj}q_{jk} = \begin{cases} q_{kj}q_{jk} & |j-i| \, or \, |k-i| > 1, \\ 1 & j = i-1, k = i+1. \end{cases}$$

Then V_i has a braiding of the above form too, and $(-m_{ij})$ corresponds to the finite Cartan matrix of type A_θ, so it is a standard braiding of type A_θ. Thus this is the complete family of standard braidings of type A_θ. □

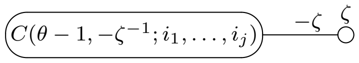
Proposition 3.10. *Let V a diagonal braided vector space. Then V is standard of type B_θ if and only if its generalized Dynkin diagram is of one of the following forms:*

(a) 

, $\zeta \in \mathbb{G}_3$, $q \in \mathbb{G}_N, N \geq 4$ $(\theta = 2)$;

(b) 

, $q \neq 0 - 1$, $0 \leq j \leq d-1$;

(c) 

, $\zeta\in\mathbb{G}_3$, $0\le j\le d-1$.

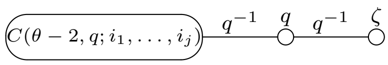
Note that the previous braiding is of Cartan type if and only if it is as in (b) and $j=0$.

Proof. First we analyze the case $\theta=2$. Let V a standard braided vector space of type B_2. There are several possibilities:

- $q_{11}^2q_{12}q_{21}=q_{22}q_{21}q_{12}=1$: this braiding is of Cartan type, with $q=q_{11}$. Note that $q\neq-1$. This braiding has the form (b) with $\theta=2, j=0$.
- $q_{11}^2q_{12}q_{21}=1,\quad q_{22}=-1$. We transform by s_2, then
$$\widetilde{q}_{11}=-q_{11}^{-1},\quad \widetilde{q}_{12}\widetilde{q}_{21}=q_{12}^{-1}q_{21}^{-1}.$$
Thus $\widetilde{q}_{11}^2\widetilde{q}_{12}\widetilde{q}_{21}=1$. It has the form (b) with $j=1$.
- $q_{11}^3=1,\quad q_{22}q_{21}q_{12}=1$. We transform by s_1,
$$\widetilde{q}_{22}=q_{11}q_{12}q_{21},\quad \widetilde{q}_{12}\widetilde{q}_{21}=q_{11}^2q_{12}^{-1}q_{21}^{-1}.$$
So $\widetilde{q}_{22}\widetilde{q}_{21}\widetilde{q}_{12}=1$, which is the case (a).
- $q_{11}^3=1,\quad q_{22}=-1$: we transform by s_1,
$$\widetilde{q}_{22}=-q_{12}^2q_{21}^2q_{11},\quad \widetilde{q}_{12}\widetilde{q}_{21}=q_{11}^2q_{12}^{-1}q_{21}^{-1}.$$
If we transform by s_2,
$$\widetilde{q}_{11}=-q_{12}q_{21}q_{11},\quad \widetilde{q}_{12}\widetilde{q}_{21}=q_{12}^{-1}q_{21}^{-1}.$$
So $q_{12}q_{21}=\pm q_{11}$, and we discard the case $q_{12}q_{21}=q_{11}$ because it was considered before. Then it has the form in case (c) with $j=0$, and is standard.

Conversely, all braidings (a), (b) and (c) are standard of type B_2.

Let now V of type B_θ, with $\theta\ge3$. Note that the first $\theta-1$ vertices determine a braiding of standard $A_{\theta-1}$ type, and the last two determine a braiding of standard B_2 type; so we have to 'glue' the possible such braidings. The possible cases are the two presented in the Proposition, and



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But if we transform by s_θ, we obtain
$$\widetilde{q}_{\theta-1,\theta-1}=\zeta q^{-1},\quad \widetilde{q}_{\theta-1,\theta-2}\widetilde{q}_{\theta-2,\theta-1}=q^{-1},$$
so $1=\widetilde{q}_{\theta-1,\theta-1}\widetilde{q}_{\theta-1,\theta-2}\widetilde{q}_{\theta-2,\theta-1}$ and we obtain $q=\pm\zeta$, or $\widetilde{q}_{\theta-1,\theta-1}=-1$. Then, $q=-\zeta$ or $q=-1$, so it is of some of the above forms.

To prove that (b), (c) are standard braidings, we use the following fact: if $m_{ij}=0$ (that is, $q_{ij}q_{ji}=1$) and we transform by s_i, then
$$\widetilde{q}_{jj}=q_{jj},\quad \widetilde{q}_{jk}\widetilde{q}_{jk}=q_{jk}q_{kj}\ (k\neq i).$$
In this case, if $|i-j|>1$, then $m_{ij}=0$; if $j=i\pm1$ we use the fact that the subdiagram determined by these two vertices is standard of type B_2 or

type A_2. So this is the complete family of all twist equivalence classes of standard braidings of type B_θ. $\square$

Proposition 3.11. *Let V a braided vector space of diagonal type. Then V is standard of type G_2 if and only if its generalized Dynkin diagram is one of the following:*

(a) $\overset{q}{\circ}\overset{q^{-3}}{\text{———}}\overset{q^{3}}{\circ}$, ord $q \geq 4$;

(b) *There exists $\zeta \in \mathbb{G}_8$ such that*

   (i) $\overset{\zeta^{2}}{\circ}\overset{\zeta}{\text{———}}\overset{\zeta^{-1}}{\circ}$, *or*

   (ii) $\overset{\zeta^{2}}{\circ}\overset{\zeta^{3}}{\text{———}}\overset{-1}{\circ}$, *or*

   (iii) $\overset{\zeta}{\circ}\overset{\zeta^{5}}{\text{———}}\overset{-1}{\circ}$.

Note that the previous braiding is of Cartan type iff it is as in (a).

Proof. Let V be a standard braiding of G_2 type. There are four possible cases:

- $q_{11}^3 q_{12} q_{21} = 1, \quad q_{22} q_{21} q_{12} = 1$: this braiding is of Cartan type, as in (a), with $q = q_{11}$. Note that if q is a root of 1, then ord $q \geq 4$ because $m_{12} = 3$.
- $q_{11}^3 q_{12} q_{21} = 1, \quad q_{22} = -1$: we transform by s_2,
$$\widetilde{q}_{11} = -q_{11}^{-2}, \quad \widetilde{q}_{12}\widetilde{q}_{21} = q_{12}^{-1} q_{21}^{-1}.$$
If $1 = \widetilde{q}_{11}^3 \widetilde{q}_{12}\widetilde{q}_{21} = -q_{11}^{-3}$, then $q_{12}q_{21} = -1$, and the braiding is of Cartan type with $q_{11} \in \mathbb{G}_6$. If not, $1 = \widetilde{q}_{11}^4 = q_{11}^{-8}$ and ord$\widetilde{q}_{11} = 4$, so ord$q_{11} = 8$. Then we can express the braiding in the form (b)-(iii).
- $q_{11}^4 = 1, \quad q_{22} q_{21} q_{12} = 1$: we transform by s_1,
$$\widetilde{q}_{22} = q_{11} q_{12}^2 q_{21}^2, \quad \widetilde{q}_{12}\widetilde{q}_{21} = -q_{12}^{-1} q_{21}^{-1}.$$
If $1 = \widetilde{q}_{22}\widetilde{q}_{21}\widetilde{q}_{12} = -q_{11} q_{12} q_{21}$, we have $q_{11}^3 q_{12} q_{21} = 1$ because $q_{11}^2 = -1$, and this is a braiding of Cartan type. So we consider now the case $-1 = \widetilde{q}_{22} = q_{11} q_{12}^2 q_{21}^2$, so we obtain $q_{22}^2 = q_{11}^{-1}$ and $q_{22} \in \mathbb{G}_8$. Then we obtain a braiding of the form (b)-(i).
- $q_{11}^4 = 1, \quad q_{22} = -1$: we transform by s_2,
$$\widetilde{q}_{11} = -q_{12} q_{21} q_{11}, \quad \widetilde{q}_{12}\widetilde{q}_{21} = q_{12}^{-1} q_{21}^{-1}.$$
If $\widetilde{q}_{11} \in \mathbb{G}_4$, then $(q_{12}q_{21})^4 = 1$. $q_{12}q_{21} \neq 1$ and $q_{12}q_{21} \neq q_{11}^{-1}$ because $m_{12} = 3$. So, $q_{12}q_{21} = -1$ or $q_{12}q_{21} = q_{11} = q_{11}^{-3}$, but these cases already were considered. So we analyze the case
$$1 = \widetilde{q}_{11}^3 \widetilde{q}_{12}\widetilde{q}_{21} = q_{11} q_{12}^2 q_{21}^2,$$
so we can express it in the form (b)-(ii) for some $\zeta \in \mathbb{G}_8$.

A simple calculation proves that this braidings are of standard type, so they are all the standard braidings of G_2 type. $\square$

4. Nichols algebras of standard braided vector spaces

In this section we study Nichols algebras associated to standard braidings. We assume that the Dynkin diagram is connected, as in Section 3. In subsection 4.1 we prove that the set $\Delta^+(\mathfrak{B}(V))$ is in bijection with Δ_C^+, the set of positive roots associated with the finite Cartan matrix C.

We describe an explicit set of generators in subsection 4.2, following [LR]. We adapt their proof since they work on enveloping algebras of simple Lie algebras. In subsection 4.3, we calculate the dimension of Nichols algebra associated to a standard braided vector space, type by type.

4.1. **PBW bases of Nichols algebras.** The next result is the analogous to [H2, Theorem 1] but for braidings of standard type.

Theorem 4.1. *Let V be a braided vector space of standard type with Cartan matrix C. Then the following are equivalent:*

(1) *The set $\Delta(\mathfrak{B}(V))$ is finite.*
(2) *The Cartan matrix C is of finite type.*

Proof. (1) $\Rightarrow$ (2) If $\Delta\left(\mathfrak{B}\left(V\right)\right)$ is finite, then $\Delta(\chi) \subseteq \Delta(\mathfrak{B}(V))$ is also finite.

▶ If C is symmetrizable, (2) holds by Lemma 3.7.

▶ Let C be non symmetrizable. We prove that either the corresponding set $\Delta(\chi)$ is not finite, or else there does not exist any standard braided vector space associated with this matrix C. The proof follows the same steps as in [H2] for the corresponding result about braided vector spaces of Cartan type. The unique step where he uses the Cartan type condition is the following, that we adapt to the standard type case. We restrict the proof to the case where the generalized Cartan matrix is not symmetrizable, and the corresponding Dynkin diagram is not simply laced cycle, such that after removing an arbitrary vertex the resulting diagram is of finite type. At this stage, as in loc. cit., we reduce to the following cases:

- For $\theta = 5$, there is only one multiple edge, because there are no two multiple edges at distance one (we remove a vertex which is not an extreme of these multiple edges and obtain a diagram of non finite type). Then we have an unique double edge,

$$C_0 = \begin{pmatrix} 2 & -2 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

  Note that, if we suppose $q_{11} = -1$, from $m_{12} = 2$ and $q_{11}^3 \neq 1$, we have $1 = q_{11}^2 q_{12} q_{21} = q_{12} q_{21}$, and then $m_{12} = m_{21} = 0$, which is not possible. Then $q_{11} \neq -1$, and from $m_{15} = 1$, we have $q_{11} q_{15} q_{51} = 1$. Following Remark 3.3 for $i = 5, j = 1$, we have $q_{55} \neq -1$, because in other case, $\widetilde{q}_{11} = -1$ when we transform by s_5, which is not possible (the new braiding is also standard). Then $q_{55}q_{51}q_{15} = q_{55}q_{54}q_{45} = 1$, and $q_{11} = q_{55}$. Now, also from Remark 3.3 but for $i = 4, j = 5$, as $\widetilde{q}_{55} \neq -1$, it follows that $q_{44} \neq -1$, and then

$$q_{44}q_{45}q_{54} = q_{44}q_{43}q_{34} = 1, \quad q_{44} = q_{55}.$$

Following,

$$q_{33}q_{34}q_{43} = q_{33}q_{32}q_{23} = q_{22}q_{23}q_{32} = q_{22}q_{21}q_{12} = 1,$$

and $q_{22} = q_{33} = q_{44} = q_{55} = q_{11}$. But then $q_{11}q_{12}q_{21} = 1$, and $m_{12} = 1$, a contradiction. Then there are no standard braidings with Cartan matrix C_0.

- For $\theta = 4$, we consider the matrix

$$C = \begin{pmatrix} 2 & -2 & 0 & -b \\ -1 & 2 & -c & 0 \\ 0 & -f & 2 & -d \\ -e & 0 & -g & 2 \end{pmatrix},$$

where $be, cf, dg = 1, 2$, because there are no triple edges. The proof is the same as in [H2], and we obtain that $\Delta(\chi)$ is infinite in this case.

- For $\theta = 3$, we consider the matrices t_i corresponding to s_i, $i = 1, 2, 3$:

$$t_1 = \begin{pmatrix} -1 & -a_{12} & -a_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 1 & 0 & 0 \\ -a_{21} & -1 & -a_{31} \\ 0 & 0 & 1 \end{pmatrix},$$

$$t_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a_{31} & -a_{32} & -1 \end{pmatrix}.$$

The proof is as in [H2], and $\Delta(\chi)$ is infinite in this case too.

(2) $\Rightarrow$ (1) Let V be a standard braided vector space with Cartan matrix of finite type. Then the matrix C is symmetrizable. We fix $\pi = \{\alpha_1, \dots, \alpha_\theta\}$ a set of simple roots corresponding to the root system Δ_C of C. We define the $\mathbb{Z}$-linear map

$$\phi : \mathbb{Z}\pi \to \mathbb{Z}^\theta, \quad \phi(\alpha_i) := \mathbf{e}_i.$$

Consider the action of s_i over Δ_C as the reflection corresponding to the simple root α_i; then ϕ is a W-module morphism. Each $\beta \in \Delta_C$ is of the form $\beta = w(\alpha_i)$, and $w = s_{i_1} \cdots s_{i_k}$, for some $i_1, \dots, i_k \in \{1, \dots, \theta\}$. Then

$$\phi(\beta) = \phi\left(s_{i_1} \cdots s_{i_k}(\alpha_i)\right) = s_{i_1} \cdots s_{i_k}\phi(\alpha_i) = s_{i_1} \cdots s_{i_k}(\mathbf{e}_i),$$

thus $\phi(\Delta_C) \subseteq \Delta\left(\mathfrak{B}(V)\right)$.

Suppose that $\Delta\left(\mathfrak{B}(V)\right) \supsetneq \phi(\Delta_C)$. In this case, we give a different proof to the one in [H2], based in the fact that C is positive definite. Let α be a root of minimum height in the non empty set $\Delta\left(\mathfrak{B}(V)\right) - \phi(\Delta_C)$. First, $\alpha \neq m\alpha_i$, for all $m \in \mathbb{N}$, and $i = 1, \dots, \theta$, because $m\alpha_i \in \Delta\left(\mathfrak{B}(V)\right) \Leftrightarrow m = \pm 1$, but $\pm\alpha_i \in \phi(\Delta_C)$. Therefore, for each s_i, as α is not a multiple of α_i, we have $s_i(\alpha) \in \Delta(\mathfrak{B}(V)) - \phi(\Delta_C)$, and then $gr(s_i(\alpha)) - gr(\alpha) \geq 0$. As $\alpha = \sum_{i=1}^{\theta} b_i\alpha_i$, we have $\sum_{i=1}^{\theta} b_i a_{ij} \leq 0$, and as $b_j \geq 0$, we have $\sum_{i,j=1}^{\theta} b_i a_{ij} b_j \leq 0$. This contradicts the fact that (a_{ij}) is definite positive, and $(b_i) \geq 0, (b_i) \neq 0$. Then $\phi(\Delta_C) = \Delta(\mathfrak{B}(V))$. □

Corollary 4.2. *Let V be a braided vector space of standard type, $\theta = \dim V$, and $C = (a_{ij})_{i,j\in\{1,\dots,\theta\}}$ the corresponding generalized Cartan matrix of finite type. Then*

(a) $\phi(\Delta_C) = \Delta(\mathfrak{B}(V))$, *where as before $\phi : \mathbb{Z}\pi \to \mathbb{Z}^\theta$ is the $\mathbb{Z}$-linear map determined by $\phi(\alpha_i) := e_i$.*

(b) *The multiplicity of each root in Δ is one.*

Proof. (a) follows from the proof of (2) ⇒ (1) of the preceding Theorem.

Using this condition, as each root is of the form $\beta = w(\alpha_i), w \in W, i \in \{1,\dots,\theta\}$, doing a certain sequence of transformations s_i's, this is the degree corresponding to a generator of the corresponding Nichols algebra, so the multiplicity (invariant by these transformations) is 1. □

4.2. **Explicit generators for a PBW basis.**

From Corollary 4.2, we restrict our attention to find one Lyndon word for each positive root of the root system associated with the corresponding finite Cartan matrix.

Proposition 4.3. [LR, Proposition 2.9] *Let l be an element of S_I. Then l is of the form $l = l_1 \dots l_k a$, where*

- *$l_i \in S_I$, for each $i = 1,\dots,k$;*
- *l_i is a beginning of l_{i-1}, for each $i > 1$;*
- *a is a letter.*

Also, if $l = uv$ is the Shirshov decomposition, then $u, v \in S_I$. □

In what follows, we describe a set of Lyndon words for each Cartan matrix of finite type C.

Consider $\alpha = \sum_{j=1}^{\theta} a_j\alpha_j \in \Delta^+$, let $l_\alpha \in S_I$ be such that $\deg l_\alpha = \alpha$. Let $l_\alpha = l_{\beta_1} \dots l_{\beta_k} x_s$ be a decomposition as above, where $s \in \{1,\dots,\theta\}$ and $\deg l_{\beta_j} = \beta_j$. Note that, as each l_{β_j} is a beginning of $l_{\beta_{j-1}}$, all the words begin with the same letter x', and as l is a Lyndon word, $x' < x_s$. Therefore, x' is the least letter of l, so

$$x' = x_i, \quad i = \min\{j : a_j \neq 0\} \qquad \Rightarrow \qquad \alpha = \sum_{j=i}^{\theta} a_j\alpha_j.$$

Then $k \leq a_i \leq 3$ – for the order given in (3.9), (3.4), (3.5), (3.6), (3.7), (3.8) ($a_i = 3$ appears only when C is of type G_2).

Now, each $l_{\beta_j} \in S_I$, so $\beta_j \in \Delta^+$; i.e., it corresponds with a term of the PBW basis. Also $\sum_{j=1}^{k} \beta_j + \alpha_s = \alpha$. If $k = 2$, we have $\beta_1 - \beta_2 =$

$\sum_{j=1}^{\theta} b_j\alpha_j, \quad b_j \geq 0$, because β_2 is a beginning of β_1 (the analogous claim is valid when the matrix is of type G_2, and $k = 3$). With these rules we define inductively Lyndon words for a PBW basis corresponding with a standard braiding for a fixed order on the letters as in [LR], but taking care that in their work they use the Serre relations. Now we have Serre quantum relations and some quantum binomial coefficients maybe are zero.

Type A_θ: In this case, the roots are of the form

$$\mathbf{u}_{i,j} := \sum_{k=i}^{j} \alpha_k, \quad 1 \leq i \leq j \leq \theta.$$

By induction on $s = j - i$, we have

$$l_{\mathbf{u}_{i,j}} = x_i x_{i+1} \dots x_j.$$

This is because when $s = 0$ we have $i = j$, and the unique possibility is $l_{\mathbf{u}_{i,i}} = x_i$. Then if we remove the last letter (when $j - i > 0$), we must obtain a Lyndon word, so the last letter must be x_j.

Type B_θ: For convenience, we use the following enumeration of vertices:

$$\circ^1 \Longleftarrow \circ^2 \text{ --- } \circ^3 \dots \qquad \circ^{\theta-1} \text{ --- } \circ^{\theta}\ . \tag{4.1}$$

The roots are of the form $\mathbf{u}_{i,j} := \sum_{k=i}^{j} \alpha_k$, or

$$\mathbf{v}_{i,j} := 2\sum_{k=1}^{i} \alpha_k + \sum_{k=i+1}^{j} \alpha_k.$$

In the first case, as above we have $l_{\mathbf{u}_{i,j}} = x_i x_{i+1} \dots x_j$. In the second case, note that if $j = i + 1$, we must have x_{i+1} as the last letter to obtain a decomposition in two words $x_1 \cdots x_i$; if $j > i + 1$, then the last letter must be x_j, so we obtain that

$$l_{\mathbf{v}_{i,j}} = x_1 x_2 \dots x_i x_1 x_2 \dots x_j.$$

Type C_θ: The roots are of the form $\mathbf{u}_{i,j} := \sum_{k=i}^{j} \alpha_k$, or

$$\mathbf{w}_{i,j} := \sum_{k=i}^{j-1} \alpha_k + 2\sum_{k=j}^{\theta-1} \alpha_k + \alpha_\theta, \quad i \leq j < \theta.$$

As before, $l_{\mathbf{u}_{i,j}} = x_i x_{i+1} \dots x_j$. Now, if $i < j$, the least letter x_i has degree 1, so if we remove the last letter, we obtain a Lyndon word; i. e., $\mathbf{w}_{i,j} - x_s$ is a root, and then $x_s = x_j$, so

$$l_{\mathbf{w}_{i,j}} = x_i x_{i+1} \dots x_{\theta-1} x_\theta x_{\theta-1} \dots x_j.$$

When $i = j$, $a_i = 2$, so there are one or two Lyndon words β_j as before. As $\mathbf{w} - x_s$ is not a root, for $s = i + 1, ..., \theta$, and $i < s$, there are two

Lyndon words $\beta_1 \geq \beta_2$, and $\beta_1 + \beta_2 = 2\sum_{k=i}^{\theta-1} \alpha_k$. The unique possibility is $\beta_1 = \beta_2 = x_i x_{i+1} \dots x_{\theta-1}$; i. e.,

$$l_{\mathbf{w}_{i,i}} = x_i x_{i+1} \dots x_{\theta-1} x_i x_{i+1} \dots x_{\theta-1} x_\theta.$$

Type D_θ: the roots are of the form $\mathbf{u}_{i,j} := \sum_{k=i}^{j} \alpha_k, \quad 1 \leq i \leq j \leq \theta$, or

$$\begin{aligned} \mathbf{z}_{i,j} &:= \sum_{k=i}^{j-1} \alpha_k + 2\sum_{k=j}^{\theta-2} \alpha_k + \alpha_{\theta-1} + \alpha_\theta, \quad i < j \leq \theta - 2, \\ \bar{\mathbf{z}}_i &:= \sum_{k=i}^{\theta-2} \alpha_k + \alpha_\theta, \quad 1 \leq i \leq \theta - 2. \end{aligned}$$

As above, $l_{\mathbf{u}_{i,j}} = x_i x_{i+1} \dots x_j$ if $j \leq n - 1$. When the roots are of type $\bar{\mathbf{z}}_i$, as $\bar{\mathbf{z}}_i - x_s$ must be a root (if x_s is the last letter), we have $s = \theta$, and then $l_{\bar{\mathbf{z}}_i} = x_i x_{i+1} \dots x_{\theta-2} x_\theta$ is the unique possibility.

Now, when $\alpha = \mathbf{u}_{i,\theta}$, the last letter is $x_{\theta-1}$ or x_θ: if it is x_θ, we have $l_{\mathbf{u}_{i,\theta}} = x_i x_{i+1} \dots x_{\theta-1} x_\theta$. As $m_{\theta-1,\theta} = 0$, we have $x_{\theta-1} x_\theta = q_{\theta-1,\theta} x_\theta x_{\theta-1}$, so

$$x_i x_{i+1} \dots x_{\theta-1} x_\theta \equiv x_i x_{i+1} \dots x_{\theta-2} x_\theta x_{\theta-1} \mod I,$$

and then $x_i x_{i+1} \dots x_{\theta-1} x_\theta \notin S_I$. So, $l_{\mathbf{u}_{i,\theta}} = x_i \dots x_{\theta-2} x_\theta x_{\theta-1}$.

In the last case, note that if $j = n - 2$, the unique possibility is β_t as before, because the least letter x_i has degree 1 and as $\alpha - \alpha_s$ is a root, $x_s = x_{\theta-2}$. Then $l_{\mathbf{z}_{i,\theta-2}} = x_i \dots x_{\theta-2} x_\theta x_{\theta-1} x_{\theta-2}$, and inductively,

$$l_{\mathbf{z}_{i,j}} = x_i \dots x_{\theta-2} x_\theta x_{\theta-1} x_{\theta-2} \dots x_j.$$

Type E_6: Note that if $\alpha = \sum_{j=1}^{6} a_j \alpha_j$ and $a_6 = 0$, then it corresponds with the Dynkin subdiagram of type D_5 determined by $1, 2, 3, 4, 5$, and we obtain l_α as above. If $a_1 = 0$ it corresponds with the Dynkin subdiagram of type D_5 determined by $2, 3, 4, 5, 6$ – the numeration is different of the one given in 3.4. Anyway, the roots are defined in a similar way, and we obtain the same list as in [LR, Fig.1]. If $a_4 = 0$, then α corresponds with the Dynkin subdiagram of type A_5 determined by $1, 2, 3, 5, 6$.

So we restrict our attention to the case $a_i \neq 0, \quad i = 1, 2, 3, 4, 5, 6$. We consider each case:

- $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$: as $a_1 = 1$, $\alpha - \alpha_s = \beta_1$ is a root, where α_s is the last letter. Then $s = 2$ or $s = 6$. In the second case, $l_{\beta_1} = x_1 x_2 x_3 x_4 x_5$, but using that $x_2 x_3 = q_{23} x_3 x_2$, we have that $x_1 x_2 x_3 x_4 x_5 \notin S_I$. So $s = 2$, and $l_\alpha = x_1 x_3 x_4 x_5 x_6 x_2$.
- $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$: from $a_1 = 1$, we note that $\alpha - \alpha_s = \beta_1$ is a root. Then $s = 4$, and $l_\alpha = x_1 x_3 x_4 x_5 x_6 x_2 x_4$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$: from $a_1 = 1$, $\alpha - \alpha_s = \beta_1$ is a root. So $s = 3$, and $l_\alpha = x_1 x_3 x_4 x_5 x_6 x_2 x_4 x_3$.
- $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$: from $a_1 = 1$, $\alpha - \alpha_s = \beta_1$ is a root. The unique possibility is $s = 5$, and $l_\alpha = x_1 x_3 x_4 x_5 x_6 x_2 x_4 x_5$.

- $\alpha = \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$: as above $a_1 = 1$, and $\alpha-\alpha_s = \beta_1$ is a root. So $s = 3$, and $l_\alpha = x_1x_3x_4x_5x_6x_2x_4x_5x_3$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$: from $a_1 = 1$, $\alpha - \alpha_s = \beta_1$ is a root. Then $s = 4$ and $l_\alpha = x_1x_3x_4x_5x_6x_2x_4x_5x_3x_4$.
- $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$: from $a_1 = 1$, $\alpha - \alpha_s = \beta_1$ is a root. So $s = 2$, and $l_\alpha = x_1x_3x_4x_5x_6x_2x_4x_5x_3x_4$.

Type E_7: If $\alpha = \sum_{j=1}^{7} a_j\alpha_j$ and $a_7 = 0$, the root corresponds to the subdiagram of type D_6 determined by $1, 2, 3, 4, 5, 6$, and we obtain l_α as above. If $a_1 = 0$, it corresponds to the subdiagram of type E_6 determined by $2, 3, 4, 5, 6, 7$. If $a_5 = 0$, then α corresponds to the subdiagram of type A_6 determined by $1, 2, 3, 4, 6, 7$.

As above, consider each case where $a_i \neq 0, \quad i = 1, 2, 3, 4, 5, 6, 7$:

- $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$: as $a_1 = 1$, $\alpha - \alpha_s = \beta_1$ is a root, if α_s is the last letter. Then $s = 2$ o $s = 7$. In the second case, $l_{\beta_1} = x_1x_2x_3x_4x_5x_6$, but from $x_2x_3 = q_{23}x_3x_2$, we have $x_1x_2x_3x_4x_5x_6x_7 \notin S_I$. So $s = 2$, and $l_\alpha = x_1x_3x_4x_5x_6x_7x_2$.
- $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$: now, $s = 4, 7$. We discard the case $s = 7$ using that $m_{47} = 0$, and then $s = 4$: $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$: as above, $s = 3, 7$, but we discard $s = 7$ using that $m_{37} = 0$, so $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_3$.
- $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$: now, $s = 5, 7$, and discard the case $s = 7$ because $m_{57} = 0$, and $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$: $s = 3, 7$, and as above we discard the case $s = 7$, so $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_3$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$: $s = 4$, and $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_3x_4$.
- $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$: $s = 2$, as above, and $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_3x_4x_2$.
- $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$:as above, the unique possibility is $s = 6$, so $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$: $s = 3$, and $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6x_3$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$: $s = 4$, and $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6x_3x_4$.
- $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$: $s = 2$, and $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6x_3x_4x_2$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$: $s = 5$, and $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6x_3x_4x_5$.
- $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$: as above, $s = 2$, and $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6x_3x_4x_5x_2$.
- $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$: $s = 4$, and $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6x_3x_4x_5x_2x_4$.

- $\alpha = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$: $s = 3$, and $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6x_3x_4x_5x_2x_4x_3$.
- $\alpha = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$: now, there are one or two words β_j. As $\alpha - \alpha_s \in \Delta^+$ iff $s = 1$ and x_1 is not the last letter (because it is the least letter), there are two words β_j. So looking at the roots we obtain $s = 7$, and
$$l_\alpha = (x_1x_3x_4x_5x_6x_2x_4x_5x_3x_4x_2)(x_1x_3x_4x_5x_6)x_7$$

Type E_8: Consider $\alpha = \sum_{j=1}^{8} a_j\alpha_j$; if $a_8 = 0$, the root corresponds to the subdiagram of type D_7 determined by $1, 2, 3, 4, 5, 6, 7$, and we obtain l_α as in that case. If $a_1 = 0$, it corresponds to the subdiagram of type E_7 determined by $2, 3, 4, 5, 6, 7, 8$. If $a_6 = 0$, then α corresponds to a subdiagram of type A_7 determined by $1, 2, 3, 4, 5, 7, 8$.

So, we consider the case $a_i \neq 0, \quad i = 1, 2, 3, 4, 5, 6, 7, 8$, and solve it case by case in a similar way as for E_7, by induction on the height.

Type F_4: Now, $\alpha = \sum_{j=1}^{4} a_j\alpha_j$. If $a_4 = 0$, then it corresponds to the subdiagram of type B_3 determined by $1, 2, 3$, so we obtain l_α as before. If $a_1 = 0$, α corresponds to the subdiagram of type C_3 determined by $2, 3, 4$.

So consider the case $a_i \neq 0, \quad i = 1, 2, 3, 4$:

- $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$: $a_1 = 1$, so $\alpha - \alpha_s = \beta_1$ is a root, where α_s is the last letter. Then $s = 4$, and $l_\alpha = x_1x_2x_3x_4$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$: $a_1 = 1$, so $\alpha - \alpha_s = \beta_1$ is a root. Now, $s = 3$ or $s = 4$. If $s = 4$, then $l_\alpha = x_1x_2x_3^2x_4$. But $m_{34} = 2$, so
$$x_3^2x_4 \equiv q_{34}(1 + q_{33})x_3x_4x_3 - q_{33}q_{34}x_4x_3^2 \mod I,$$
and $x_1x_2x_3^2x_4 \notin S_I$, a contradiction. So $s = 3$, and we have that $l_\alpha = x_1x_2x_3x_4x_3$.
- $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$: $a_1 = 1$, and as above, $s = 2$ or $s = 4$: if $s = 4$, then $l_\alpha = x_1x_2x_3^2x_2x_4$, but it is not an element of S_I, because $x_2x_4 \equiv q_{24}x_2x_4 \mod I$. Then $s = 2$, and $l_\alpha = x_1x_2x_3x_4x_3x_2$.
- $\alpha = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$: $a_1 = 1$, so $s = 3$, and we have that $l_\alpha = x_1x_2x_3x_4x_3x_2x_3$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4$: $a_1 = 1$, so $s = 4$, and $l_\alpha = x_1x_2x_3x_4x_3x_4$.
- $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4$: $a_1 = 1$, so $s = 2$ or $s = 4$, but we discard the case $s = 4$ since $x_2x_4 \equiv q_{24}x_2x_4 \mod I$. So, $l_\alpha = x_1x_2x_3x_4x_3x_4x_2$.
- $\alpha = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$: $a_1 = 1$, so $s = 3$, and
$$l_\alpha = x_1x_2x_3x_4x_3x_4x_2x_3.$$
- $\alpha = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$: $a_1 = 1$, so $s = 3$, and
$$l_\alpha = x_1x_2x_3x_4x_3x_4x_2x_3^2.$$
- $\alpha = \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$: $a_1 = 1$, so $s = 2$, and
$$l_\alpha = x_1x_2x_3x_4x_3x_4x_2x_3^2x_2.$$

- $\alpha = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$: $a_1 = 2$, then there are one or two Lyndon words β_j. If there is only one, $\beta_1 = \alpha - \alpha_s \in \Delta^+$. The unique possibility is $s = 1$, but it contradicts that l_α is a Lyndon word. Then there exist $\beta_1, \beta_2 \in \Delta^+$ such that $\beta_1 + \beta_2 = \alpha - \alpha_s$, and β_2 is a beginning of β_1. So $s = 2$ and $\beta_1 = \beta_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$, i.e., $l_\alpha = x_1x_2x_3x_4x_3x_1x_2x_3x_4x_3x_2$.

Type G_2: the roots are $\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2$:

$$l_{\alpha_1} = x_1, \qquad l_{\alpha_2} = x_2, \qquad l_{m\alpha_1+\alpha_2} = x_1^m x_2, \quad m = 1, 2, 3.$$

If $\alpha = 3\alpha_1 + 2\alpha_2$, the last letter is x_2. If we suppose $\beta_1 = 3\alpha_1 + \alpha_2$, then $l_\alpha = x_1^3 x_2^2$, but

$$(adx_2)^2 x_1 = x_2^2 x_1 - q_{21}(1 + q_{22})x_2x_1x_2 + q_{22}q_{21}x_1x_2^2 \equiv 0 \qquad \text{mod } I,$$

so we have

$$x_1^3x_2^2 \equiv (q_{22}^{-1} + 1)x_1^2x_2x_1x_2 - q_{22}^{-1}q_{21}^{-1}x_1^2x_2^2x_1 \qquad \text{mod } I,$$

and then $l_\alpha = x_1^3x_2^2 \notin S_I$ because $q_{22}^{-1}q_{21}^{-1} \neq 0$, so there are at least two words β_j. Analogously, if we suppose that there are three words β_j, as $\beta_1 \geq \beta_2 \geq \beta_3$ and $\beta_1 + \beta_2 + \beta_3 = 3\alpha_1 + \alpha_2$, we have $l_{\beta_1} = l_{\beta_2} = x_1 > l_{\beta_3} = x_1x_2$, and also $l_\alpha = x_1^3x_2^2 \notin S_I$. So there are two Lyndon words of degree $\beta_1 \geq \beta_2$, so the unique possibility is $\beta_1 = 2\alpha_1 + \alpha_2, \quad \beta_2 = \alpha_1$; i. e., $l_\alpha = x_1^2x_2x_1x_2$.

4.3. Dimension of Nichols algebras of standard braidings.

We begin by standard braidings of types $C_\theta, D_\theta, E_6, E_7, E_8, F_4$, which are just of Cartan type.

Proposition 4.4. *Let V a braided vector space of Cartan type, where $q_{44} \in \mathbb{G}_N$ if V is of type F_4, or $q_{11} \in \mathbb{G}_N$ otherwise, for some $N \in \mathbb{N}$. Then, for the associated Nichols algebra $\mathfrak{B}(V)$, we give*

$$\textbf{Type } C_\theta\textbf{:} \quad \dim \mathfrak{B}(V) = \begin{cases} N^{\theta^2} & N \text{ odd}, \\ N^{\theta^2}/2^\theta & N \text{ even}; \end{cases}$$

$$\textbf{Type } F_4\textbf{:} \quad \dim \mathfrak{B}(V) = \begin{cases} N^{24} & N \text{ odd}, \\ N^{24}/2^{12} & N \text{ even}; \end{cases}$$

$$\textbf{Types } D_\theta, E_6, E_7, E_8\textbf{:} \quad \dim \mathfrak{B}(V) = N^{|\Delta^+|}.$$

Note that the last case corresponds to simply-laced Dynkin diagrams.

Proof. Note that if N is odd, then $\text{ord}q^2 = \text{ord}q = N$, but if N is even, we have $\text{ord}q^2 = N/2$. Also, as the braiding is of Cartan type,

$$q_{s_i(\alpha)} = \chi\left(s_i(\alpha), s_i(\alpha)\right) = \widetilde{\chi}(\alpha, \alpha) = \chi(\alpha, \alpha) = q_\alpha.$$

Using this, we just have to determine how many roots there are in the orbit of each simply root.

When V is of type C_θ, $q_{ii} = q$, except for $q_{\theta\theta} = q^2$. The roots in the orbit of α_θ by the action of the Weyl group are $q_{\mathbf{w}_{ii}}$ for $1 \leq i < \theta$, and the others are in the orbit of α_j, for some $j < \theta$. Then, there are θ roots such that $q_\alpha = q^2$, and $q_\alpha = q$ for the rest.

When V is of type F_4, we have $q_{11} = q_{22} = q^2$, and $q_{33} = q_{44} = q$. There are exactly 12 roots in the union of orbits corresponding to α_1 and α_2, and the other 12 in the union of orbits corresponding to α_3 and α_4. So

$$\left|\left\{\alpha \in \Delta^+ : q_\alpha = q\right\}\right| = \left|\left\{\alpha \in \Delta^+ : q_\alpha = q^2\right\}\right| = 12.$$

When V is of type D or E, all $q_\alpha = q$ because $q_{ii} = q$, for all $1 \le i \le \theta$.

The formula for the dimension is a consequence of the theory of PBW bases above and Corollary 4.2. □

Now we treat the types A_θ, B_θ and G_2.

Proposition 4.5. *Let V be a standard braided vector space of type A_θ as in Proposition 3.9. Then the associated Nichols algebra $\mathfrak{B}(V)$ is of finite dimension if and only if q is a root of unit of order $N \ge 2$. In such case,*

$$\dim \mathfrak{B}(V) = 2^{\binom{\theta+1}{2}-\binom{t}{2}-\binom{\theta+1-t}{2}} N^{\binom{t}{2}+\binom{\theta+1-t}{2}}, \tag{4.2}$$

where $t = \theta - \sum_{k=1}^{j}(-1)^{j-k} i_k$.

Proof. q is a root of unit of order $N \ge 2$ because the height of each PBW generator is finite. To calculate the dimension, recall that from Corollary 4.2, we have to determine q_α for $\alpha \in \Delta_C$. As before, $\mathbf{u}_{ij} = \sum_{k=i}^{j} \mathbf{e}_k, \quad i \le j$, and we have

$$\Delta(\mathfrak{B}(V)) = \{\mathbf{u}_{ij} : 1 \le i \le j \le \theta\}.$$

If $1 \le i \le j \le \theta$, we define $\quad \kappa_{ij} := \sharp\{k \in \{i, \dots, j\} : q_{kk} = -1\}$.

We prove by induction on $j - i$ that

- if κ_{ij} is odd, then $q_{\mathbf{u}_{ij}} = -1$;
- if κ_{ij} is even, then $q_{\mathbf{u}_{ij}} = q_{i,i+1}^{-1} q_{i+1,i}^{-1}$.

If $j - i = 0$, then $q_{\mathbf{u}_{ii}} = q_{ii}$; in this case, $\kappa_{ii} = 1$ if $q_{ii} = -1$ or $\kappa_{ii} = 0$ if $q_{ii} = (q_{i,i+1} q_{i+1,i})^{-1} \ne -1$. Now, assume this is valid for certain j, and calculate it for $j + 1$:

$$\begin{aligned} q_{\mathbf{u}_{i,j+1}} &= \chi(\mathbf{u}_{ij} + \mathbf{e}_{j+1}, \mathbf{u}_{ij} + \mathbf{e}_{j+1}) = q_{\mathbf{u}_{ij}} \chi(\mathbf{u}_{ij}, \mathbf{e}_{j+1}) \chi(\mathbf{e}_{j+1}, \mathbf{u}_{ij}) q_{j+1,j+1} \\ &= q_{\mathbf{u}_{ij}} q_{j,j+1} q_{j+1,j} q_{j+1,j+1} \\ &= \begin{cases} q_{\mathbf{u}_{ij}} & q_{j+1,j+1} \ne -1 \ (\kappa_{i,j+1} = \kappa_{ij}), \\ (-1) q q^{-1} = -1 & q_{j+1,j+1} = -1, \ \kappa_{ij} \text{ even}, \\ (-1) q (-1) = q & q_{j+1,j+1} = -1, \ \kappa_{ij} \text{ odd}. \end{cases} \end{aligned}$$

So this proves the inductive step, and to calculate the dimension of $\mathfrak{B}(V)$ we have to calculate the number of $\mathbf{u}_{ij}$ such that

$$q_{\mathbf{u}_{ij}} = q_{i,i+1}^{-1} q_{i+1,i}^{-1} = q^{\pm 1},$$

this is, $\sharp\left\{\kappa_{ij} : i \le j, \kappa_{ij} \text{ even}\right\}$.

We consider an $1 \times (\theta+1)$ board, numbered from 1 to $\theta+1$, and paint its squares of white or black: the square $\theta+1$ is white, and then the i-th square is the same color of the $i+1$-th square if $q_{ii} \neq -1$, or different color if $q_{ii} = -1$. All the possible colorations of this board are in bijective correspondence with the choices of $1 \leq i_1 < \ldots < i_j \leq \theta$ for all j (the positions where we put a -1 in the corresponding q_{ii} of the braiding), and the number of white squares is

$$t = (\theta - i_j) + (i_{j-1} - i_{j-2}) + \ldots = \theta - \sum_{k=1}^{j} (-1)^{j-k} i_k$$

So $\sharp \{\kappa_{ij} : i \leq j, \kappa_{ij} \text{ even}\}$ is the number of pairs (a, b), $\quad 1 \leq a < b \leq \theta+1$ $(a = i$ and $b = j+1)$ such that the squares in positions a and b are of the same color, that is,

$$\binom{t}{2} + \binom{\theta+1-t}{2}.$$

Then we obtain the formula (4.2) for the dimension of $\mathfrak{B}(V)$. $\square$

Proposition 4.6. *Let V be a standard braided vector space of type B_θ as in Proposition 3.10. Then the associated Nichols algebra $\mathfrak{B}(V)$ is of finite dimension if and only if q is a root of unit of order $N \geq 2$. In such cases,*

- *if the braiding is as in* (a) *of Proposition 3.10,*

$$\dim \mathfrak{B}(V) = 3^2 N^2 \qquad \text{if 3 divides } N, \tag{4.3}$$

$$\dim \mathfrak{B}(V) = 3^3 N^2 \qquad \text{if 3 does not divide } N; \tag{4.4}$$

- *if the braiding is as in* (b)*, then $0 \leq j \leq d-1$, and*

$$\dim \mathfrak{B}(V) = 2^{2t(\theta-t)+\theta} k^{\theta^2-2t\theta+2t^2} \qquad \text{if } N = 2k, \tag{4.5}$$

$$\dim \mathfrak{B}(V) = 2^{(2t+1)(\theta-t)+1} N^{\theta^2-2t\theta+2t^2} \qquad \text{if } N \text{ is odd}; \tag{4.6}$$

- *if the braiding is as in* (c)*, then*

$$\dim \mathfrak{B}(V) = 2^{\theta(\theta-1)} 3^{\theta^2-2t\theta+2t^2}. \tag{4.7}$$

Here, $t = \theta - \sum_{k=1}^{j} (-1)^{j-k} i_k$.

Proof. It is clear that q should be a root of 1.

Now, we proceed to calculate $\dim \mathfrak{B}(V)$. From Corollary 4.2, we have to determined q_α for $\alpha \in \Delta_C$, and multiply their orders. As before, $\mathbf{u}_{ij} = \sum_{k=i}^{j} \mathbf{e}_k$, $\quad 1 \leq i \leq j \leq \theta$ and $\mathbf{v}_{ij} = 2\sum_{k=1}^{i} e_k + \sum_{k=i+1}^{j} e_k = 2e_{1,i} + e_{i+1,j}$, $\quad 1 \leq i < j$, so

$$\Delta(\mathfrak{B}(V)) = \{\mathbf{u}_{ij} : 1 \leq i \leq j \leq \theta\} \cup \{\mathbf{v}_{ij} : 1 \leq i < j \leq \theta\}.$$

We calculate $q_{\mathbf{u}_{ij}}$, $1 < i \leq j \leq \theta$ as above, because they correspond with a braiding of standard $A_{\theta-1}$ type, and

$$\begin{aligned} q_{\mathbf{v}_{ij}} &= \chi(\mathbf{v}_{ij}, \mathbf{v}_{ij}) = \chi(\mathbf{u}_{1i}, \mathbf{u}_{1i})^4 \chi(\mathbf{u}_{1i}, \mathbf{u}_{i+1,j})^2 \chi(\mathbf{u}_{i+1,j}, \mathbf{u}_{1i})^2 q_{\mathbf{u}_{i+1,j}} \\ &= q_{11}^4 q_{12}^2 q_{21}^2 \left(\prod_{k=2}^{i} q_{kk}^2 q_{k-1,k} q_{k-1,k} q_{k+1,k} q_{k+1,k} \right)^2 q_{\mathbf{u}_{i+1,j}} = q_{\mathbf{u}_{i+1,j}}, \end{aligned}$$

where we use that

- $q_{ij} q_{ji} = 1$ if $|i-j| > 1$,
- $q_{11}^4 q_{12}^2 q_{21}^2 = 1$, and
- $q_{kk}^2 q_{k-1,k} q_{k-1,k} q_{k+1,k} q_{k+1,k} = 1$ if $2 \leq k \leq \theta - 1$.

To calculate the other q_α's, we analyze each case:

(a) Note that $q_{\mathbf{e}_1} = \zeta$, $q_{\mathbf{e}_1+\mathbf{e}_2} = \zeta$, $q_{2\mathbf{e}_1+\mathbf{e}_2} = \zeta q^{-1}$, $q_{\mathbf{e}_2} = q$, so there are two possibilities: $\dim \mathfrak{B}(V) = 3^2 N^2$ if 3 divides N, and $\dim \mathfrak{B}(V) = 3^3 N^2$ if 3 does not divide N.

(b) We have that

$$q_{\mathbf{u}_{1k}} = q^{-1} q_{\mathbf{u}_{2k}} = \begin{cases} q^2 q^{-1} = q & \kappa_{2k} \text{ even}, \\ -q^{-1} & \kappa_{2k} \text{ odd}; \end{cases}$$

and also $q_{11} = q$. We have that κ_{2k} is even iff $j \in \{i_j + 1, \theta\}$, or $i \in \{i_{j-2} + 1, i_{j-1}\}$, and so on. Then there are

$$t = (\theta - i_j) + (i_{j-1} - i_{j-2}) + \ldots = \theta - \sum_{k=1}^{j} (-1)^{j-k} i_k$$

numbers (the corresponding with the number in the above Proposition) such that $\kappa_{i,\theta-1}$ is even. There are $2\left(\binom{t}{2} + \binom{\theta-t}{2}\right)$ roots such that $q_\alpha = q^2$, $2\left(\binom{\theta}{2} - \binom{t}{2} - \binom{\theta-t}{2}\right)$ roots such that $q_\alpha = -1$, $t+1$ roots such that $q_\alpha = q$ and $\theta - 1 - t$ roots such that $q_\alpha = -q^{-1}$.

Note that if $N = 2k$, then $\operatorname{ord}(-q^{-1}) = 2k$ and $\operatorname{ord}(q^2) = k$, so

$$\begin{aligned} \dim \mathfrak{B}(V) &= 2^{(\theta-1)\theta - t(t-1) - (\theta-t)(\theta-t-1)} k^{t(t-1) + (\theta-t)(\theta-t-1)} (2k)^\theta \\ &= 2^{2t(\theta-t)+\theta} k^{\theta^2 - 2t\theta + 2t^2}; \end{aligned}$$

if N is odd, then $\operatorname{ord}(-q^{-1}) = 2N$ and $\operatorname{ord}(q^2) = N$, so

$$\begin{aligned} \dim \mathfrak{B}(V) &= 2^{\theta(\theta-1) - t(t-1) - (\theta-t)(\theta-1-t)} N^{t(t-1) + (\theta-t)(\theta-1-t) + t + 1} \\ & \quad (2N)^{\theta-1-t} = 2^{(2t+1)(\theta-t)+1} N^{\theta^2 - 2t\theta + 2t^2}. \end{aligned}$$

(c) In a similar way,

$$q_{\mathbf{u}_{1i}} = (-\zeta^2) q_{\mathbf{u}_{2i}} = \begin{cases} (-\zeta^2)^2 = \zeta & \kappa_{2i} \text{ even}, \\ (-1)(-\zeta^2) = \zeta^2 & \kappa_{2i} \text{ odd}; \end{cases}$$

and also $q_{11} = \zeta$. There are $2\left(\binom{t}{2}+\binom{\theta-t}{2}\right)$ roots such that $q_\alpha = -\zeta^2$, $2\left(\binom{\theta}{2}-\binom{t}{2}-\binom{\theta-t}{2}\right)$ roots such that $q_\alpha=-1$, $t+1$ roots such that $q_\alpha=\zeta$ and $\theta-1-t$ roots such that $q_\alpha=\zeta^2$. As $\mathrm{ord}\zeta=\mathrm{ord}\zeta^2=3$ and $\mathrm{ord}(-\zeta^2)=6$, we have

$$\begin{aligned}\dim \mathfrak{B}(V) &= 2^{\theta(\theta-1)-t(t-1)-(\theta-t)(\theta-1-t)}6^{t(t-1)+(\theta-t)(\theta-1-t)}3^{\theta}\\ &= 2^{\theta(\theta-1)}3^{\theta^2-2t\theta+2t^2}.\end{aligned}$$

So, the proof is completed. □

Proposition 4.7. *Let V be a standard braided vector space of type G_2 as in Proposition 3.11. Then the associated Nichols algebra $\mathfrak{B}(V)$ is of finite dimension if and only if q is a root of unit of order $N \geq 4$. Then*

- *in case* (a) *of Proposition 3.11,*

$$\dim \mathfrak{B}(V) = N^6 \qquad \textit{if 3 does not divide } N,$$
$$\dim \mathfrak{B}(V) = 27k^6 \qquad \textit{if } N=3k;$$

- *in case* (b)*,* $\dim \mathfrak{B}(V) = 2^{12}$.

Proof. For (a) note that q is a root of 1, because x_1 has finite height, and

- $q_\alpha = q$ if $\alpha \in \{e_1, e_1+e_2, 2e_1+e_2\}$,
- $q_\alpha = q^3$ if $\alpha \in \{e_2, 3e_1+e_2, 3e_1+2e_2\}$,

so the dimension is $\dim \mathfrak{B}(V) = N^6$ if 3 does not divide N, and $\dim \mathfrak{B}(V) = 27k^6$ if $N=3k$. For (b) we calculate

type	q_{x_2}	$q_{x_1x_2}$	$q_{x_1^3x_2^2}$	$q_{x_1^2x_2}$	$q_{x_1^3x_2}$	q_{x_1}	$\dim \mathfrak{B}(V)$
$\overset{\zeta^2}{\circ}\overset{\zeta}{\text{———}}\overset{\zeta^{-1}}{\circ}$	8	4	2	8	2	4	2^{12}
$\overset{\zeta^2}{\circ}\overset{\zeta^3}{\text{———}}\overset{-1}{\circ}$	2	8	2	4	8	4	2^{12}
$\overset{\zeta}{\circ}\overset{\zeta^5}{\text{———}}\overset{-1}{\circ}$	2	4	8	4	2	8	2^{12}

so the proof is complete. □

5. Presentation by generators and relations of Nichols algebras of standard braided vector spaces

In this section we give a presentation by generators and relations of Nichols algebras of standard braided vector spaces. To do this, we give some technical results about relations and PBW-bases in Subsection 5.1; also we calculate the coproduct of some hyperwords in $T(V)$. In Subsections 5.2, 5.3 and 5.4 we express the braided commutator of two PBW-generators as combination of elements of the PBW-basis under some assumptions. Then, we obtain the desired presentation with a proof similar to the ones in [AD] and [AS5]. In Subsection 5.5 we solve the problem when the braiding is of Cartan type using the transformation in Subsection 2.3.

There is a procedure to describe a (non-minimal) set of relations for Nichols algebras of rank 2 in [H4, Th. 4].

5.1. **Some general relations.**

Let V be a standard braided vector space with connected Dynkin diagram. Let $x_1, \dots, x_\theta$ be an ordered basis of V, and $\{x_\alpha : \alpha \in \Delta^+(\mathfrak{B}(V))\}$ a set of PBW generators. Here, $x_\alpha \in \mathfrak{B}(V)$ is, by abuse of notation, the image by the canonical projection of $x_\alpha \in T(V)$, the hyperword corresponding to a Lyndon word l_α. We denote

$$q_\alpha := \chi(\alpha, \alpha), \qquad N_\alpha := \mathrm{ord} q_\alpha, \quad \alpha \in \Delta^+(\mathfrak{B}(V)).$$

Note that each x_α is homogeneous and has the same degree as l_α. Also,

$$x_\alpha \in T(V)^{\chi_\alpha}_{g_\alpha}, \tag{5.1}$$

where if $\alpha = b_1\mathbf{e}_1 + \cdots + b_\theta \mathbf{e}_\theta$, then $g_\alpha = g_1^{b_1} \dots g_\theta^{b_\theta}$, $\chi_\alpha = \chi_1^{b_1} \dots \chi_\theta^{b_\theta}$.

Proposition 5.1. *If the matrix of the braiding is symmetric, then the PBW basis is orthogonal with respect to the bilinear form in Proposition 1.4.*

Proof. We prove by induction on $k := \max\{\ell(u), \ell(v)\}$ that $(u|v) = 0$, where $u \neq v$ are products of PBW generators (we also allow powers greater than the corresponding heights). If $k = 1$, then $u = 1$ or x_i, $v = x_j$, for some $i, j \in \{1, \dots, \theta\}$, and $(x_i|x_j) = \delta_{ij}$.

Suppose it is valid when the length of both words is least than k, and let $u, v \in B_{I(V)}, u \neq v$ be hyperwords such that one (or both) has length k. If both are hyperletters, they have different degrees $\alpha \neq \beta \in \mathbb{Z}^\theta$, so $u = x_\alpha$, $v = x_\beta$, and $(x_\alpha|x_\beta) = 0$, since the homogeneous components are orthogonal for $(|)$.

Suppose that $u = x_\alpha$ and $v = x_{\beta_1}^{h_1} \dots x_{\beta_m}^{h_m}$, for some $x_{\beta_1} > \dots > x_{\beta_m}$. If they have different $\mathbb{Z}^\theta$-degree, they are orthogonal. Then, we assume that $\alpha = \sum_{j=1}^m h_m \beta_m$. By [B, Ch. VI, Prop. 19], we can reorder the β_i's, using h_i copies of β_i, in such form that each partial sum is a root. Using [R2, Prop. 21], the order induced by the Lyndon words l_α is convex, so $\beta_n < \alpha$. Using Lemma 1.9 and (1.8),

$$\begin{aligned}(u|v) =& (x_\alpha|w)(1|x_{\beta_m}) + (1|w)(x_{\alpha_n}|x_{\beta_m}) \\ &+ \sum_{l_1 \geq \cdots \geq l_p > \alpha, l_i \in L} (x_{l_1,\dots,l_p}|w)([l_1]_c \cdots [l_p]_c|x_{\beta_m})\end{aligned}$$

where $v = wx_{\beta_n}$. Note that $(1|x_{\beta_m}) = (1|w) = 0$. Also, $[l_1]_c \cdots [l_p]_c$ is a linear combination of greater hyperwords of the same degree and an element of $I(V)$. By inductive hypothesis and the fact that $I(V)$ is the radical of the bilinear form, $([l_1]_c \cdots [l_p]_c|x_{\beta_m}) = 0$.

Consider now

$$u = x_{\alpha_1}^{j_1} \dots x_{\alpha_n}^{j_n}, \ x_{\alpha_1} > \dots > x_{\alpha_n}, \quad v = x_{\beta_1}^{h_1} \dots x_{\beta_m}^{h_m}, \ x_{\beta_1} > \dots > x_{\beta_m},$$

and suppose that $x_{\alpha_n} \leq x_{\beta_m}$ (if not, use that the bilinear form is symmetric). Using Lemma 1.9 and (1.8),

$$(u|v) = (w|1)(x_{\alpha_n}|v) + \sum_{i=0}^{h_m} \binom{h_m}{i}_{q_{\beta_m}} (w|x_{\beta_1}^{h_1} \dots x_{\beta_{m-1}}^{h_{m-1}} x_{\beta_m}^{i})(x_{\alpha_n}|x_{\beta_m}^{h_m - i})$$
$$+ \sum_{\substack{l_1 \geq \dots \geq l_p > l, l_i \in L \\ 0 \leq j \leq m}} (w|x_{l_1,\dots,l_p}^{(j)})(x_{\alpha_n}|\,[l_1]_c \dots [l_p]_c\, [x_{\beta_m}]_c^j)$$

where $w = x_{\alpha_1}^{h_1} \dots x_{\alpha_m}^{h_m - 1}$. Note that for the first summand, $(w|1) = 0$. In the last sum, $(x_{\alpha_n}|\,[l_1]_c \dots [l_p]_c\, [x_{\beta_m}]_c^j) = 0$, because by the previous results, $[l_1]_c \dots [l_p]_c\, [x_{\beta_m}]_c^j$ is a combination of hyperwords of the PBW basis greater or equal than it and an element of $I(V)$, then we use induction hypothesis and the fact that $I(V)$ is the radical of this bilinear form. As also $x_{\alpha_n}, x_{\beta_m}^{h_m - i}$ are different elements of the PBW basis for $h_m - i \neq 1$, we have that

$$(u|v) = (h_m)_{q_{\beta_m}} (w|x_{\beta_1}^{h_1} \dots x_{\beta_{m-1}}^{h_{m-1}} x_{\beta_m}^{h_m - 1})(x_{\alpha_n}|x_{\beta_m}).$$

Then it is zero if $\alpha_n \neq \beta_m$, but also if $\alpha_n = \beta_m$, because in that case w, $x_{\beta_1}^{h_1} \dots x_{\beta_{m-1}}^{h_{m-1}} x_{\beta_m}^{h_m - 1}$ are different products of PBW generators, and we use induction hypothesis. □

Corollary 5.2. *If $\alpha \in \Delta^+(\mathfrak{B}(V))$, then*

$$x_\alpha^{N_\alpha} = 0. \tag{5.2}$$

Proof. Let (q_{ij}) be symmetric. If $u = x_{\alpha_1}^{j_1} \dots x_{\alpha_n}^{j_n}, \quad x_{\alpha_1} > \dots > x_{\alpha_n}$, then

$$(u|u) = \prod_{i=1}^{n} (j_i)_{q_{\alpha_i}}! (x_{\alpha_i}|x_{\alpha_i})^{j_i}, \tag{5.3}$$

where $(x_\alpha|x_\alpha) \neq 0$ for all $\alpha \in \Delta^+(\mathfrak{B}(V))$.

If we consider $u = x_\alpha^{N_\alpha}$, we have that $(u|v) = 0$, for each element v of the PBW basis, because they are ordered products of x_α different of u, and $(u|u) = 0$ since $q_\alpha \in \mathbb{G}_{N_\alpha}$. Also, $(I(V)|x_\alpha^{N_\alpha}) = 0$, because it is the radical of this bilinear form, so $(T(V)|x_\alpha^{N_\alpha}) = 0$, and then $x_\alpha^{N_\alpha} \in I(V)$. That is, we have $x_\alpha^{N_\alpha} = 0$ in $\mathfrak{B}(V)$.

For the general case, we recall that a diagonal braiding is twist equivalent to a braiding with a symmetric matrix, see [AS3, Theorem 4.5]. Also, there exists a linear isomorphism between the corresponding Nichols algebras. The corresponding x_α are related by a non-zero scalar, because they are an iteration of braided commutators between the hyperwords. □

We shall need some technical results about graded algebras intermediated between $T(V)$ and $\mathfrak{B}(V)$.

Lemma 5.3. *Let $i, \neq j \in \{1, \dots, \theta\}$. Let $\mathfrak{B}$ be a graded algebra provided with an inclusion of braided vector spaces $V \hookrightarrow \mathfrak{B}^1$. Assume that:*

- *there exist skew derivations D_i of $\mathfrak{B}$ as in Proposition 2.1;*
- $x_i^N = 0$ *if* $N := \operatorname{ord} q_{ii} < \min\{n \in \mathbb{N} : q_{ii}^n q_{ij} q_{ji} = 1\} - 1$.

For each $m \in \mathbb{N}$, $x_i^m x_j$ is a linear combination of greater hyperwords (for a fixed order such that $x_i < x_j$) if and only if

$$(\operatorname{ad} x_i)^{m_{ij}+1} x_j = 0, \quad i \neq j. \tag{5.4}$$

Proof. If $(ad_c x_i)^m x_j = 0$, there exist $a_k \in \mathsf{k}$ such that

$$0 = [x_i^m x_j]_c = (\operatorname{ad}_c x_i)^m x_j = x_i^m x_j + \sum_{k=0}^{m-1} a_k x_i^k x_j x_i^{m-k}.$$

Conversely, suppose that there exist $m \in \mathbb{N}$ such that $x_i^m x_j$ is a linear combination of greater hyperwords. Let

$$n = \min\{m \in \mathbb{N} : x_i^m x_j \text{ is a linear combination of greater hyperwords}\}.$$

If $x_i^n = 0$, then q_{ii} is a root of 1, because of the derivations. In this case, if N is the order of q_{ii}, then $x_i^N = 0$ and $x_i^{N-1} \neq 0$. Also, $(\operatorname{ad}_c x_i)^N x_j = 0$. Hence, we can assume $x_i^n \neq 0$ and $(n)_{q_{ii}}! \neq 0$.

Note that $[x_i^{n-k} x_j x_i^k]_c = [x_i^{n-k} x_j]_c x_i^k$. As $\mathfrak{B}$ is graded, $x_i^n x_j$ is a linear combination of $x_i^{n-k} x_j x_i^k$, $0 \leq k < n$ Hence, there exist $\alpha_k \in \mathsf{k}$ such that

$$[x_i^n x_j]_c = \sum_{k=1}^{n} \alpha_k \left[x_i^{n-k} x_j\right]_c x_i^k.$$

Applying D_i we obtain

$$0 = D_i([x_i^n x_j]_c) = \sum_{k=1}^{n} \alpha_k D_i\left(\left[x_i^{n-k} x_j\right]_c x_i^k\right) = \sum_{k=1}^{n} \alpha_k (k)_{q_{ii}} \left[x_i^{n-k} x_j\right]_c x_i^k.$$

By the hypothesis about n, $\alpha_1 = 0$. As $(n)_{q_{ii}}! \neq 0$, applying D_i several times we conclude that $\alpha_k = 0$ for $k = 2, \dots, n$. Then, $[x_i^n x_j]_c = 0$. □

Recall that 5.4 holds in $\mathfrak{B}(V)$, for $1 \leq i \neq j \leq \theta$

The second lemma is related to Dynkin diagrams of a standard braiding which have two consecutive simple edges.

Lemma 5.4. *Let $\mathfrak{B}$ be a graded algebra provided with an inclusion of braided vector spaces $V \hookrightarrow \mathfrak{B}^1$. Assume that:*

- *there exist skew derivations D_i in $\mathfrak{B}$ as in Proposition 2.1;*
- *there exist different $j, k, l \in \{1, \dots, \theta\}$ such that $m_{kj} = m_{kl} = 1$, $m_{jl} = 0$;*
- $(\operatorname{ad} x_k)^2 x_j = (\operatorname{ad} x_k)^2 x_l = (\operatorname{ad} x_j) x_l = 0$ *hold in $\mathfrak{B}$;*
- $x_k^2 = 0$ *if $q_{kk} q_{kj} q_{jk} \neq 1$ or $q_{kk} q_{kl} q_{lk} \neq 1$.*

(1) If we order the letters $x_1, \dots, x_\theta$ such that $x_j < x_k < x_l$, then $x_j x_k x_l x_k$ is a linear combination of greater words if and only if

$$[(\operatorname{ad} x_j)(\operatorname{ad} x_k) x_l, x_k]_c = 0. \tag{5.5}$$

(2) If V is standard and $q_{kk} \neq -1$, then (5.5) *holds in* $\mathfrak{B}$.

(3) If V is standard and $\dim \mathfrak{B}(V) < \infty$*, then* (5.5) *holds in* $\mathfrak{B} = \mathfrak{B}(V)$.

Proof. (1) ($\Leftarrow$) If (5.5) holds, then $x_j x_k x_l x_k$ is a linear combination of greater words, by Remark 1.7, and

$$[x_j x_k x_l x_k]_c = \left[[x_j x_k x_l]_c, x_k\right]_c = [(\operatorname{ad} x_j)(\operatorname{ad} x_k) x_l, x_k]_c .$$

($\Rightarrow$) If $x_j x_k x_l x_k$ is a linear combination of greater words, then the hyperword $[x_j x_k x_l x_k]_c$ is a linear combination of hyperwords corresponding to words greater than $x_j x_k x_l x_k$ (of the same degree, because $\mathfrak{B}$ is homogeneous); this follows by Remark 1.7. As $(\operatorname{ad} x_k)^2 x_j = (\operatorname{ad} x_k)^2 x_l = (\operatorname{ad} x_j) x_l = 0$, we do not consider hyperwords with $x_j x_k^2$, $x_k^2 x_l$ and $x_j x_l$ as factors of the corresponding words. Then, $[x_j x_k x_l x_k]_c$ is a linear combination of

$$\begin{aligned}
[x_k x_l x_k x_j]_c &= [x_k x_l]_c x_k x_j, & [x_l x_k x_j x_k]_c &= x_l x_k [x_j x_k]_c , \\
[x_k x_j x_k x_l]_c &= x_k [x_j x_k x_l]_c , & \left[x_l x_k^2 x_j\right]_c &= x_l x_k^2 x_j .
\end{aligned}$$

As $D_j([x_j x_k x_l x_k]_c) = D_j(x_k [x_j x_k x_l]_c) = D_j(x_l x_k [x_j x_k]_c) = 0$, in that linear combination there are no hyperwords ending in x_j; indeed,

$$D_j([x_k x_l]_c x_k x_j) = [x_k x_l]_c x_k, \quad D_j(x_l x_k^2 x_j) = x_l x_k^2,$$

and $[x_k x_l]_c x_k$, $x_l x_k^2$ are linearly independent. Therefore, there exist $\alpha, \beta \in \mathsf{k}$ such that

$$[x_j x_k x_l x_k]_c = \alpha x_l x_k [x_j x_k]_c + \beta x_k [x_j x_k x_l]_c .$$

Applying D_l, we have

$$0 = \alpha q_{kj} q_{kk} x_l [x_j x_k]_c + \alpha(1 - q_{kj} q_{jk}) x_l x_k x_j + \beta q_{kk} q_{kj} q_{kl} [x_j x_k x_l]_c .$$

Now, $x_l [x_j x_k]_c$, $x_l x_k x_j$ and $[x_j x_k x_l]_c$ are linearly independent by Lemma 2.7, so $\alpha = \beta = 0$.

(2) We assume that some quantum Serre relations hold in $\mathfrak{B}$; using them:

$$\begin{aligned}
x_j x_k x_l x_k &= q_{kl}^{-1}(1+q_{kk})^{-1} x_j x_k^2 x_l + q_{kk} q_{kj} (1+q_{kk})^{-1} x_j x_l x_k^2 \\
&= q_{kk}^{-1} q_{kj}^{-1} q_{kl}^{-1} x_k x_j x_k x_l + q_{kk}^{-1} q_{kj}^{-1} q_{kl}^{-1} (1+q_{kk})^{-1} x_k^2 x_j x_l \\
&\quad + q_{kk} q_{kl} q_{jk} (1+q_{kk})^{-1} x_l x_j x_k^2 .
\end{aligned}$$

It follows that $x_k x_j x_k x_l \notin G_I$, for an order such that $x_j < x_k < x_l$. Also, $x_j x_l x_k^2 \notin G_I$, since $(ad_c x_j) x_l = 0$, and (5.5) is valid by the previous item.

(3) If V is a standard braided vector space satisfying the above conditions, then we consider V_k as the braided vector space obtained transforming by s_k, then $\widetilde{m}_{jl} = 0$. Therefore, $\mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B}(V_k))$, so $s_k(\mathbf{e}_j + \mathbf{e}_l) = 2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B}(V))$. It follows that $x_j x_k x_l x_k$ is a linear combination of greater words, since it is a Lyndon word when we consider an order such that $x_j < x_k < x_l$. $\square$

We prove now two relations related to the double edge in a Dynkin diagram of standard braiding of type B_θ.

Lemma 5.5. *Let $\mathfrak{B}$ be a graded algebra provided with an inclusion of braided vector spaces $V \hookrightarrow \mathfrak{B}^1$. Assume that:*

- *there exist $j \neq k \in \{1, \dots, \theta\}$ such that $m_{kj} = 2, m_{jk} = 1$;*
- *there exist skew derivations as in Proposition 2.1;*
- *the following relations hold in $\mathfrak{B}$:*

$$(\operatorname{ad} x_k)^3 x_j = (\operatorname{ad} x_j)^2 x_k = 0; \tag{5.6}$$
$$x_k^3 = x_j^2 = 0, \qquad \text{if } q_{kk}^3 = q_{jj}^2 = 1.$$

(1) If we order the letters $x_1, \dots, x_\theta$ such that $x_k < x_j$, then $x_k^2 x_j x_k x_j$ is a linear combination of greater words if and only if

$$\left[(\operatorname{ad} x_k)^2 x_j, (\operatorname{ad} x_k) x_j\right]_c = 0. \tag{5.7}$$

(2) If V is standard, $q_{jj} \neq -1$ and $q_{kk}^2 q_{kj} q_{jk} = 1$, then (5.7) *holds in $\mathfrak{B}$.*
(3) If V is standard and $\dim \mathfrak{B}(V) < \infty$*, then* (5.7) *holds in $\mathfrak{B} = \mathfrak{B}(V)$.*

Proof. (1) ($\Leftarrow$) If (5.7) holds in $\mathfrak{B}$, then $x_k^2 x_j x_k x_j$ is a linear combination of greater words. It follows from (1.7), and

$$\left[x_k^2 x_j x_k x_j\right]_c = \left[\left[x_k^2 x_j\right]_c, [x_k x_j]_c\right]_c = \left[(\operatorname{ad} x_k)^2 x_j, (\operatorname{ad} x_k) x_j\right]_c.$$

($\Rightarrow$) If $x_k^2 x_j x_k x_j$ is a linear combination of greater words, then $[x_k^2 x_j x_k x_j]_c$ is a linear combination of hyperwords corresponding to words greater than $x_k^2 x_j x_k x_j$ (of the same degree, because $\mathfrak{B}$ is homogeneous).

First, there are not hyperwords whose corresponding words have factors $x_k^3 x_j$, $x_k x_j^2$, by 5.6. As $\left[x_k^2 x_j x_k x_j\right]_c \in \ker D_k$, and

$$D_k(x_j [x_k^2 x_j]_c x_k) = x_j [x_k^2 x_j]_c,$$
$$D_k([x_k x_j]_c^2 x_k) = [x_k x_j]_c^2,$$
$$D_k(x_j [x_k x_j]_c x_k^2) = (1 + q_{kk}) x_j [x_k x_j]_c x_k,$$

in that linear combination there are no hyperwords ending in x_k, except $x_j^2 x_k^3$ if $q_{kk} \in \mathbb{G}_3$. We consider $q_{jj} \neq -1$ if $q_{kk} \in \mathbb{G}_3$, since otherwise $x_j^2 x_k^3 = 0$ by hypothesis. Then, there exists $\alpha, \alpha' \in \mathsf{k}$ such that

$$\left[x_k^2 x_j x_k x_j\right]_c = \alpha \left[x_k x_j x_k^2 x_j\right]_c + \alpha' x_j^2 x_k^3 = \alpha \left[x_k x_j\right]_c \left[x_k^2 x_j\right]_c + \alpha' x_j^2 x_k^3.$$

We prove by direct calculation that $D_j(\left[x_k^2 x_j x_k x_j\right]_c) = 0$. Then, applying D_j to the previous equality,

$$\begin{aligned} 0 =& \alpha'(1 + q_{jj}) x_j x_k^3 + \chi(\mathbf{e}_k + \mathbf{e}_j, 2\mathbf{e}_k + \mathbf{e}_j) \alpha (\operatorname{ad} x_k)^2 (x_j) x_k \\ &+ (1 - q_{kj} q_{jk})(1 - q_{kk} q_{kj} q_{jk}) \alpha (\operatorname{ad} x_k)(x_j) x_k^2, \end{aligned}$$

where we use that $(\operatorname{ad} x_k)^3 (x_j) = 0$ and

$$x_k (\operatorname{ad} x_k)^m (x_j) = (\operatorname{ad} x_k)^{m+1} (x_j) + q_{kk}^m q_{kj} (\operatorname{ad} x_k)^m (x_j) x_k.$$

As $(1-q_{kj}q_{jk})(1-q_{kk}q_{kj}q_{jk}) \neq 0$ and $(\operatorname{ad} x_k)^2(x_j)x_k$, $(\operatorname{ad} x_k)(x_j)x_k^2$, $x_jx_k^3$ are linearly independent, it follows that $\alpha = \alpha' = 0$.

(2) Using $(\operatorname{ad} x_j)^2 x_k = 0$ in the first equality and $(\operatorname{ad} x_k)^3 x_j = 0$ in the last expression,

$$\begin{aligned} x_k^2x_jx_kx_j &= (1+q_{jj})^{-1}q_{jk}^{-1}x_k^2x_j^2x_k + (1+q_{jj})^{-1}q_{jk}q_{jj}x_k^3x_j^2 \\ &\in (3)_{q_{kk}}(1+q_{jj})^{-1}q_{kj}q_{jk}q_{jj}x_k^2x_jx_kx_j + \mathsf{k}\mathbb{X}_{>x_k^2x_jx_kx_j}. \end{aligned}$$

Suppose that $(3)_{q_{kk}}(1+q_{jj})^{-1}q_{kj}q_{jk}q_{jj} = 1$; that is, $(3)_{q_{kk}} = (1+q_{jj})$. Then, $q_{jj} = q_{kk} + q_{kk}^2$, so

$$1 = q_{jj}q_{kj}q_{jk} = q_{kk}q_{kj}q_{jk} + q_{kk}^2q_{kj}q_{jk} = q_{kk}q_{kj}q_{jk} + 1,$$

which is a contradiction since $q_{kk}q_{kj}q_{jk} \in \mathsf{k}^\times$. It follows that $x_k^2x_jx_kx_j$ is a linear combination of greater words, so (5.7) follows by previous item.

(3) If V is a standard braided vector space, and we consider V_j as the braided vector space obtained transforming by s_j, then $\widetilde{m}_{kj} = 2$. Therefore, $3\mathbf{e}_k + \mathbf{e}_j \notin \Delta^+(\mathfrak{B}(V_k))$, so $s_j(3\mathbf{e}_k + \mathbf{e}_j) = 3\mathbf{e}_k + 2\mathbf{e}_j \notin \Delta^+(\mathfrak{B}(V))$. As $x_k^2x_jx_kx_j$ is a Lyndon word of degree $3\mathbf{e}_k + 2\mathbf{e}_j$ if $x_k < x_j$, then it is a linear combination of greater words. ☐

Lemma 5.6. *Let $\mathfrak{B}$ be a graded algebra provided with an inclusion of braided vector spaces $V \hookrightarrow \mathfrak{B}^1$. Assume that*

- *there exist different $j,k,l \in \{1,\dots,\theta\}$ such that $m_{kj} = 2$, $m_{jk} = m_{jl} = m_{lj} = 1$, $m_{kl} = 0$;*
- *there exist skew derivations D_i in $\mathfrak{B}$ as in Proposition 2.1;*
- *the following relations hold in $\mathfrak{B}$:* (5.5), (5.7),

$$(\operatorname{ad} x_k)^3 x_j = (\operatorname{ad} x_j)^2 x_k (\operatorname{ad} x_j)^2 x_l = (\operatorname{ad} x_k) x_l = 0;$$

$$x_k^3 = x_j^2 = 0, \qquad \textit{if } q_{kk}^3 = q_{jj}^2 = 1. \tag{5.8}$$

(1) If we order the letters $x_1,\dots,x_\theta$ such that $x_k < x_j < x_l$, then $x_k^2x_jx_lx_kx_j$ is a linear combination of greater words if and only if

$$\left[(\operatorname{ad} x_k)^2(\operatorname{ad} x_j)x_l, (\operatorname{ad} x_k)x_j\right]_c = 0. \tag{5.9}$$

(2) If V is a standard braided vector space and $q_{kk} \notin \mathbb{G}_3$, $q_{jj} \neq -1$, then (5.9) *holds in $\mathfrak{B}$.*

(3) If V is standard and $\dim \mathfrak{B}(V) < \infty$, *then* (5.9) *holds in* $\mathfrak{B}(V)$.

Proof. (1) ($\Leftarrow$) As in last two Lemmata, if (5.9) is valid, then $x_k^2x_jx_lx_kx_j$ is a linear combination of greater words, by (1.7), and

$$[x_k^2x_jx_lx_kx_j]_c = \left[(\operatorname{ad} x_k)^2(\operatorname{ad} x_j)x_l, (\operatorname{ad} x_k)x_j\right]_c.$$

($\Rightarrow$) Suppose that $x_k^2x_jx_lx_kx_j$ is a linear combination of greater words. Then, $[x_k^2x_jx_lx_kx_j]_c$ is a linear combination of hyperwords corresponding to words greater than $x_k^2x_jx_lx_kx_j$ (of the same degree, because $\mathfrak{B}$ is homogeneous). We discard those words which have x_kx_l, $x_k^3x_j$, $x_kx_j^2$, $x_j^2x_l$, $x_kx_jx_lx_j$ and $x_k^2x_jx_kx_j$ by the hypothesis about $\mathfrak{B}$.

Also, as $D_k(\left[x_k^2x_jx_lx_kx_j\right]_c)=0$, the coefficients of those hyperwords corresponding to words ending in x_k are 0 as in Lemma 5.5, except $[x_jx_l]_c\,x_jx_k^3$, $x_lx_j^2x_k^3$, if $q_{kk}\in\mathbb{G}_3$. Then,

$$\begin{aligned}\left[x_k^2x_jx_lx_kx_j\right]_c =&\alpha\,[x_kx_j]_c\left[x_k^2x_jx_l\right]_c+\beta\,[x_kx_jx_l]_c\left[x_k^2x_j\right]_c\\ &+\gamma x_l\,[x_kx_j]_c\left[x_k^2x_j\right]_c+\mu\,[x_jx_l]_c\,x_jx_k^3+\nu x_lx_j^2x_k^3.\end{aligned}$$

By direct calculation,

$$D_j(\left[x_k^2x_jx_lx_kx_j\right]_c)=D_j(\left[x_k^2x_jx_l\right]_c)=D_j([x_kx_jx_l]_c)=0,$$

so applying D_j to the previous equality,

$$\begin{aligned}0=&\alpha q_{jk}^2q_{jj}q_{jl}x_j\left[x_k^2x_jx_l\right]_c+\beta(1-q_{kk}q_{kj}q_{jk})(1-q_{kk}^2q_{kj}q_{jk})\,[x_kx_jx_l]_c\,x_k^2\\ &+\gamma(1-q_{kk}q_{kj}q_{jk})(1-q_{kk}^2q_{kj}q_{jk})x_l\,[x_kx_j]_c\,x_k^2+\gamma q_{jk}^2q_{jj}x_lx_j\left[x_k^2x_j\right]_c\\ &+\mu\,[x_jx_l]_c\,x_k^3+\nu(1+q_{jj})x_lx_jx_k^3,\end{aligned}$$

Note that $\nu=0$ if $q_{jj}\neq-1$; otherwise, $x_j^2=0$ by hypothesis, so we can discard this last summand. The other hyperwords appearing in this expression are linearly independent, since the corresponding words are linearly independent by Lemma 2.7. Then, $\alpha=\beta=\gamma=\mu=0$.

(2) If $q_{kk}\notin\mathbb{G}_3$ and $q_{jj}\neq-1$, then $x_k^2x_jx_lx_kx_j$ is a linear combination of greater words, using the quantum Serre relations in a similar way that in Lemma 5.6, so we apply the previous item.

(3) If V is a standard braided vector space, and we consider V_k as the braided vector space obtained transforming by s_k, then $\widetilde{m}_{kj}=2$. Therefore, $\mathbf{e}_k+2\mathbf{e}_j+\mathbf{e}_l\notin\Delta^+(\mathfrak{B}(V_k))$ by Lemma 5.5, so $s_k(\mathbf{e}_k+2\mathbf{e}_j+\mathbf{e}_l)=3\mathbf{e}_k+2\mathbf{e}_j+\mathbf{e}_l\notin\Delta^+(\mathfrak{B}(V))$. As $x_k^2x_jx_lx_kx_j$ is a Lyndon word, it follows that it is a linear combination of greater words, and we apply (1). $\square$

We give now explicit formulas for the comultiplication of previous hyperwords.

Lemma 5.7. *Consider the structure of graded braided Hopf algebra of $T(V)$, given in subsection 2.1. Then, for all $k\neq j$,*

$$\begin{aligned}\Delta((\operatorname{ad}x_k)^{m_{kj}+1}x_j) &= (\operatorname{ad}x_k)^{m_{kj}+1}x_j\otimes1+1\otimes(\operatorname{ad}x_k)^{m_{kj}+1}x_j\\ &\quad+\prod_{1\leq t\leq m_{kj}}(1-q_{kk}^tq_{kj}q_{jk})x_k^{m_{ij}+1}\otimes x_j.\end{aligned}\tag{5.10}$$

Proof. By the definition of m_{kj} and (2.5), $F_k((\operatorname{ad} x_k)^{m_{kj}+1}x_j) = 0$. Also, $F_l((\operatorname{ad} x_k)^{m_{kj}+1}x_j)$ for $l \neq k$ by (2.6) and the properties of F_l, so we have

$$\Delta_{1,m_{kj}}((\operatorname{ad} x_k)^{m_{kj}+1}x_j) = \sum_{l=1}^{\theta} x_l \otimes F_l((\operatorname{ad} x_k)^{m_{kj}+1}x_j) = 0.$$

Now, $D_k(\left[x_k^i x_j\right]_c x_k^{s-i}) = 0$ from (2.3), and from (2.4)

$$D_j(\left[x_k^i x_j\right]_c x_k^{s-i}) = \prod_{1\leq t\leq m_{kj}} (1 - q_{kk}^t q_{kj} q_{jk}) x_k^{m_{ij}+1},$$

so we deduce that

$$\Delta_{m_{kj},1}((\operatorname{ad} x_k)^{m_{kj}+1}x_j) = \prod_{1\leq t\leq m_{kj}} (1 - q_{kk}^t q_{kj} q_{jk}) x_k^{m_{ij}+1} \otimes x_j.$$

As hyperwords form a basis of $T(V)$, we can express for each $1 < s < m_{kj}$,

$$\begin{aligned}\Delta_{m_{kj}+1-s,s}((\operatorname{ad} x_k)^{m_{kj}+1}x_j) = &\sum_{t=0}^{m_{kj}+1-s} \epsilon_{st} \left[x_k^t x_j\right]_c x_k^{m_{kj}+1-s-t} \otimes x_k^s \\ &+ \sum_{p=0}^{s} \rho_{sp} x_k^{m_{kj}+1-s} \otimes \left[x_k^{s-p} x_j\right]_c x_k^p,\end{aligned}$$

for some $\epsilon_{st}, \rho_{sp} \in \mathsf{k}$. Then, for each $0 \leq t \leq m_{kj} + 1 - s$,

$$\begin{aligned} 0 =& ((\operatorname{ad} x_k)^{m_{kj}+1}x_j | \left[x_k^t x_j\right]_c x_k^{m_{kj}+1-t} x_k^s) \\ =& \left(((\operatorname{ad} x_k)^{m_{kj}+1}x_j)_{(1)} | \left[x_k^t x_j\right]_c x_k^{m_{kj}+1-t-s} \right) \left(((\operatorname{ad} x_k)^{m_{kj}+1}x_j)_{(2)} | x_k^s \right) \\ =& \epsilon_{st} \left(\left[x_k^t x_j\right]_c x_k^{m_{kj}+1-t-s} | \left[x_k^t x_j\right]_c x_k^{m_{kj}+1-t-s} \right) (x_k^s | x_k^s) \\ =& \epsilon_{st} (m_{kj}+1-s-t)_{q_{kk}}! (s)_{q_{kk}}! (\left[x_k^t x_j\right]_c | \left[x_k^t x_j\right]_c), \end{aligned}$$

where we use that $(\operatorname{ad} x_k)^{m_{kj}+1}x_j \in I(V)$ for the first equality, (1.8) for the second, (1.10) and the orthogonality between increasing products of hyperwords for the third, and (5.3) for the last. As

$$(m_{kj}+1-s-t)_{q_{kk}}! (s)_{q_{kk}}! (\left[x_k^t x_j\right]_c | \left[x_k^t x_j\right]_c) \neq 0,$$

we conclude that ϵ_{st} for all $0 \leq t \leq m_{kj} + 1 - s$. In a similar way, $\rho_{sp} = 0$ for all $0 \leq p \leq s$, so we obtain (5.10). □

Lemma 5.8. *Let $\mathfrak{B}$ be a braided graded Hopf algebra provided with an inclusion of braided vector spaces $V \hookrightarrow \mathcal{P}(\mathfrak{B})$. Assume that*

- *there exist $1 \leq j \neq k \neq l \leq \theta$ such that $m_{kj} = m_{kl} = 1$, $m_{jl} = 0$;*
- *the following relations hold in $\mathfrak{B}$:*

$$(\operatorname{ad} x_k)^2 x_j = (\operatorname{ad} x_k)^2 x_l = (\operatorname{ad} x_j) x_l = 0;$$

$$x_k^2 = 0 \text{ if } q_{kk}q_{kj}q_{jk} \neq 1 \text{ or } q_{kk}q_{kl}q_{lk} \neq 1.$$

Then, $\quad u := [(\operatorname{ad} x_j)(\operatorname{ad} x_k)x_l, x_k]_c \in \mathcal{P}(\mathfrak{B})$.

Proof. From (2.3), $D_j(u)=0$. Also, $D_k\left((\operatorname{ad}x_j)(\operatorname{ad}x_k)x_l\right)=0$, so

$$D_k(u)=\left(1-q_{kk}^2q_{jk}q_{kj}q_{kl}q_{lk}\right)(\operatorname{ad}x_j)(\operatorname{ad}x_k)x_l=0.$$

From (2.4) and the properties of D_l we have

$$\begin{aligned}D_l(u)=&q_{lk}(1-q_{kl}q_{lk})[x_jx_k]_cx_k-q_{jk}q_{kk}q_{lk}(1-q_{kl}q_{lk})x_k[x_jx_k]_c\\ =&q_{lk}(1-q_{lk}q_{kl})[[x_jx_k]_c,x_k]_c=0.\end{aligned}$$

Then, $\Delta_{31}(u)=0$. Now, from (2.6) and the properties of F_k,F_l we have $F_k(u)=F_l(u)=0$. Using (2.5), we have

$$\begin{aligned}F_j(u)=&(1-q_{jk}q_{kj})[x_kx_l]_cx_k-q_{jk}q_{kk}q_{lk}q_{kj}(1-q_{jk}q_{kj})x_k[x_kx_l]_c\\ =&(1-q_{lk}q_{kl})(1-q_{kj}q_{jk}q_{kk}^2q_{lk}q_{jk})[x_kx_l]_cx_k=0.\end{aligned}$$

Then, we also have $\Delta_{13}(u)=0$.

Also, we have

$$\Delta(u)=\Delta((\operatorname{ad}x_j)(\operatorname{ad}x_k)x_l)\Delta(x_k)-q_{\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_j,\mathbf{e}_j}\Delta(x_k)\Delta((\operatorname{ad}x_j)(\operatorname{ad}x_k)x_l),$$

and looking at the terms in $\mathfrak{B}^2\otimes\mathfrak{B}^2$,

$$\begin{aligned}\Delta_{2,2}(u)=&(1-q_{lk}q_{kl})[x_jx_k]_c\otimes\left(x_lx_k-q_{kj}q_{jk}q_{kk}^2q_{lk}x_kx_l\right)\\ &+(1-q_{kj}q_{jk})q_{lk}q_{kk}\left(x_jx_k-q_{jk}x_kx_j\right)\otimes[x_kx_l]_c\\ =&\left(1-q_{kj}q_{jk}-\left(1-q_{lk}q_{kl}\right)q_{kk}q_{jk}q_{kj}\right)q_{lk}q_{kk}[x_jx_k]_c\otimes[x_kx_l]_c.\end{aligned}$$

Now, we calculate

$$\begin{aligned}1-q_{kj}q_{jk}-&(1-q_{lk}q_{kl})q_{kk}q_{jk}q_{kj}=1-q_{kj}q_{jk}-q_{kk}q_{jk}q_{kj}+q_{kk}^{-1}\\ &=q_{kk}^{-1}(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})=0,\end{aligned}$$

so $u\in\mathcal{P}(\mathfrak{B})$. □

Lemma 5.9. *Let $\mathfrak{B}$ be a braided graded Hopf algebra provided with an inclusion of braided vector spaces $V\hookrightarrow\mathcal{P}(\mathfrak{B})$. Assume that*

- *there exist $1\le k\neq j\le\theta$ such that $m_{kj}=2, m_{jk}=1$;*
- *the following relations hold in $\mathfrak{B}$:*
  - $(\operatorname{ad}x_s)^{m_{st}+1}x_t=0$, *for all $1\le s\neq t\le\theta$;*
  - $x_s^{m_{st}+1}=0$ *for each s such that $q_{ss}^{m_{st}}q_{st}q_{ts}\neq1$, for some $t\neq s$.*

(a) *If $v:=\left[(\operatorname{ad}x_k)^2x_j,(\operatorname{ad}x_k)x_j\right]_c$, then there exists $b\in\mathsf{k}$ such that*

$$\Delta(v)=v\otimes1+1\otimes v+b(1-q_{kk}^2q_{kj}^2q_{jk}^2q_{jj})x_k^3\otimes x_j^2. \tag{5.11}$$

(b) *Assume that there exists $l\neq j,k$ such that $m_{jl}=m_{lj}=1$, $m_{kl}=m_{lk}=0$, and that (5.5) is valid in $\mathfrak{B}$. Call*

$$w:=\left[(\operatorname{ad}x_k)^2(\operatorname{ad}x_j)x_l,(\operatorname{ad}x_k)x_j\right]_c,$$

then there exist constants $b_1,b_2\in\mathsf{k}$, such that

$$\begin{aligned}\Delta(w)=&\ w\otimes1+1\otimes w+b_1v\otimes x_l\\ &+b_2(1-q_{kk}^2q_{kj}q_{jk})x_k^3\otimes\left((\operatorname{ad}x_j)x_l\right)x_j.\end{aligned} \tag{5.12}$$

Proof. (a) Note that $F_j(v) = 0$, since v is a braided commutator of two elements in $\ker F_j$. Also, using (1.4),

$$[(\operatorname{ad} x_k)^2 x_j, x_j]_c = q_{kj}(q_{jj} - q_{kk})[x_k x_j]_c^2,$$

so we calculate

$$\begin{aligned} F_k(v) =&(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})[x_kx_j]_c^2 + q_{kk}^2 q_{jk}(1-q_{kj}q_{jk})[x_k^2x_j]_c x_j \\ &- q_{kk}^2 q_{kj}^2 q_{jk} q_{jj}(1-q_{kj}q_{jk}) x_j [x_k^2 x_j]_c \\ &- q_{kk}^3 q_{kj}^2 q_{jk}^2 q_{jj}(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})[x_kx_j]_c^2 \\ =&(q_{kk}^2 q_{jk} q_{kj}(1-q_{kj}q_{jk})(q_{jj}-q_{kk}) + (1+q_{kk}) \\ &(1-q_{kk}q_{kj}q_{jk})(1-q_{kk}^3 q_{kj}^2 q_{jk}^2 q_{jj})[x_kx_j]_c^2 = 0 \end{aligned}$$

since the coefficient of $[x_kx_j]_c^2$ is zero for each possible braiding. Then,

$$\Delta_{1,4}(v) = x_k \otimes F_k(v) = 0.$$

Also, $D_k(v) = 0$, and we calculate

$$\begin{aligned} D_j(v) =&(1-q_{kj}q_{jk})\big([x_k^2x_j]x_k + (1-q_{kk}q_{kj}q_{jk})q_{jk}q_{jj}x_k^2[x_kx_j]_c \\ &-q_{kk}^2 q_{kj}^2 q_{jk} q_{jj}(1-q_{kk}q_{kj}q_{jk})[x_kx_j]_c x_k^2 - q_{kk}^2 q_{kj}^2 q_{jk}^3 q_{jj}^2 x_k [x_k^2 x_j]\big) \\ =& \big(1 + (1+q_{kk})(1-q_{kk}q_{kj}q_{jk})q_{kk}q_{kj}q_{jk}q_{jj} - q_{kk}^4 q_{kj}^3 q_{jk}^3 q_{jj}^2\big) \\ &(1-q_{kj}q_{jk})[x_k^2x_j]x_k, \end{aligned}$$

where we reorder the hyperwords and use that $(\operatorname{ad} x_k)^3 x_j = 0$; also,

$$1 + (1+q_{kk})(1-q_{kk}q_{kj}q_{jk})q_{kk}q_{kj}q_{jk}q_{jj} - q_{kk}^4 q_{kj}^3 q_{jk}^3 q_{jj}^2 = 0, \tag{5.13}$$

by calculation for each possible braiding. Then,

$$\Delta_{4,1}(v) = D_j(v) \otimes x_j = 0.$$

To finish, we use that

$$\begin{aligned} \Delta(v) =&\Delta((\operatorname{ad}_c x_k)^2 x_j)\Delta((\operatorname{ad}_c x_k)x_j) \\ &- \chi(2e_k + e_j, e_k + e_j)\Delta((\operatorname{ad}_c x_k)x_j)\Delta((\operatorname{ad}_c x_k)^2 x_j). \end{aligned}$$

Looking at the terms in $\mathfrak{B}^3 \otimes \mathfrak{B}^2$ and $\mathfrak{B}^2 \otimes \mathfrak{B}^3$, and using the definition of braided commutator, we obtain

$$\begin{aligned} \Delta_{32}(v) =&(1-q_{kk}^4 q_{kj}^3 q_{jk}^3 q_{jj}^2)[x_k^2x_j]_c \otimes [x_kx_j]_c + (1+q_{kk})(1-q_{kk}q_{kj}q_{jk}) \\ &q_{kk}q_{kj}q_{jk}q_{jj}\left(x_k\,[x_kx_j]_c - q_{kk}q_{kj}\,[x_kx_j]_c\, x_k\right) \otimes [x_kx_j]_c \\ &+ 1 - q_{kj}q_{jk})^2(1-q_{kk}^2 q_{kj} q_{jk})(1-q_{kk}^2 q_{kj}^2 q_{jk}^2 q_{jj}) x_k^3 \otimes x_j^2 \\ =& \big(1 + (1+q_{kk})(1-q_{kk}q_{kj}q_{jk})q_{kk}q_{kj}q_{jk}q_{jj} - q_{kk}^4 q_{kj}^3 q_{jk}^3 q_{jj}^2\big)\,[x_k^2x_j]_c \\ &\otimes [x_kx_j]_c + b_1(1-q_{kk}^2 q_{kj}^2 q_{jk}^2 q_{jj}) x_k^3 \otimes x_j^2. \end{aligned}$$

Also,

$$\begin{aligned}\Delta_{23}(v) =&(1-q_{kk}q_{kj}q_{jk})(1-q_{kj}q_{jk})x_k^2\otimes\big((1+q_{kk})q_{kk}q_{jk}\left[x_kx_j\right]_c x_j\\ &-(1+q_{kk})q_{kk}^2q_{kj}^2q_{jk}^2q_{jj}x_j\left[x_kx_j\right]_c + x_j\left[x_kx_j\right]_c\\ &-q_{kk}^4q_{kj}^2q_{jk}^3q_{jj}\left[x_kx_j\right]_c x_j\big)\\ =&\big(1-q_{kk}^4q_{kj}^3q_{jk}^3q_{jj}^2+(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})q_{kk}q_{kj}q_{jk}q_{jj}\big)\\ &(1-q_{kk}q_{kj}q_{jk})(1-q_{kj}q_{jk})x_k^2\otimes x_j\left[x_kx_j\right]_c.\end{aligned}$$

Using (5.13), we obtain (5.11).

(b) We call $y=(\operatorname{ad}x_k)^2(\operatorname{ad}x_j)x_l,\ z=(\operatorname{ad}x_k)x_j$. Observe that $\Delta(w)=\Delta(y)\Delta(z)-\chi(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l,\mathbf{e}_k+\mathbf{e}_j)\Delta(z)\Delta(y)$, and

$$\begin{aligned}\Delta(y)=&\,y\otimes 1+(1-q_{jl}q_{lj})(\operatorname{ad}x_k)^2x_j\otimes x_l\\ &+(1-q_{kj}q_{jk})(1-q_{kk}q_{kj}q_{jk})x_k^2\otimes(\operatorname{ad}x_j)x_l\\ &+(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})x_k\otimes(\operatorname{ad}x_k)(\operatorname{ad}x_j)x_l+1\otimes y,\\ \Delta(z)=&\,z\otimes 1+(1-q_{kj}q_{jk})x_k\otimes x_j+1\otimes z.\end{aligned}$$

From (2.3), $D_k(w)=0$, and from (2.4),

$$\begin{aligned}D_l(w)&=(1-q_{lj}q_{jl})q_{lk}q_{lj}\left[(\operatorname{ad}x_k)^2x_j,(\operatorname{ad}x_k)x_j\right]_c,\\ D_j(w)&=-(1-q_{kj}q_{jk})q_{kk}^{-2}q_{kj}^{-1}q_{kl}^{-1}(\operatorname{ad}x_k)^3(\operatorname{ad}x_j)x_l\\ &=-(1-q_{kj}q_{jk})q_{kk}^{-2}q_{kj}^{-1}q_{kl}^{-1}[(\operatorname{ad}x_k)^3x_j,x_l]_c=0.\end{aligned}$$

where we use that $\left[x_k,x_l\right]_c=0$ and (1.4) for the last equality. It follows that

$$\Delta_{51}(w)=(1-q_{lj}q_{jl})q_{lk}q_{lj}\left[(\operatorname{ad}x_k)^2x_j,(\operatorname{ad}x_k)x_j\right]_c\otimes x_l.$$

Also, $F_j(z)=F_j(y)=F_l(z)=F_l(y)=0$ by (2.6) and the properties of these skew derivations, so $F_j(w)=F_l(w)=0$. We calculate

$$\begin{aligned}F_k(w)=&(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})[x_kx_jx_l]_c[x_kx_j]_c+q_{kk}^2q_{jk}q_{lk}(1-q_{kj}q_{jk})\\ &[x_k^2x_jx_l]_cx_j-\chi(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l,\mathbf{e}_k+\mathbf{e}_j)\big((1-q_{kj}q_{jk})x_j[x_k^2x_jx_l]_c\\ &+(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})q_{kk}q_{jk}[x_kx_jx_l]_c[x_kx_j]_c)\\ =&q_{kk}^2q_{jk}q_{lk}(1-q_{kj}q_{jk})\left[[x_k^2x_jx_l]_c,x_j\right]_c-(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})\\ &q_{kk}^3q_{kj}^2q_{jk}^2q_{jj}q_{lj}q_{lk}\left[[x_kx_j]_c,[x_kx_jx_l]_c\right]_c\\ =&q_{kk}^2q_{kj}q_{jk}q_{jj}q_{lj}q_{lk}\left(1-q_{kj}q_{jk}-(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})q_{kk}q_{kj}q_{jk}\right)\\ &[[x_kx_j]_c,[x_kx_jx_l]_c]_c=0\end{aligned}$$

where we use (1.4) and (5.5) in the third equality, and calculate that

$$1-q_{kj}q_{jk}-(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})q_{kk}q_{kj}q_{jk}=0, \tag{5.14}$$

for each possible standard braiding. It follows that $\Delta_{15}(w)=0$.

We calculate each of other terms of $\Delta(w)$ by direct calculation. First,

$$\begin{aligned}\Delta_{42}(w) =& \left(1-\chi(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l,\mathbf{e}_k+\mathbf{e}_j)\chi(\mathbf{e}_k+\mathbf{e}_j,2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l)\right) y\otimes z\\ &+(1-q_{kj}q_{jk})(1-q_{lj}q_{jl})\big(q_{lk}[x_k^2x_j]_cx_k\otimes x_lx_j\\ &\quad -\chi(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l,\mathbf{e}_k+\mathbf{e}_j)q_{jk}^2q_{jj}x_k[x_k^2x_j]_c\otimes x_jx_l\big)\\ &+(1-q_{kj}q_{jk})(1-q_{kk}q_{kj}q_{jk})\big(\chi(\mathbf{e}_j+\mathbf{e}_l,\mathbf{e}_k+\mathbf{e}_j)x_k^2z\\ &\quad -\chi(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l,\mathbf{e}_k+\mathbf{e}_j)zx_k^2\big)\otimes[x_jx_l]_c\\ =&(1-q_{kj}q_{jk})q_{lk}\left(1-q_{jk}q_{kj}+(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})q_{kk}q_{kj}q_{jk}\right)\\ &[x_k^2x_j]_cx_k\otimes[x_jx_l]_c=0.\end{aligned}$$

In a similar way we calculate

$$\begin{aligned}\Delta_{33}(w) =&(1-q_{lj}q_{jl})[x_k^2x_j]\otimes\left(x_lz-\chi(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l,\mathbf{e}_k+\mathbf{e}_j)\right.\\ &\left.\chi(\mathbf{e}_k+\mathbf{e}_j,\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l)zx_l\right)+(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})\\ &\chi(\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l,\mathbf{e}_k+\mathbf{e}_j)\left(x_kz-q_{kk}q_{kj}zx_k\right)\otimes[x_kx_jx_l]_c\\ &+(1-q_{kk}q_{kj}q_{jk})(1-q_{kj}q_{jk})^2x_k^3\otimes\left(\chi(\mathbf{e}_j+\mathbf{e}_l,\mathbf{e}_k)[x_jx_l]_cx_j\right.\\ &\left.-\chi(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l,\mathbf{e}_k+\mathbf{e}_j)\chi(\mathbf{e}_j,2\mathbf{e}_k)x_j[x_jx_l]_c\right)\\ =&\left((1+q_{kk})(1-q_{kk}q_{kj}q_{jk})-q_{kk}q_{kj}q_{jk}q_{jj}(1-q_{lj}q_{jl})\right)\\ &\chi(\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l,\mathbf{e}_k+\mathbf{e}_j)[x_k^2x_j]_c\otimes[x_kx_jx_l]_c\\ &+(1-q_{kk}q_{kj}q_{jk})(1-q_{kj}q_{jk})^2(1-q_{kk}^2q_{kj}q_{jk})x_k^3\otimes[x_jx_l]_cx_j,\end{aligned}$$

and the coefficient of $[x_k^2x_j]_c\otimes[x_kx_jx_l]_c$ is zero (we calculate it for each possible standard braiding). Also,

$$\begin{aligned}\Delta_{24}(w) =&(1-q_{kk}q_{kj}q_{jk})(1-q_{kj}q_{jk})x_k^2\otimes\left((1+q_{kk})\chi(\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l,\mathbf{e}_k)\right.\\ &[x_kx_jx_l]_cx_j-(1+q_{kk})\chi(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l,\mathbf{e}_k+\mathbf{e}_j)q_{jk}x_j[x_kx_jx_l]_c\\ &\left.-\chi(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l,\mathbf{e}_k+\mathbf{e}_j)\chi(\mathbf{e}_k+\mathbf{e}_j,2\mathbf{e}_k)\left[[x_kx_j]_c,[x_jx_l]_c\right]_c\right)\\ =&(1-q_{kk}q_{kj}q_{jk})(1-q_{kj}q_{jk})\chi(\mathbf{e}_j+\mathbf{e}_l,\mathbf{e}_k+\mathbf{e}_j)q_{kj}\\ &\left(q_{kk}(1-q_{kk}q_{kj}q_{jk})-q_{jj}(1-q_{jl}q_{lj})\right)x_k^2\otimes x_j[x_kx_jx_l]_c=0\end{aligned}$$

From the above calculations, we obtain (5.12). $\square$

5.2. **Presentation when the type is A_θ.**

In this subsection we shall consider V a standard braided vector space of type A_θ, and $\mathfrak{B}$ a $\mathbb{Z}^\theta$-graded algebra, provided with an inclusion of vector spaces $V\hookrightarrow\mathfrak{B}^1=\oplus_{1\le j\le\theta}\mathfrak{B}^{\mathbf{e}_j}$. We can extend the braiding to $\mathfrak{B}$ by

$$c(u\otimes v)=\chi(\alpha,\beta)v\otimes u,\quad u\in\mathfrak{B}^\alpha,v\in\mathfrak{B}^\beta,\ \alpha,\beta\in\mathbb{N}^\theta.$$

We assume that

$$\begin{aligned} x_i^2 &= 0 && \text{if } q_{ii} = -1, \\ \operatorname{ad}_c x_i(x_j) &= 0 && \text{if } |j-i| > 1, \\ (\operatorname{ad}_c x_i)^2(x_j)_c &= 0 && \text{if } |j-i| = 1, \\ [(\operatorname{ad}_c x_i)(ad_c x_{i+1})x_{i+2}, x_{i+1}]_c &= 0 && 2 \leq i \leq \theta - 1, \end{aligned}$$

are valid on $\mathfrak{B}$. Using the same notation as in Subsection 4.2,

$$x_{\mathbf{e}_i} = x_i, \qquad x_{\mathbf{u}_{i,j}} := \left[x_i, x_{\mathbf{u}_{i+1,j}}\right]_c \quad (i < j).$$

Lemma 5.10. *Let $1 \leq i \leq j < p \leq r \leq \theta$. The following relations hold in $\mathfrak{B}$:*

$$\left[x_{\mathbf{u}_{ij}}, x_{\mathbf{u}_{pr}}\right]_c = 0, \quad p - j \geq 2; \tag{5.15}$$

$$\left[x_{\mathbf{u}_{ij}}, x_{\mathbf{u}_{j+1,r}}\right]_c = x_{\mathbf{u}_{ir}}. \tag{5.16}$$

Proof. Note that $x_{\mathbf{u}_{pr}}$ belongs to the subalgebra generated by $x_p, \dots, x_r$, and $\left[x_{\mathbf{u}_{ij}}, x_s\right]_c = 0$, for each $p \leq s \leq r$. Then, (5.15) is deduced from this fact.

We prove (5.16) by induction on $j - i$: if $i = j$, it is exactly the definition of $x_{\mathbf{u}_{ir}}$. To prove the inductive step, we use the inductive hypothesis, (5.15) and (1.4) (the braided Jacobi identity) to obtain

$$\begin{aligned} \left[x_{\mathbf{u}_{i,j+1}}, x_{\mathbf{u}_{j+2,r}}\right]_c &= \left[\left[x_{\mathbf{u}_{ij}}, x_{i+1}\right]_c, x_{\mathbf{u}_{j+2,r}}\right]_c = \left[x_{\mathbf{u}_{ij}}, \left[x_{i+1}, x_{\mathbf{u}_{j+2,r}}\right]_c\right]_c \\ &= \left[x_{\mathbf{u}_{ij}}, x_{\mathbf{u}_{j+1,r}}\right]_c = x_{\mathbf{u}_{ir}}, \end{aligned}$$

and (5.16) is also proved. □

Lemma 5.11. *If $i < p \leq r < j$, the following relation holds in $\mathfrak{B}$:*

$$\left[x_{\mathbf{u}_{ij}}, x_{\mathbf{u}_{pr}}\right]_c = 0. \tag{5.17}$$

Proof. When $p = r = j - 1$ and $i = j - 2$, note that this is exactly

$$[(\operatorname{ad}_c x_i)(ad_c x_{i+1})x_{i+2}, x_{i+1}]_c = 0.$$

Then, we have by (1.4)

$$[x_{\mathbf{u}_{i-1,j}}, x_{j-1}]_c = [[x_{i-1}, x_{\mathbf{u}_{i,j}}]_c, x_{j-1}]_c = [x_{i-1}, [x_{\mathbf{u}_{i,j}}, x_{j-1}]_c]_c.$$

We assume that $j - i > 2$, so $[x_{i-1}, x_{j-1}]_c = 0$ by hypothesis on $\mathfrak{B}$. Then, we prove the case $p = r = j - 1$ by induction on $p - i$.

Using (1.4) and (5.16), we also have

$$\begin{aligned} [x_{\mathbf{u}_{i,j+1}}, x_p]_c &= [[x_{\mathbf{u}_{i,j}}, x_{j+1}]_c, x_p]_c = [x_{\mathbf{u}_{i,j}}, [x_{j+1}, x_p]_c]_c \\ &\quad + q_{j+1,p}[x_{\mathbf{u}_{i,j}}, x_{j-1}]_c x_{j+1} - \chi(\mathbf{u}_{i,j}, \mathbf{e}_{j+1}) x_{j+1}[x_{\mathbf{u}_{i,j}}, x_{j-1}]_c, \end{aligned}$$

so using that $[x_{j+1}, x_p]_c = 0$ if $j > p$, by induction on $j - p$ we prove (5.17) for the case $p = r$.

For the general case, we use (1.4) one more time as follows

$$[x_{\mathbf{u}_{i,j}}, x_{\mathbf{u}_{p,r+1}}]_c = [x_{\mathbf{u}_{i,j}}, [x_{\mathbf{u}_{pr}}, x_{r+1}]_c]_c = [[x_{\mathbf{u}_{i,j}}, x_{\mathbf{u}_{pr}}]_c, x_{r+1}]_c$$
$$- \chi(\mathbf{u}_{pr}, \mathbf{e}_{r+1})[x_{\mathbf{u}_{ij}}, x_{r+1}]_c x_{\mathbf{u}_{pr}} + \chi(\mathbf{u}_{ij}, \mathbf{u}_{pr}) x_{\mathbf{u}_{pr}} [x_{\mathbf{u}_{ij}}, x_{r+1}]_c,$$

and we prove (5.17) by induction on $r - p$. □

Lemma 5.12. *The following relations hold in $\mathfrak{B}$:*

$$\left[x_{\mathbf{u}_{ij}}, x_{\mathbf{u}_{ip}}\right]_c = 0, \quad i \le j < p; \tag{5.18}$$
$$\left[x_{\mathbf{u}_{ij}}, x_{\mathbf{u}_{pj}}\right]_c = 0, \quad i < p \le j. \tag{5.19}$$

Proof. To prove (5.18), note that if $i = j = p - 1$, we have

$$\left[x_{\mathbf{u}_{ii}}, x_{\mathbf{u}_{i,i+1}}\right]_c = [x_i, [x_i, x_{i+1}]_c]_c = (\operatorname{ad} x_i)^2 x_{i+1} = 0.$$

As $[x_i, x_{\mathbf{u}_{i+2,p}}]_c = 0$ for each $p > i + 1$ by (5.15), we use (1.4), the previous case and (5.16) to obtain

$$\left[x_{\mathbf{u}_{ii}}, x_{\mathbf{u}_{ip}}\right]_c = \left[x_{\mathbf{u}_{ii}}, [x_{\mathbf{u}_{i,i+1}}, x_{\mathbf{u}_{i+2,p}}]_c\right]_c = 0.$$

Now, if $i < j < p$, from (5.15) and the relations between the q_{st} we obtain

$$\left[x_{\mathbf{u}_{i+1,j}}, x_{\mathbf{u}_{ip}}\right]_c = -\chi(\mathbf{u}_{ip}, \mathbf{u}_{i+1,j}) \left[x_{\mathbf{u}_{ip}}, x_{\mathbf{u}_{i+1,j}}\right]_c = 0.$$

Using (1.4) and the previous case we conclude

$$\left[x_{\mathbf{u}_{ij}}, x_{\mathbf{u}_{ip}}\right]_c = \left[\left[x_{\mathbf{u}_{ii}}, x_{\mathbf{u}_{i+1,j}}\right]_c, x_{\mathbf{u}_{ip}}\right]_c = 0.$$

The proof of (5.19) is analogous. □

Lemma 5.13. *If $i < p \le r < j$, the following relation holds in $\mathfrak{B}$:*

$$\left[x_{\mathbf{u}_{ir}}, x_{\mathbf{u}_{pj}}\right]_c = \chi(\mathbf{u}_{ir}, \mathbf{u}_{pr}) \left(1 - q_{r,r+1} q_{r+1,r}\right) x_{\mathbf{u}_{pr}} x_{\mathbf{u}_{ij}}. \tag{5.20}$$

Proof. We calculate

$$\begin{aligned}
\left[x_{\mathbf{u}_{ir}}, x_{\mathbf{u}_{pj}}\right]_c &= \left[x_{\mathbf{u}_{ir}}, \left[x_{\mathbf{u}_{pr}}, x_{\mathbf{u}_{r+1,j}}\right]_c\right]_c \\
&= \chi(\mathbf{u}_{ir}, \mathbf{u}_{pr}) x_{\mathbf{u}_{pr}} x_{\mathbf{u}_{ij}} - \chi(\mathbf{u}_{pr}, \mathbf{u}_{r+1,j}) x_{\mathbf{u}_{ij}} x_{\mathbf{u}_{pr}} \\
&= \left(\chi(\mathbf{u}_{ir}, \mathbf{u}_{pr}) - \chi(\mathbf{u}_{ij}, \mathbf{u}_{pr}) \chi(\mathbf{u}_{pr}, \mathbf{u}_{r+1,j})\right) x_{\mathbf{u}_{pr}} x_{\mathbf{u}_{ij}} \\
&= \chi(\mathbf{u}_{ir}, \mathbf{u}_{pr}) \left(1 - \chi(\mathbf{u}_{pr}, \mathbf{u}_{r+1,j}) \chi(\mathbf{u}_{r+1,j}, \mathbf{u}_{pr})\right) x_{\mathbf{u}_{pr}} x_{\mathbf{u}_{ij}},
\end{aligned}$$

where we use (5.16) in the first equality, (1.4) in the second, (5.19) in the third and the relation between the q_{ij} in the last. □

We prove the main Theorem of this subsection, namely, the presentation by generators and relations of the Nichols algebra associated to V.

Theorem 5.14. *Let V be a standard braided vector space of type A_θ, $\theta = \dim V$, and $C = (a_{ij})_{i,j \in \{1,\dots,\theta\}}$ the corresponding Cartan matrix of type A_θ.*

The Nichols algebra $\mathfrak{B}(V)$ is presented by generators x_i, $1 \le i \le \theta$, and relations

$$\begin{aligned}
x_\alpha^{N_\alpha} &= 0, \quad \alpha \in \Delta^+; \\
ad_c(x_i)^{1-a_{ij}}(x_j) &= 0, \quad i \ne j; \\
[(\operatorname{ad} x_{j-1})(\operatorname{ad} x_j) x_{j+1}, x_j]_c &= 0, \quad 1 < j < \theta,\ q_{jj} = -1.
\end{aligned}$$

Moreover, the following elements constitute a basis of $\mathfrak{B}(V)$*:*

$$x_{\beta_1}^{h_1} x_{\beta_2}^{h_2} \dots x_{\beta_P}^{h_P}, \qquad 0 \le h_j < N_{\beta_j}, \text{ if } \beta_j \in S_I, \quad 1 \le j \le P. \tag{5.21}$$

Proof. From Corollary 4.2 and the definitions of x_α's, we know that the last statement about the PBW basis is true.

Now, let $\mathfrak{B}$ be the algebra presented by generators $x_1, \dots, x_\theta$ and the relations of the Theorem. From Lemmata 5.3, 5.4 and Proposition 5.2 we have a canonical epimorphism $\phi : \mathfrak{B} \to \mathfrak{B}(V)$. Note that last relation also holds in $\mathfrak{B}$ for $q_{jj} \neq 1$, by Lemma 5.4, (2).

The proof is similar to the ones of [AD, Lemma 3.7] and [AS5, Lemma 6.12]. Consider the subspace $\mathcal{I}$ of $\mathfrak{B}$ generated by the elements in (5.21). Using Lemmata 5.10, 5.11, 5.12 and 5.13 we prove that $\mathcal{I}$ is an ideal. But $1 \in \mathcal{I}$, so $\mathcal{I} = \mathfrak{B}$.

The image of the elements in (5.21) by ϕ are a basis of $\mathfrak{B}(V)$, so ϕ is an isomorphism. □

Note that the presentation and the dimension of $\mathfrak{B}(V)$ agree with the results presented in [AD] and [AS5].

5.3. Presentation when the type is B_θ.

Now, we shall consider V a standard braided vector space of type B_θ, and $\mathfrak{B}$ a $\mathbb{Z}^\theta$-graded algebra, provided with an inclusion of vector spaces $V \hookrightarrow \mathfrak{B}^1 = \oplus_{1 \le j \le \theta} \mathfrak{B}^{\mathbf{e}_j}$. Then, we can extend the braiding to $\mathfrak{B}$. We assume that the following relations hold in $\mathfrak{B}$

$$\begin{aligned} x_i^2 &= 0 & &\text{if } q_{ii} = -1, \\ x_1^3 &= 0 & &\text{if } q_{11} \in \mathbb{G}_3, \\ (\operatorname{ad}_c x_i) x_j &= 0, & &|j-i| > 1; \\ (\operatorname{ad}_c x_i)^2 x_j &= 0, & &|j-i| = 1, \ i \neq 1; \\ [(\operatorname{ad}_c x_i)(ad_c x_{i+1}) x_{i+2}, x_{i+1}]_c &= 0, & &2 \le i \le \theta; \\ (\operatorname{ad}_c x_1)^3 x_2 &= 0; \\ \left[(\operatorname{ad}_c x_1)^2 x_2, (ad_c x_1) x_2\right]_c &= 0, \\ \left[(\operatorname{ad} x_1)^2 (\operatorname{ad} x_2) x_3, (\operatorname{ad} x_1) x_2\right]_c &= 0. \end{aligned}$$

Using the same notation as in Subsection 4.2,

$$x_{\mathbf{v}_{ij}} = \left[x_{\mathbf{u}_{1i}}, x_{\mathbf{u}_{1j}}\right]_c, \qquad 1 \le i < j \le \theta.$$

From the proof of relations corresponding the A_θ case, relations (5.15), (5.16), (5.17), (5.19) and (5.20) are valid for $i \ge 1$, but for relation (5.18) we must consider $i > 1$.

Lemma 5.15. *Let $1 \leq s < t$, $1 < k \leq j$. The following relations hold in $\mathfrak{B}$:*

$$[x_{\mathbf{v}_{st}}, x_{\mathbf{u}_{kj}}]_c \begin{cases} = 0 & t+1 < k; \\ = x_{\mathbf{v}_{sj}} & t+1 = k < j; \\ = 0 & s+1 < k \leq j \leq t; \\ = \chi(\mathbf{v}_{st}, \mathbf{u}_{kt})(1 - q_{t,t+1}q_{t+1,t}) x_{\mathbf{u}_{kt}} x_{\mathbf{v}_{sj}} & s+1 < k \leq t < j; \\ = \chi(\mathbf{u}_{1t}, \mathbf{u}_{s+1,j}) x_{\mathbf{v}_{jt}} & s+1 = k \leq j < t; \\ = (\chi(\mathbf{u}_{1t}, \mathbf{u}_{s+1,t}) - \chi(\mathbf{u}_{1s}, \mathbf{u}_{1t})) x_{\mathbf{u}_{1t}}^2 & s+1 = k, j = t; \\ \in \mathsf{k} x_{\mathbf{v}_{tj}} + \mathsf{k} x_{\mathbf{u}_{1j}} x_{\mathbf{u}_{1t}} + \mathsf{k} x_{\mathbf{u}_{s+1,j}} x_{\mathbf{v}_{sj}} & s+1 = k \leq t < j; \\ = \gamma_{st}^{kj} x_{\mathbf{u}_{ks}} x_{\mathbf{v}_{jt}} & k \leq s < j \leq t; \\ \in \mathsf{k} x_{\mathbf{u}_{ks}} x_{\mathbf{v}_{tj}} + \mathsf{k} x_{\mathbf{u}_{ks}} x_{\mathbf{u}_{1j}} x_{\mathbf{u}_{1t}} + \mathsf{k} x_{\mathbf{u}_{kt}} x_{\mathbf{v}_{sj}} & k \leq s < t < j; \\ 0 & k \leq j \leq s, \end{cases}$$

where $\gamma_{st}^{kj} = \chi(\mathbf{u}_{1t}, \mathbf{u}_{kj}) \chi(\mathbf{u}_{1s}, \mathbf{u}_{ks})(1 - q_{s,s+1}q_{s+1,s})$.

Proof. The first, the third and the last cases follow from the fact that

$$[x_{\mathbf{u}_{1s}}, x_{\mathbf{u}_{kj}}]_c = [x_{\mathbf{u}_{1t}}, x_{\mathbf{u}_{kj}}]_c = 0$$

using (5.15), (5.17), (5.18) or (5.19)(depending on each case), and (1.4).

For the second, use that $\left[x_{\mathbf{u}_{1s}}, x_{\mathbf{u}_{t+1,j}}\right]_c = 0$, (5.16) and (1.4), to obtain

$$\begin{aligned} x_{\mathbf{v}_{sj}} &= \left[x_{\mathbf{u}_{1s}}, x_{\mathbf{u}_{1j}}\right]_c = \left[x_{\mathbf{u}_{1s}}, \left[x_{\mathbf{u}_{1t}}, x_{\mathbf{u}_{t+1,j}}\right]_c\right]_c \\ &= \left[\left[x_{\mathbf{u}_{1s}}, x_{\mathbf{u}_{1t}}\right]_c, x_{\mathbf{u}_{t+1,j}}\right]_c = \left[x_{\mathbf{v}_{st}}, x_{\mathbf{u}_{t+1,j}}\right]_c. \end{aligned}$$

For the fourth, use (1.4) and the third case to calculate

$$\begin{aligned} [x_{\mathbf{v}_{st}}, x_{\mathbf{u}_{kj}}]_c =& [x_{\mathbf{v}_{st}}, [x_{\mathbf{u}_{kt}}, x_{\mathbf{u}_{t+1,j}}]_c]_c \\ =& \chi(\mathbf{v}_{st}, \mathbf{u}_{kt}) x_{\mathbf{u}_{kt}} x_{\mathbf{v}_{sj}} - \chi(\mathbf{u}_{kt}, \mathbf{u}_{t+1,j}) x_{\mathbf{v}_{sj}} x_{\mathbf{u}_{kt}} \\ =& \chi(\mathbf{v}_{st}, \mathbf{u}_{kt})(1 - \chi(\mathbf{u}_{kt}, \mathbf{u}_{t+1,j})\chi(\mathbf{u}_{t+1,j}, \mathbf{u}_{kt})) x_{\mathbf{u}_{kt}} x_{\mathbf{v}_{sj}}. \end{aligned}$$

For the fifth, note that $\chi(\mathbf{u}_{1t}, \mathbf{u}_{s+1,j})^{-1} = \chi(\mathbf{u}_{s+1,j}, \mathbf{u}_{1t})$. Then, use (5.16), (5.17) and (1.4) to prove that

$$\begin{aligned} [x_{\mathbf{v}_{st}}, x_{\mathbf{u}_{s+1,j}}]_c =& [[x_{\mathbf{u}_{1s}}, x_{\mathbf{u}_{1t}}]_c, x_{\mathbf{u}_{s+1,j}}]_c \\ =& \chi(\mathbf{u}_{1t}, \mathbf{u}_{s+1,j}) x_{\mathbf{u}_{1j}} x_{\mathbf{u}_{1t}} - \chi(\mathbf{u}_{1s}, \mathbf{u}_{1t}) x_{\mathbf{u}_{1t}} x_{\mathbf{u}_{1s}} \\ =& \chi(\mathbf{u}_{1t}, \mathbf{u}_{s+1,j})(x_{\mathbf{u}_{1j}} x_{\mathbf{u}_{1t}} - \chi(\mathbf{u}_{1j}, \mathbf{u}_{1t}) x_{\mathbf{u}_{1t}} x_{\mathbf{u}_{1s}}). \end{aligned}$$

The sixth case follows in a similar way.

For the seventh case, we use (1.4), (1.5) and the previous case to calculate

$$\begin{aligned}
[x_{\mathbf{v}_{st}}, x_{\mathbf{u}_{s+1,j}}]_c = & [x_{\mathbf{v}_{st}}, [x_{\mathbf{u}_{s+1,t}}, x_{\mathbf{u}_{t+1,j}}]_c]_c \\
= & (\chi(\mathbf{u}_{1t}, \mathbf{u}_{s+1,t}) - \chi(\mathbf{u}_{1s}, \mathbf{u}_{1t}))[x^2_{\mathbf{u}_{1t}}, x_{\mathbf{u}_{t+1,j}}] \\
& + \chi(\mathbf{v}_{st}, \mathbf{u}_{s+1,t}) x_{\mathbf{u}_{s+1,t}} x_{\mathbf{v}_{sj}} - \chi(\mathbf{u}_{s+1,t}, \mathbf{u}_{t+1,j}) x_{\mathbf{v}_{sj}} x_{\mathbf{u}_{s+1,t}} \\
= & (\chi(\mathbf{u}_{1t}, \mathbf{u}_{s+1,t}) - \chi(\mathbf{u}_{1s}, \mathbf{u}_{1t}))((x_{\mathbf{v}_{tj}} + \chi(\mathbf{u}_{1t}, \mathbf{u}_{1j}) x_{\mathbf{u}_{1j}} x_{\mathbf{u}_{1t}}) \\
& + \chi(\mathbf{u}_{1t}, \mathbf{u}_{t+1,j}) x_{\mathbf{u}_{1j}} x_{\mathbf{u}_{1t}}) - \chi(\mathbf{u}_{s+1,t}, \mathbf{u}_{t+1,j}) x_{\mathbf{v}_{tj}} \\
& + (\chi(\mathbf{v}_{st}, \mathbf{u}_{s+1,t}) - \chi(\mathbf{u}_{s+1,t}, \mathbf{u}_{t+1,j}) \chi(\mathbf{v}_{sj}, \mathbf{u}_{s+1,t})) x_{\mathbf{u}_{s+1,t}} x_{\mathbf{v}_{sj}}.
\end{aligned}$$

We use the previous cases, (5.17) and (5.20) to calculate for the eighth case

$$\begin{aligned}
[x_{\mathbf{v}_{st}}, x_{\mathbf{u}_{kj}}]_c = & [[x_{\mathbf{u}_{1s}}, x_{\mathbf{u}_{1t}}]_c, x_{\mathbf{u}_{kj}}]_c \\
= & \chi(\mathbf{u}_{1t}, \mathbf{u}_{kj})(\chi(\mathbf{u}_{1s}, \mathbf{u}_{ks})(1 - q_{s,s+1} q_{s+1,s}) x_{\mathbf{u}_{ks}} x_{\mathbf{u}_{1j}}) x_{\mathbf{u}_{1t}} \\
& - \chi(\mathbf{u}_{1s}, \mathbf{u}_{1t}) x_{\mathbf{u}_{1t}} (\chi(\mathbf{u}_{1s}, \mathbf{u}_{ks})(1 - q_{s,s+1} q_{s+1,s}) x_{\mathbf{u}_{ks}} x_{\mathbf{u}_{1j}}) \\
= & \gamma_{st}^{kj} x_{\mathbf{u}_{ks}} (x_{\mathbf{u}_{1j}} x_{\mathbf{u}_{1t}} - \chi(\mathbf{u}_{1j}, \mathbf{u}_{1t}) x_{\mathbf{u}_{1t}} x_{\mathbf{u}_{1j}}).
\end{aligned}$$

To finish, we prove the ninth case in a similar way as follow

$$\begin{aligned}
[x_{\mathbf{v}_{st}}, x_{\mathbf{u}_{kj}}]_c = & [x_{\mathbf{v}_{st}}, [x_{\mathbf{u}_{kt}}, x_{\mathbf{u}_{t+1,j}}]_c]_c \\
= & \gamma_{st}^{kt} (1 - q_{\mathbf{v}_{1t}}) [x_{\mathbf{u}_{ks}} x^2_{\mathbf{u}_{1t}}, x_{\mathbf{u}_{t+1,j}}] + \chi(\mathbf{v}_{st}, \mathbf{u}_{k,t}) x_{\mathbf{u}_{kt}} x_{\mathbf{v}_{sj}} \\
& - \chi(\mathbf{u}_{kt}, \mathbf{u}_{t+1,j}) x_{\mathbf{v}_{sj}} x_{\mathbf{u}_{kt}}.
\end{aligned}$$

□

We consider the remaining commutator $\left[x_{\mathbf{v}_{st}}, x_{\mathbf{u}_{jk}}\right]_c$: when $j = 1$.

Lemma 5.16. *Let $s < t$ in $\{1, \dots, \theta\}$. The following relations hold in $\mathfrak{B}$*

$$[x_{\mathbf{v}_{st}}, x_{\mathbf{u}_{1k}}]_c = 0, \quad s < k \leq t; \tag{5.22}$$

$$[x_{\mathbf{u}_{1s}}, x_{\mathbf{v}_{st}}]_c = 0. \tag{5.23}$$

Proof. By hypothesis we have

$$[x_{\mathbf{v}_{12}}, x_{\mathbf{u}_{12}}]_c = \left[(\operatorname{ad}_c x_1)^2 x_2, (\operatorname{ad}_c x_1) x_2\right]_c = 0,$$

$$[x_{\mathbf{v}_{13}}, x_{\mathbf{u}_{12}}]_c = \left[(\operatorname{ad}_c x_1)^2 (\operatorname{ad}_c x_2) x_3, (\operatorname{ad}_c x_1) x_2\right]_c = 0.$$

For $t \geq 4$, $[x_{\mathbf{u}_{4t}}, x_{\mathbf{u}_{12}}]_c = 0$ by (5.15), and using (1.4),

$$[x_{\mathbf{v}_{1t}}, x_{\mathbf{u}_{12}}]_c = [[x_{\mathbf{v}_{13}}, x_{\mathbf{u}_{4t}}]_c, x_{\mathbf{u}_{12}}]_c = 0.$$

For each $k \leq t$ we have $[x_{\mathbf{u}_{1t}}, x_{\mathbf{u}_{3k}}]_c = [x_1, x_{\mathbf{u}_{3k}}]_c = 0$, so $[x_{\mathbf{v}_{1t}}, x_{\mathbf{u}_{3k}}]_c = 0$. Using (1.4) and (5.16) we have

$$[x_{\mathbf{v}_{1t}}, x_{\mathbf{u}_{1k}}]_c = \left[x_{\mathbf{v}_{1t}}, [x_{\mathbf{u}_{12}}, x_{\mathbf{u}_{3k}}]_c\right]_c = 0.$$

Now, consider $2 \leq s \leq k$. As $[x_{\mathbf{v}_{1t}}, x_{\mathbf{u}_{1k}}]_c = [x_{\mathbf{u}_{2s}}, x_{\mathbf{u}_{1k}}]_c = 0$ by previous results and (5.17), we conclude from (1.5) and Lemma 5.15 that (5.22) is valid in the general case.

To prove (5.23), for $s = 1, t = 2$ we have

$$[x_{\mathbf{u}_{11}}, x_{\mathbf{v}_{12}}]_c = [x_1, x_{\mathbf{v}_{12}}]_c = (\operatorname{ad}_c x_1)^3 x_2 = 0.$$

Using that $[x_1, x_{\mathbf{u}_{3t}}]_c = 0$ if $t \geq 3$ and (1.4), we deduce that

$$[x_{\mathbf{u}_{11}}, x_{\mathbf{v}_{1t}}]_c = [x_1, [x_{\mathbf{v}_{12}}, x_{\mathbf{u}_{3t}}]_c]_c = 0.$$

If $1 < s < t$, by the previous case we have

$$[x_{\mathbf{u}_{1s}}, x_{\mathbf{v}_{1t}}]_c = -\chi(x_{\mathbf{u}_{1s}}, x_{\mathbf{v}_{1t}})\,[x_{\mathbf{v}_{1t}}, x_{\mathbf{u}_{1s}}]_c = 0.$$

By (5.19), $[x_{\mathbf{u}_{1s}}, x_{\mathbf{u}_{2s}}]_c = 0$. Also, $[x_{\mathbf{v}_{1t}}, x_{\mathbf{u}_{2s}}]_c = \chi(\mathbf{u}_{1t}, \mathbf{u}_{2s}) x_{\mathbf{v}_{st}}$, by Lemma 5.15. Then, (5.23) follows by (1.4) and the last three equalities. □

Lemma 5.17. *Let $s < k < t$. The following relations hold in $\mathfrak{B}$:*

$$(5.24)\quad [x_{\mathbf{v}_{sk}}, x_{\mathbf{u}_{1t}}]_c = \chi(\mathbf{v}_{sk}, \mathbf{u}_{1k})(1 - q_{k,k+1}q_{k+1,k}) x_{\mathbf{u}_{1k}} x_{\mathbf{v}_{st}},$$
$$(5.25)\quad [x_{\mathbf{u}_{1s}}, x_{\mathbf{v}_{kt}}]_c = \chi(\mathbf{u}_{1s}, \mathbf{u}_{1k})(1 + q_{\mathbf{u}_{1k}})(1 - q_{k,k+1}q_{k+1,k}) x_{\mathbf{u}_{1k}} x_{\mathbf{v}_{st}}.$$

Proof. The proof follows by (1.4), the second case of Lemma 5.15 and (5.23),

$$\begin{aligned}
[x_{\mathbf{v}_{sk}}, x_{\mathbf{u}_{1t}}]_c &= \Big[x_{\mathbf{v}_{sk}}, \big[x_{\mathbf{u}_{1k}}, x_{\mathbf{u}_{k+1,t}}\big]_c\Big]_c \\
&= \chi(\mathbf{v}_{sk}, \mathbf{u}_{1k}) x_{\mathbf{u}_{1k}} x_{\mathbf{v}_{st}} - \chi(\mathbf{u}_{1k}, \mathbf{u}_{k+1,t}) x_{\mathbf{v}_{st}} x_{\mathbf{u}_{1k}} \\
&= \chi(\mathbf{v}_{sk}, \mathbf{u}_{1k})\left(1 - \chi(\mathbf{u}_{1k}, \mathbf{u}_{k+1,t})\chi(\mathbf{u}_{k+1,t}, \mathbf{u}_{1k})\right) x_{\mathbf{u}_{1k}} x_{\mathbf{v}_{st}}; \\
[x_{\mathbf{u}_{1s}}, x_{\mathbf{v}_{kt}}]_c &= [x_{\mathbf{u}_{1s}}, [x_{\mathbf{u}_{1k}}, x_{\mathbf{u}_{1t}}]_c]_c = [x_{\mathbf{v}_{sk}}, x_{\mathbf{u}_{1t}}]_c \\
&\quad + \chi(\mathbf{u}_{1s}, \mathbf{u}_{1k}) x_{\mathbf{u}_{1k}} x_{\mathbf{v}_{st}} - \chi(\mathbf{u}_{1k}, \mathbf{u}_{1t}) x_{\mathbf{v}_{st}} x_{\mathbf{u}_{1k}} \\
&= \chi(\mathbf{u}_{1s}, \mathbf{u}_{1k})(q_{\mathbf{u}_{1k}}(1 - q_{k,k+1}q_{k+1,k}) + 1 - q_{k,k+1}q_{k+1,k}) x_{\mathbf{u}_{1k}} x_{\mathbf{v}_{st}}.
\end{aligned}$$

□

We deal with the expression of the commutator of two words of type $x_{\mathbf{v}_{st}}$.

Lemma 5.18. *Let $s < t, k < j$, $s \leq k$, with $k \neq s$ or $j \neq t$. The following relations hold in $\mathfrak{B}$:*

$$\big[x_{\mathbf{v}_{st}}, x_{\mathbf{v}_{kj}}\big]_c \begin{cases} = 0, & k < j \leq t, \\ & k = s, t < j \\ = \chi(\mathbf{v}_{st}, \mathbf{v}_{kt})(1 - q_{t,t+1}q_{t+1,t}) x_{\mathbf{v}_{kt}} x_{\mathbf{v}_{sj}}, & k < t < j; \\ = \chi(\mathbf{v}_{st}, \mathbf{u}_{1t})^2 (1 - q_{t,t+1}q_{t+1,t}) & \\ \quad (1 - q_{\mathbf{u}_{1t}} q_{t,t+1} q_{t+1,t}) x_{\mathbf{u}_{1t}}^2 x_{\mathbf{v}_{sj}}, & k = t < j; \\ \in \mathsf{k} x_{\mathbf{v}_{tj}} x_{\mathbf{v}_{sk}} + \mathsf{k} x_{\mathbf{v}_{tk}} x_{\mathbf{v}_{sj}} + \mathsf{k} x_{\mathbf{u}_{1k}} x_{\mathbf{u}_{1t}} x_{\mathbf{v}_{sj}}, & t < k < j. \end{cases}$$

Proof. The first and the second cases follow from (1.4) and (5.22), (5.23), respectively. For the third case, we use the previous one and (1.4),

$$\begin{aligned}
[x_{\mathbf{v}_{st}}, x_{\mathbf{v}_{kj}}]_c = & \left[x_{\mathbf{v}_{st}}, \left[x_{\mathbf{u}_{1k}}, x_{\mathbf{u}_{1j}}\right]_c\right]_c \\
=&\chi(\mathbf{v}_{st}, \mathbf{u}_{1k}) x_{\mathbf{u}_{1k}} (\chi(\mathbf{v}_{st}, \mathbf{u}_{1t})(1 - q_{t,t+1}q_{t+1,t}) x_{\mathbf{u}_{1t}} x_{\mathbf{v}_{sj}}) \\
& - \chi(\mathbf{u}_{1k}, \mathbf{u}_{1j})(\chi(\mathbf{v}_{st}, \mathbf{u}_{1t})(1 - q_{t,t+1}q_{t+1,t}) x_{\mathbf{u}_{1t}} x_{\mathbf{v}_{sj}}) x_{\mathbf{u}_{1k}} \\
=&(1 - q_{t,t+1}q_{t+1,t}) \big(\chi(\mathbf{v}_{st}, \mathbf{u}_{1k})\chi(\mathbf{v}_{st}, \mathbf{u}_{1t}) x_{\mathbf{u}_{1k}} x_{\mathbf{u}_{1t}} x_{\mathbf{v}_{sj}} \\
& - \chi(\mathbf{u}_{1k}, \mathbf{u}_{1j})\chi(\mathbf{v}_{st}, \mathbf{u}_{1t})\chi(\mathbf{v}_{sj}, \mathbf{u}_{1k}) x_{\mathbf{u}_{1t}} x_{\mathbf{u}_{1k}} x_{\mathbf{v}_{sj}}\big) \\
=&\chi(\mathbf{v}_{st}, \mathbf{u}_{1k})\chi(\mathbf{v}_{st}, \mathbf{u}_{1t})(1 - q_{t,t+1}q_{t+1,t}) \\
& (x_{\mathbf{u}_{1k}} x_{\mathbf{u}_{1k}} - \chi(\mathbf{u}_{1k}, \mathbf{u}_{1t}) x_{\mathbf{u}_{1k}} x_{\mathbf{u}_{1k}}) x_{\mathbf{v}_{sj}}.
\end{aligned}$$

The fourth case is similar to the previous one.

To prove the last case, use (1.4) and Remark 5.17 to express

$$\begin{aligned}
[x_{\mathbf{v}_{st}}, x_{\mathbf{v}_{kj}}]_c = & \left[x_{\mathbf{v}_{st}}, \left[x_{\mathbf{u}_{1k}}, x_{\mathbf{u}_{1j}}\right]_c\right]_c \\
=&[\chi(\mathbf{v}_{st}, \mathbf{u}_{1t})(1 - q_{t,t+1}q_{t+1,t}) x_{\mathbf{u}_{1t}} x_{\mathbf{v}_{sk}}, x_{\mathbf{u}_{1j}}]_c \\
& + \chi(\mathbf{v}_{st}, \mathbf{u}_{1k}) x_{\mathbf{u}_{1k}} (\chi(\mathbf{v}_{st}, \mathbf{u}_{1t})(1 - q_{t,t+1}q_{t+1,t}) x_{\mathbf{u}_{1t}} x_{\mathbf{v}_{sj}}) \\
& - \chi(\mathbf{u}_{1k}, \mathbf{u}_{1j})(\chi(\mathbf{v}_{st}, \mathbf{u}_{1t})(1 - q_{t,t+1}q_{t+1,t}) x_{\mathbf{u}_{1t}} x_{\mathbf{v}_{sj}}) x_{\mathbf{u}_{1k}}.
\end{aligned}$$

and the proof finish using (1.5) and the previous identities. □

Theorem 5.19. *Let V be a standard braided vector space of type B_θ, $\theta = \dim V$, and $C = (a_{ij})_{i,j\in\{1,\dots,\theta\}}$ the corresponding Cartan matrix of type B_θ.*

The Nichols algebra $\mathfrak{B}(V)$ is presented by generators x_i, $1 \le i \le \theta$, and relations

$$\begin{aligned}
x_\alpha^{N_\alpha} &= 0, & & \alpha \in \Delta^+; \\
ad_c(x_i)^{1-a_{ij}}(x_j) &= 0, & & i \neq j; \\
[(\operatorname{ad} x_{j-1})(\operatorname{ad} x_j) x_{j+1}, x_j]_c &= 0, & & 1 < j < \theta,\ q_{jj} = -1; \\
\big[(\operatorname{ad} x_1)^2 x_2, (\operatorname{ad} x_1) x_2\big]_c &= 0, & & q_{11} \in \mathbb{G}_3 \text{ or } q_{22} = -1; \\
\big[(\operatorname{ad} x_1)^2 (\operatorname{ad} x_2) x_3, (\operatorname{ad} x_1) x_2\big]_c &= 0, & & q_{11} \in \mathbb{G}_3 \text{ or } q_{22} = -1.
\end{aligned}$$

Moreover, the following elements constitute a basis of $\mathfrak{B}(V)$:

$$x_{\beta_1}^{h_1} x_{\beta_2}^{h_2} \dots x_{\beta_P}^{h_P}, \qquad 0 \le h_j \le N_{\beta_j} - 1,\ \text{if } \beta_j \in S_I, \quad 1 \le j \le P. \tag{5.26}$$

Proof. The proof is analogous to the corresponding of Theorem 5.14, since by previous Lemmata we express the commutator of two generators $x_\alpha < x_\beta$ as a linear combination of monotone hyperwords, whose greater hyperletter is great or equal than x_β. □

5.4. **Presentation when the type is G_2.**

We consider now standard braidings of G_2 type, with $m_{12} = 3, m_{21} = 1$.

Lemma 5.20. *Let $\mathfrak{B} := T(V)/I$, for some $I \in \mathfrak{S}$, such that in $\mathfrak{B}$ hold*

$$(\operatorname{ad} x_1)^4 x_2 = (\operatorname{ad} x_2)^2 x_1 = 0; \tag{5.27}$$

$$x_i^{N_i} = 0, \qquad i = 1, 2, \ N_i := \operatorname{ord} q_{ii}. \tag{5.28}$$

(a) $\left[x_1^3 x_2 x_1 x_2\right]_c = 0 \Longleftrightarrow 4e_1 + 2e_2 \notin \Delta^+(\mathfrak{B})$;

*Assume now that **(a)** holds in $\mathfrak{B}$. Then*

(b) $\left[(\operatorname{ad} x_1)^3 x_2, (\operatorname{ad} x_1)^2 x_2\right]_c = 0 \Longleftrightarrow 5e_1 + 2e_2 \notin \Delta^+(\mathfrak{B})$;

(c) $\left[\left[x_1^2 x_2 x_1 x_2\right]_c, [x_1 x_2]_c\right]_c = 0 \Longleftrightarrow 4e_1 + 3e_2 \notin \Delta^+(\mathfrak{B})$;

*Assume also that **(b)**, **(c)** holds in $\mathfrak{B}$, then*

(d) $\left[\left[x_1^2 x_2\right]_c, \left[x_1^2 x_2 x_1 x_2\right]_c\right]_c = 0 \Longleftrightarrow 5e_1 + 3e_2 \notin \Delta^+(\mathfrak{B})$.

In particular, all these relations hold when V is a standard braiding and $\mathfrak{B} = \mathfrak{B}(V)$ is finite dimensional.

Proof. Order the letters $x_1 < x_2$, and consider a PBW basis as in Theorem 1.12. We denote $\gamma_k := \prod_{0 \le j \le k-1} (1 - q_{11}^j q_{12} q_{21})$.

(a) If the first assertion is true, then $4e_1 + 2e_2 \notin \Delta^+(\mathfrak{B})$ since there are no possible Lyndon words in S_I: $x_1^3 x_2 x_1 x_2$ is the unique Lyndon word such that $x_1^3 x_2, x_1 x_2^2$ are not factors, and it is not in S_I because of the hypothesis.

Reciprocally, if $4e_1 + 2e_2 \notin \Delta^+(\mathfrak{B})$, then $\left[x_1^3 x_2 x_1 x_2\right]_c$ is a linear combination of greater hyperwords, and $\left[x_1 x_2 x_1^3 x_2\right]_c, \left[x_1^2 x_1^2 x_2\right]$ are the unique greater hyperwords that are not in S_I and do not end in x_1 (we discard words ending in x_1 since $\left[x_1^3 x_2 x_1 x_2\right]_c$ is in $\ker D_1$). So, taking their Shirshov decomposition, there exist $\alpha, \beta \in \mathsf{k}$ such that

$$\left[x_1^3 x_2 x_1 x_2\right]_c - \alpha \left[x_1 x_2\right]_c \left[x_1^3 x_2\right]_c - \beta \left[x_1^2 x_2\right]_c^2 = 0. \tag{5.29}$$

Note that $\left[x_1^3 x_2 x_1 x_2\right]_c = \operatorname{ad} x_1 \left(\left[x_1^2 x_2 x_1 x_2\right]_c\right)$, so by direct calculation,

$$D_2 \left(\left[x_1^2 x_2 x_1 x_2\right]_c\right) = 0.$$

Then, we apply D_2 to both sides of equality (5.29) and express the result as a linear combination of $\left[x_1^3 x_2\right]_c x_1$, $\left[x_1^2 x_2\right]_c x_1^2$ and $[x_1 x_2]_c x_1^3$, then the coefficient of $[x_1 x_2]_c x_1^3$ is

$$\alpha (1 - q_{12} q_{21})(1 - q_{11} q_{12} q_{21}),$$

so $\alpha = 0$. Then, note also that

$$D_1^2 D_2 \left(\left[x_1^3 x_2 x_1 x_2\right]_c\right) = 0,$$

but

$$D_1^2 D_2 \left(\left[x_1^2 x_2\right]_c^2\right) = (1 - q_{12} q_{21})(1 - q_{11} q_{12} q_{21})(1 + q_{11})(q_{2e_1+e_2} + 1) \left[x_1^2 x_2\right]_c.$$

Looking at the proof of Proposition 4.7, $q_{2e_1+e_2} \neq -1$, so $\beta = 0$.

(b) Under the conditions ***(a)***, (5.27) and (5.28), the unique possible Lyndon word of degree $5e_1 + 2e_j$ is $x_1^3x_2x_1^2x_2$, and

$$\left[x_1^2x_2x_1x_2x_1x_2\right]_c = \left[(\operatorname{ad} x_1)^3x_2, (\operatorname{ad} x_1)^2x_2\right]_c.$$

Then we proceed as before. One implication is clear. For the other, if $5e_1 + 2e_j \notin \Delta^+(\mathfrak{B})$, then there exists $\alpha \in \mathsf{k}$ such that

$$\left[(\operatorname{ad} x_1)^3x_2, (\operatorname{ad} x_1)^2x_2\right]_c = \alpha(\operatorname{ad} x_1)^2x_2(\operatorname{ad} x_1)^3x_2.$$

Then, we apply D_2 and express the equality as a linear combination of $(\operatorname{ad} x_1)^3x_2x_1^2$ and $(\operatorname{ad} x_1)^2x_2x_1^3$ (we use that $(\operatorname{ad} x_1)^4x_2 = 0$ by hypothesis); the coefficient of $(\operatorname{ad} x_1)^2x_2x_1^3$ is $\alpha\gamma_3$, so $\alpha = 0$.

(c) The proof is similar. Since we consider Lyndon words without $x_1^3x_2$, $x_1x_2^2$ as factor, the unique possible Lyndon word of degree $4e_1 + 3e_j$ is $x_1^2x_2x_1x_2x_1x_2$, and

$$\left[x_1^2x_2x_1x_2x_1x_2\right]_c = \left[\left[x_1^2x_2x_1x_2\right]_c, [x_1x_2]_c\right]_c.$$

If $4e_1 + 3e_j \notin \Delta^+(\mathfrak{B})$, then there exist $\alpha_i \in \mathsf{k}$ such that

$$\begin{aligned}\left[x_1^2x_2(x_1x_2)^2\right]_c \quad &= \quad \alpha_1 [x_1x_2]_c \left[x_1^2x_2x_1x_2\right]_c + \alpha_2 [x_1x_2]_c^2 \left[x_1^2x_2\right]_c \\ &\quad +\alpha_3x_2 \left[x_1^2x_2\right]_c^2 + \alpha_4x_2 [x_1x_2]_c \left[x_1^3x_2\right]_c,\end{aligned}$$

since we discard words greater than $x_1^2x_2x_1x_2x_1x_2$ ending in x_1 as above; we also discard words with factors $x_1^4x_2$, $x_1x_2^2$, $x_1^3x_2x_1^2x_2$, by the hypothesis about $\mathfrak{B}$. We apply D_2 to this equality. Using the definition of braided commutator, express it as a linear combination of elements of PBW basis, which have degree $4e_1 + 2e_2$.

The coefficient of $x_2 [x_1x_2]_c x_1^3$ is $\alpha_4\gamma_3$ since this PBW generator appears only in the expression of $D_2(x_2 [x_1x_2]_c \left[x_1^3x_2\right]_c)$. Then, $\alpha_4 = 0$.

Using this fact, the coefficient of $x_2 \left[x_1^3x_2\right]_c x_1$ is

$$\alpha_3\gamma_2(1 + q_{11})q_{11}^2q_{12}q_{21}^2q_{22},$$

since it appears only in the expression of $D_j(x_2 \left[x_1^2x_2\right]_c^2)$. Then, $\alpha_3 = 0$.

Now, look at the coefficient of $[x_1x_2]_c^2x_1^2$. It is $\alpha_2\gamma_2$, so $\alpha_2 = 0$. Then, we calculate the coefficient of $[x_1^2x_2]_c^2$:

$$\alpha_1\gamma_2 \left(\chi(\mathbf{e}_1, 5\mathbf{e}_1 + \mathbf{e}_2) - \chi(2\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2)\right) = \alpha_1\gamma_2q_{11}q_{12} \left(q_{11}^3 - q_{22}q_{12}q_{21}\right).$$

As $q_{11}^3 \neq q_{22}q_{12}q_{21}$ for each standard braiding, we conclude $\alpha_1 = 0$.

(d) If ***(b)***, ***(c)*** holds, then the unique possible Lyndon word which does not have factors $x_1^4x_2$ and $x_1x_2^2$ of degree $5e_1 + 3e_2$ is $x_1^2x_2x_1^2x_2x_1x_2$, and

$$\left[x_1^2x_2x_1^2x_2x_1x_2\right]_c = \left[\left[x_1^2x_2\right]_c, \left[x_1^2x_2x_1x_2\right]_c\right]_c.$$

Then this hyperword is not in S_I iff there exist $\nu_i \in \mathsf{k}$ such that

$$\begin{aligned}\left[x_1^2x_2x_1^2x_2x_1x_2\right]_c =&\nu_1 \left[x_1^2x_2x_1x_2\right]_c \left[x_1^2x_2\right]_c + \nu_2 [x_1x_2]_c \left[x_1^2x_2\right]_c^2 \\ &+ \nu_3 [x_1x_2]_c^2 \left[x_1^3x_2\right]_c + \nu_4x_2 \left[x_1^2x_2\right]_c \left[x_1^3x_2\right]_c.\end{aligned} \tag{5.30}$$

Apply D_2 and note that $D_2(\left[x_1^2x_2x_1^2x_2x_1x_2\right]_c) = 0$ under the hypothesis of $\mathfrak{B}$. Then, express the resulting sum as a linear combination of elements of the PBW basis, which have degree $5e_1 + 2e_2$.

The hyperword $x_2[x_1^2x_2]x_1^3$ appears only for $D_2(x_2\left[x_1^2x_2\right]_c\left[x_1^3x_2\right]_c)$, and its coefficient is $\nu_4\gamma_3$, and as $\gamma_3 \neq 0$ we conclude that $\nu_4 = 0$.

Analogously, $[x_1x_2]_c^2\,x_1^3$ appears only for $[x_1x_2]_c^2\left[x_1^3x_2\right]_c$ (due to $\nu_4 = 0$). Its coefficient is $\nu_3\gamma_3$, so $\nu_3 = 0$.

Note that $D_1^2D_2(\left[x_1^2x_2x_1x_2\right]_c) = 0$. We apply $D_1^2D_2$ to the expression (5.30), and obtain

$$0 = \nu_1\gamma_2(1+q_{11})\left[x_1^2x_2x_1x_2\right]_c + \nu_2\gamma_2(1+q_{11})(1+q_{2\mathbf{e}_1+\mathbf{e}_2})[x_1x_2]_c[x_1^2x_2]_c.$$

$\left[x_1^2x_2x_1x_2\right]_c$ and $[x_1x_2]_c[x_1^2x_2]_c$ are linearly independent, since they are linearly independent in $\mathfrak{B}(V)$, and we have a surjection $\mathfrak{B} \to \mathfrak{B}(V)$. Then,

$$\nu_1\gamma_2(1+q_{11}) = \nu_2\gamma_2(1+q_{11})(1+q_{2\mathbf{e}_1+\mathbf{e}_2}) = 0.$$

But for standard braidings of type G_2 we note that $q_{11}, q_{2\mathbf{e}_1+\mathbf{e}_2} \neq -1$ and $\gamma_2 \neq 0$, so $\nu_1 = \nu_2 = 0$.

The last statement is true since

$$\Delta^+(\mathfrak{B}(V)) = \{e_1, e_1+e_2, 2e_1+e_2, 3e_1+e_2, 3e_1+2e_2, e_2\},$$

if the braiding is standard of type G_2. □

Remark 5.21. Let V be a standard braided vector space of G_2 type, and $\mathfrak{B}$ a braided graded Hopf algebra satisfying the hypothesis of the Lemma above. In a similar way to Lemma 5.5, if $q_{11} \notin \mathbb{G}_4$, $q_{22} \neq -1$, then $5e_1 + 2e_2, 4e_1 + 2e_2 4e_1 + 3e_2, 5e_1 + 3e_2 \notin \Delta^+(\mathfrak{B})$.

It follows because $x_1^3x_2x_1^2x_2, x_1^2x_2x_1x_2x_1x_2, x_1^2x_2x_1^2x_2x_1x_2 \notin S_I$, using the quantum Serre relations as in cited Lemma.

Theorem 5.22. *Let V be a standard braided vector space of type G_2.*

The Nichols algebra $\mathfrak{B}(V)$ is presented by generators x_1, x_2, and relations

$$x_\alpha^{N_\alpha} = 0, \quad \alpha \in \Delta^+, \tag{5.31}$$

$$ad_c(x_1)^4(x_2) = ad_c(x_2)^2(x_1) = 0, \tag{5.32}$$

and if $q_{11} \in \mathbb{G}_4$ or $q_{22} = -1$,

$$\left[(\operatorname{ad} x_1)^3x_2, (\operatorname{ad} x_1)^2x_2\right]_c = 0, \tag{5.33}$$

$$\left[x_1, \left[x_1^2x_2x_1x_2\right]_c\right]_c = 0, \tag{5.34}$$

$$\left[\left[x_1^2x_2x_1x_2\right]_c, [x_1x_2]_c\right]_c = 0, \tag{5.35}$$

$$\left[\left[x_1^2x_2\right]_c, \left[x_1^2x_2x_1x_2\right]_c\right]_c = 0. \tag{5.36}$$

Moreover, the following elements constitute a basis of $\mathfrak{B}(V)$:

$$x_2^{h_{e_2}}\,[x_1x_2]_c^{h_{e_1+e_2}}\left[x_1^2x_2x_1x_2\right]_c^{h_{3e_1+2e_2}}\left[x_1^2x_2\right]_c^{h_{2e_1+e_2}}\left[x_1^3x_2\right]_c^{h_{3e_1+e_2}}x_1^{h_{e_1}}, \quad 0 \le h_\alpha \le N_\alpha - 1. \tag{5.37}$$

Proof. The statement about the PBW basis follows from Corollary 4.2 and the definitions of x_α's.

Now, let $\mathfrak{B}$ be the algebra presented by generators x_1, x_2 and relations (5.32), (5.31), (5.33), (5.34), (5.35) and (5.36). From Lemma 5.20 and Proposition 5.2, we have a canonical epimorphism of algebras $\phi : \mathfrak{B} \to \mathfrak{B}(V)$.

Consider the subspace $\mathfrak{I}$ of $\mathfrak{B}$ generated by the elements in (5.37). We prove by induction in the sum S of the h_α's of a such product M that $x_1 M \in \mathfrak{I}$; moreover, we prove that it is a linear combination of products which first hyperletter is least or equal than the first hyperletter of M. If $S = 0$, we have $M = 1$. Then

- If $M = x_1^{N_1}$, then $x_1 M = x_1^{N_1+1}$, that is zero if $N_1 = \operatorname{ord} x_1 - 1$.
- If $M = \left[x_1^3 x_2\right]_c M'$, then we use that $x_1 \left[x_1^3 x_2\right]_c = q_{11}^3 q_{12} \left[x_1^3 x_2\right]_c x_1$ to prove that $x_1 M \in \mathfrak{I}$, and it is zero or begins with $\left[x_1^3 x_2\right]_c$.
- If $M = \left[x_1^2 x_2\right]_c M'$, then we use that

$$x_1 \left[x_1^2 x_2\right]_c = \left[x_1^3 x_2\right]_c + q_{11}^2 q_{12} \left[x_1^2 x_2\right]_c x_1.$$

So, we use the inductive step and relation (5.33) to prove that $x_1 M \in \mathfrak{I}$, and it is zero or a linear combination of hyperwords that begin with an hyperletter least or equal than $\left[x_1^2 x_2\right]_c$.

- If $M = \left[x_1^2 x_2 x_1 x_2\right]_c M'$, then we deduce from (5.34)

$$x_1 \left[x_1^2 x_2 x_1 x_2\right]_c = \chi(\mathbf{e}_1, 3\mathbf{e}_1 + 2\mathbf{e}_2) \left[x_1^2 x_2 x_1 x_2\right]_c x_1,$$

and using also relations (5.35), (5.36), we prove that $x_1 M \in \mathfrak{I}$, and it is zero or a linear combination of hyperwords that begin with an hyperletter least or equal than $\left[x_1^2 x_2 x_1 x_2\right]_c$.

- If $M = \left[x_1 x_2\right]_c M'$, then observe that

$$x_1 \left[x_1 x_2\right]_c = \left[x_1^2 x_2\right]_c + q_{11} q_{12} \left[x_1 x_2\right]_c x_1.$$

In this case, using inductive step, relations (5.34), (5.35), and that

$$\begin{aligned} \left[x_1^2 x_2\right]_c \left[x_1 x_2\right]_c = {} & \left[\left[x_1^2 x_2\right]_c, \left[x_1 x_2\right]_c\right]_c \\ & + \chi(2\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2) \left[x_1 x_2\right]_c \left[x_1^2 x_2\right]_c, \end{aligned}$$

by definition of braided commutator, we prove that $x_1 M \in \mathfrak{I}$, and it is zero or a linear combination of hyperwords that begin with an hyperletter least or equal than $\left[x_1 x_2\right]_c$.

- If $M = x_2 M'$, then we use that $x_1 x_2 = \left[x_1 x_2\right]_c + q_{12} x_2 x_1$, and also $\left[\left[x_1 x_2\right]_c, x_2\right]_c = 0$ to prove that $x_1 M \in \mathfrak{I}$, and it is zero or a linear combination of hyperwords.

Now, $x_2 M$ is a product of non increasing hyperwords or is zero, for each element in (5.37), so $\mathfrak{I}$ is an ideal of $\mathfrak{B}$ containing 1, and then $\mathfrak{I} = \mathfrak{B}$. As the elements in (5.37) are a basis of $\mathfrak{B}(V)$, we have that ϕ is an isomorphism. $\square$

5.5. **Presentation when the braiding is of Cartan type.**

In this subsection, we present the Nichols algebra of a diagonal braiding vector space of Cartan type with matrix (q_{ij}), by generators and relations. This was established in [AS3, Th. 4.5] assuming that q_{ii} has odd order and that order is not divisible by 3 if i belongs to a component of type G_2. The proof in loc. cit. combines a reduction to symmetric (q_{ij}) by twisting, with results from [AJS] and [dCP]. We also note that some particular instances were already proved earlier in this section.

Fix a standard braided vector space V with connected Dynkin diagram and a natural $i \in \{1, \dots, \theta\}$. Suppose that $\mathfrak{B}$ is a quotient by an ideal $I \in \mathfrak{S}$ of $T(V)$. *We assume moreover that*

$$\begin{cases} (5.4) \text{ holds in } \mathfrak{B}, \text{ if } 1 \leq i \neq j \leq \theta; \\ (5.5) \text{ holds in } \mathfrak{B}, \text{ if } m_{kj} = m_{kl} = 1, \ m_{jl} = 0; \\ (5.7) \text{ holds in } \mathfrak{B}, \text{ if } m_{kj} = 2, \ m_{jk} = 1; \\ (5.9) \text{ holds in } \mathfrak{B}, \text{ if } m_{kj} = 2, \ m_{jk} == m_{jl} = 1, \ m_{kl} = 0; \\ V \text{ is not of type } G_2. \end{cases} \tag{5.38}$$

Note that if (5.4) holds in an algebra with derivations D_i, then (2.11) holds also, by Lemma 2.7. By 2.6, we have an algebra $s_i(\mathfrak{B})$ provided with skew derivations D_i. We call $\tilde{x}_k = (ad_c x_i)^{m_{ik}}(x_k)\#1 \in s_i(\mathfrak{B})$, for $k \neq i$, and $\tilde{x}_i = 1\#y$. They generate $s_i(\mathfrak{B})^1$ as vector space.

Under these conditions, we prove:

Lemma 5.23. *The graded algebra $s_i(\mathfrak{B})$ satisfies* (5.38).

Proof. Step I: we prove that $s_i(\mathfrak{B})$ satisfies (5.4).

Proof. Each $m\mathbf{e}_k + e_j$, $0 \leq m \leq m_{kj}$ is an element of $\Delta(\mathfrak{B}(V_i))$, so $s_i(m\mathbf{e}_k + \mathbf{e}_j) \in \Delta(\mathfrak{B}(V))$. As we have a surjective morphism of braided graded Hopf algebras $\mathfrak{B} \to \mathfrak{B}(V)$, we have $\Delta(\mathfrak{B}(V)) \subseteq \Delta(\mathfrak{B})$.

From Lemma 5.3, $(\mathrm{ad}_c\, \tilde{x}_k)^m \tilde{x}_j = 0$ if and only if $x_k^m x_j$ is a linear combination of greater words, for an order in which $x_k < x_j$. Then, by the relation between the Hilbert series of $\mathfrak{B}$ and $s_i(\mathfrak{B})$ established in Theorem 2.6, (5.4) for $s_i(\mathfrak{B})$ is equivalent to

$$s_i\left((m_{kj}+1)\mathbf{e}_k + \mathbf{e}_j\right) \notin \Delta^+(\mathfrak{B}).$$

When $k = i \neq j$, this says $-\mathbf{e}_i + \mathbf{e}_j \notin \Delta^+(\mathfrak{B})$ so (5.4) holds.

To prove (5.4) for $s_i(\mathfrak{B})$ when $j = i$, we prove that

$$(m_{ki}+1)\mathbf{e}_k + ((m_{ki}+1)m_{ik} - 1)\mathbf{e}_i \notin \Delta^+(\mathfrak{B}).$$

We analyze several cases.

- If $m_{ki} = m_{ik} = 0$, we have $\mathbf{e}_k - \mathbf{e}_i \notin \Delta^+(\mathfrak{B})$.
- If $m_{ki} = m_{ik} = 1$, then $2\mathbf{e}_k + \mathbf{e}_i \notin \Delta^+(\mathfrak{B})$, because $(\mathrm{ad}\, x_k)^2 x_i = 0$.
- If $m_{ki} = 1, m_{ik} = 2$, then $2\mathbf{e}_k + 3\mathbf{e}_i \notin \Delta^+(\mathfrak{B})$, since we apply Lemma 5.5 to $\mathfrak{B}$ and it satisfies relation (5.7) by hypothesis.

- If $m_{ki}=2, m_{ik}=1$, then $3\mathbf{e}_k+2\mathbf{e}_i\notin\Delta^+(\mathfrak{B})$, as before.

Then (5.4) holds, for each $k\neq i$.

Now, consider $\theta\geq 3$, and $k,j\neq i$.

If $m_{ik}=m_{ij}=0$, then $s_i(m\mathbf{e}_k+\mathbf{e}_j)=m\mathbf{e}_k+\mathbf{e}_j$, and $(m_{kj}+1)\mathbf{e}_k+\mathbf{e}_j\notin\Delta^+(\mathfrak{B})$, since the quantum Serre relation holds in $\mathfrak{B}$.

If $m_{ik}=1, m_{ij}=0$, then $s_i(m\mathbf{e}_k+\mathbf{e}_j)=m\mathbf{e}_i+m\mathbf{e}_k+\mathbf{e}_j$. If we consider $x_j<x_i<x_k$ and look at the possible Lyndon words in S_I, from (5.4), it has no factors $x_i^2x_k, x_jx_i$, so the unique possibility is $x_j(x_kx_i)^m$.

- If $m_{kj}=0$, then $x_jx_kx_i=q_{jk}x_kx_jx_i$, so $x_jx_kx_i\notin S_I$.
- If $m_{kj}=1$, then $x_jx_kx_lx_k\notin S_I$ when $m_{ki}=1$, since (5.5) is valid in $\mathfrak{B}$, or when $m_{ki}=2$ we have $q_{kk}\neq -1$ and

$$\begin{aligned} x_j(x_kx_i)^2 =&(1+q_{kk})^{-1}q_{ki}^{-1}\ x_jx_k^2x_i^2+(1+q_{kk})^{-1}q_{ki}q_{kk}^2\ x_jx_ix_k^2x_i\\ =&q_{ki}^{-1}q_{kj}^{-1}q_{kk}^{-2}\ x_kx_jx_kx_i^2+(1+q_{kk})^{-1}q_{ki}^{-1}q_{kj}^{-2}q_{kk}^{-2}\ x_k^2x_jx_i^2\\ &+(1+q_{kk})^{-1}q_{ki}q_{kk}^2q_{ji}\ x_ix_jx_k^2x_i, \end{aligned}$$

so in both cases, $x_j(x_kx_i)^2\notin S_I$.

- If $m_{kj}=2$, then $m_{ki}=m_{jk}=1$ and $q_{kk}\neq -1$. The proof is similar to the previous case.

If $m_{ik}=2, m_{ij}=0$, then $s_i(m\mathbf{e}_k+\mathbf{e}_j)=2m\mathbf{e}_i+m\mathbf{e}_k+\mathbf{e}_j$ and $m_{kj}=0,1$. When $m_{kj}=0$, the proof is clear as above. When $m_{kj}=1$, for $j<k<i$ and considering only the quantum Serre relations, the unique possible Lyndon word is $x_j(x_kx_i^2)^2$. But from $[[x_i^2x_k]_c,[x_ix_k]_c]_c=0$, we deduce that such word is not in S_I.

If $m_{ik}=0, m_{ij}=1$, then $s_i(m\mathbf{e}_k+\mathbf{e}_j)=\mathbf{e}_i+m\mathbf{e}_k+\mathbf{e}_j$. If $k<i<j$, note that from $x_kx_i, x_k^{m_{kj}+1}x_j\notin S_I$, there are not Lyndon words of degree $\mathbf{e}_i+(m_{kj}+1)\mathbf{e}_k+\mathbf{e}_j$ in S_I.

If $m_{ik}=0, m_{ij}=2$, then $s_i(m\mathbf{e}_k+\mathbf{e}_j)=2\mathbf{e}_i+m\mathbf{e}_k+\mathbf{e}_j$, and the proof is analogous to the previous case.

If $m_{ik}=m_{ij}=1$, then $m_{kj}=0$, and $s_i(\mathbf{e}_k+\mathbf{e}_j)=2\mathbf{e}_i+\mathbf{e}_k+\mathbf{e}_j$, which is not in $\Delta^+(\mathfrak{B})$ from Lemma 5.4.

If $m_{ik}=2, m_{ij}=1$ (it is analogous to $m_{ik}=1, m_{ij}=2$), then $m_{kj}=0$ and $s_i(\mathbf{e}_k+\mathbf{e}_j)=3\mathbf{e}_i+\mathbf{e}_k+\mathbf{e}_j$. In this way, $q_{ii}\neq -1$ and if $x_k<x_i<x_j$, then the unique Lyndon word without $x_i^2x_j$, $x_kx_i^3$ as factors is

$$\begin{aligned} x_kx_i^2x_jx_i =&(1+q_{ii})^{-1}q_{ij}^{-1}x_kx_i^3x_j+(1+q_{ii})^{-1}q_{ii}^2q_{ij}x_kx_ix_jx_i^2\\ \in&\mathsf{k}(x_ix_kx_i^2x_j)+\mathsf{k}(x_i^2x_kx_ix_j)+\mathsf{k}(x_i^3x_kx_j)+\mathsf{k}(x_kx_ix_jx_i^2), \end{aligned}$$

using the quantum Serre relations, and then there are not Lyndon words of degree $3\mathbf{e}_i+\mathbf{e}_k+\mathbf{e}_j$ in S_I.

So, (5.4) holds, for each $k,j\neq i, k\neq j$. $\square$

STEP II: $s_i(\mathfrak{B})$ satisfies (5.5).

Proof. Consider $m_{kj} = m_{kl} = 1$. We prove case-by-case that

$$s_i(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l) \notin \Delta^+(\mathfrak{B}).$$

• $m_{ij} = m_{ik} = m_{il} = 0$, then $s_i(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l) = 2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l$, so it follows from Lemma 5.4, because $2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B})$.

• $m_{ij} \neq 0$ (analogously, $m_{il} \neq 0$), so $m_{ik} = m_{il} = 0$, because there are no cycles in the Dynkin diagram. Then $s_i(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l) = 2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l + m_{ij}\mathbf{e}_i$, and if we consider $x_k < x_l < x_j < x_i$, using that $x_k x_i = q_{ki} x_i x_k, x_j x_l = q_{jl} x_l x_j$ and $x_l x_i = q_{li} x_i x_l$, and also that $x_k^2 x_l, x_k^2 x_j \notin S_I$, we conclude that all possible Lyndon words of degree $2e_k + e_j + e_l + m_{ij} e_i$ are not elements of S_I, except $x_k x_l x_k x_j x_i^{m_{ij}}$, but also it is not an element of S_I, because $x_k x_l x_k x_j \notin S_I$. Then, $2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l + m_{ij}\mathbf{e}_i \notin \Delta^+(\mathfrak{B})$.

• $m_{ik} = 1$, and therefore $m_{ij} = m_{il} = 0$, then, $s_i(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l) = 2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l + 2m_{ik}\mathbf{e}_i$, and if we consider $x_l < x_i < x_k < x_j$, using that $x_j x_i = q_{ji} x_i x_j, x_j x_l = q_{jl} x_l x_j$ and $x_l x_i = q_{li} x_i x_l$, and also that $x_k^2 x_l, x_k^2 x_j \notin S_I$, we discard as before all possible Lyndon words of degree $2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l + 2m_{ik}\mathbf{e}_i$, except $x_l x_k x_j x_k x_i^{2m_{ij}}$, but it is not an element of S_I, because $x_k x_l x_k x_j \notin S_I$. Then $2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l + 2m_{ij}\mathbf{e}_i \notin \Delta^+(\mathfrak{B})$.

• $i = j$ (or analogously, $i = l$): $s_j(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l) = 2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B})$ if $m_{jk} = 1$ by Lemma 5.4, or $s_j(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l) = 2\mathbf{e}_k + 3\mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B})$ if $m_{jk} = 2$, by Lemma 5.5.

• $i = k$: $s_k(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l) = \mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B})$, since $m_{jl} = 0$.

Also, if $\mathbf{u} \in \{\mathbf{e}_k + \mathbf{e}_j, \mathbf{e}_k + \mathbf{e}_l, \mathbf{e}_k, \mathbf{e}_j, \mathbf{e}_l\}$, then $\mathbf{u} \in \Delta(\mathfrak{B}(V_i))$, so $s_i(\mathbf{u}) \in \Delta(\mathfrak{B}(V))$. The canonical surjective algebra morphisms from $T(V)$ to $\mathfrak{B}$ and $\mathfrak{B}(V)$ induce a surjective algebra morphism $\mathfrak{B} \to \mathfrak{B}(V)$, so $\Delta(\mathfrak{B}(V)) \subseteq \Delta(\mathfrak{B})$; in particular, each $s_i(\mathbf{u}) \in \Delta(\mathfrak{B})$.

Consider a basis as in Proposition 1.11 for an order such that $x_j < x_k < x_l$. From Lemma 2.7, $x_j x_k$, $x_k x_l$, $x_j x_k x_l$ are elements of this basis, since they are not linear combination of greater words modulo I_i, the ideal of $T(V_i)$ such that $s_i(\mathfrak{B}) = T(V_i)/I_i$. In the same way, $(x_k x_l)(x_j x_k)$, $x_l x_k (x_j x_k)$, $(x_k x_l) x_k x_j$, $x_k (x_j x_k x_l)$, $x_l x_k^2 x_j$ (if $x_k^2 \neq 0$) are elements of such basis, where the parenthesis indicates the Lyndon decomposition as non increasing products of Lyndon words. Also, $x_j x_l$, $x_j x_k^2$, $x_k^2 x_l$ are not in such basis by (5.4). By the relations between the Hilbert series in Theorem 2.6 and the fact that $2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l \notin s_i\left(\Delta^+(\mathfrak{B})\right)$, we note that $x_j x_k x_l x_k$ is not an element of such basis. Then, this word is a linear combination of greater words. By Lemma 5.4, this implies that (5.5) holds in $s_i(\mathfrak{B})$. □

Step III: $s_i(\mathfrak{B})$ satisfies (5.7).

Proof. As before, we prove first that $s_i(3\mathbf{e}_k + 2\mathbf{e}_j) \notin \Delta^+(\mathfrak{B})$ case by case:

• $m_{ik} = m_{ij} = 0$, then $s_i(3\mathbf{e}_k + 2\mathbf{e}_j) = 3\mathbf{e}_k + 2\mathbf{e}_j \notin \Delta^+(\mathfrak{B})$ by hypothesis.

• $m_{ik} = 0, m_{ij} = 1$, then $s_i(3\mathbf{e}_k + 2\mathbf{e}_j) = 2\mathbf{e}_i + 3\mathbf{e}_k + 2\mathbf{e}_j$. If consider the order in the letters $x_k < x_i < x_j$, a Lyndon word of degree $2e_i + 3e_k + 2e_j$

in S_I begins with x_k, and x_kx_i is not a factor, because $x_kx_i = q_{ki}x_ix_j$. Then the possible Lyndon words with these conditions are $x_k^2x_jx_ix_kx_jx_i$ and $x_k^2x_jx_kx_jx_i^2$; the first is not in S_I because from (5.5) for j,k,i we can express $x_jx_ix_kx_j$ as a linear combination of greater words, and the second is not in S_I because $x_k^2x_jx_kx_j \notin S_I$.

• $m_{ik} = 1, m_{ij} = 0$, then $s_i(3\mathbf{e}_k + 2\mathbf{e}_j) = 3\mathbf{e}_i + 3\mathbf{e}_k + 2\mathbf{e}_j$. If consider the order in the letters $x_j < x_i < x_k$, a Lyndon word of degree $3e_i + 3e_k + 2e_j$ in S_I begins with x_j, and x_jx_i is not a factor. Using that also $x_i^2x_k, x_j^2x_k \notin S_I$, the possible Lyndon word with these conditions is $x_jx_kx_ix_jx_kx_ix_kx_i$. But from the condition in the m_{rs}'s, we are in cases C_θ or F_4, and we use that $(\operatorname{ad} x_i)^2x_k = 0$, $q_{ii} \neq -1$ to replace $x_ix_kx_i$ by a linear combination of $x_i^2x_k$ and $x_kx_i^2$, and also use $x_jx_i = q_{ji}x_ix_j$, so we conclude that $x_jx_kx_ix_jx_kx_ix_kx_i \notin S_I$.

• $i = j$: $s_j(3\mathbf{e}_k + 2\mathbf{e}_j) = 3\mathbf{e}_k + \mathbf{e}_j \notin \Delta^+(\mathfrak{B})$, since $m_{kj} = 2$.

• $i = k$: $s_k(3\mathbf{e}_k + 2\mathbf{e}_j) = \mathbf{e}_k + 2\mathbf{e}_j \notin \Delta^+(\mathfrak{B})$, since $m_{jk} = 1$.

If $\mathbf{v} \in \{\mathbf{e}_k+\mathbf{e}_j, 2\mathbf{e}_k+\mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_j\}$, then $\mathbf{v} \in \Delta(\mathfrak{B}(V_i))$, so $s_i(\mathbf{v}) \in \Delta(\mathfrak{B}(V))$. As $\Delta(\mathfrak{B}(V)) \subseteq \Delta(\mathfrak{B})$, in particular we have that each $\mathbf{v} \in s_i(\Delta(\mathfrak{B}))$.

As in *a)*, consider a basis as in Proposition 1.11 for an order such that $x_k < x_j$. In a similar way, x_kx_j, $x_k^2x_j$ are elements of this basis, but $x_k^3x_j$ and $x_kx_j^2$ are not in such basis by (5.4). From Lemma 2.7, $(x_kx_j)(x_k^2x_j)$, $x_j(x_k^2x_j)x_k$, $(x_kx_j)^2x_k$, $x_j(x_kx_j)x_k^2$, $x_j^2x_k^3$ (the last if $x_j^2, x_k^3 \neq 0$) are not linear combination of greater words modulo I_i, so they are elements of previous basis. And by the relations between the Hilbert series and the fact that $3\mathbf{e}_k+2\mathbf{e}_j \notin s_i(\Delta^+(\mathfrak{B}))$, we note that the Lyndon word $x_k^2x_jx_kx_j$ is not an element of such basis. Then, this word is a linear combination of greater words, and by Lemma 5.5, this implies that (5.7) holds in $s_i(\mathfrak{B})$. □

Step IV: $s_i(\mathfrak{B})$ satisfies (5.9).

Proof. We prove case-by-case that

$$s_i(3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l) \notin \Delta^+(\mathfrak{B}).$$

• $m_{ik} = m_{ij} = m_{il} = 0$: $s_i(3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l) = 3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l$, and it is not in $\Delta^+(\mathfrak{B})$ by Lemma 5.6.

• $i \neq j,k,l$ and $m_{ik} \neq 0$: the unique possibility is $m_{ik} = m_{ki} = 1$, so V is of type F_4. Then, $s_i(3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l) = 3\mathbf{e}_i + 3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l$. For the order $x_l < x_j < x_k < x_i$, the unique possible Lyndon word without factors $x_lx_j^2$, x_lx_k, x_lx_i, $x_j^2x_k$, x_jx_i, $x_kx_i^2$, $x_k^2x_i$ is $x_lx_jx_kx_ix_jx_kx_ix_kx_i$. Using the quantum Serre relations, and the fact that $q_{ii} = q_{kk} \neq -1$, we obtain that this Lyndon word is not in S_I. Then, $3\mathbf{e}_i + 3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B})$.

• $i \neq j,k,l$ and $m_{ij} \neq 0$: there are not standard braided vector spaces with these m_{st}'s.

• $i \neq j,k,l$ and $m_{il} \neq 0$: the unique possibility is $m_{il} = m_{li} = 1$. Then, $s_i(3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l) = 3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l + \mathbf{e}_i$ If we consider $x_k < x_j < x_l < x_i$, then the unique possible Lyndon word of such degree without factors x_kx_l, x_kx_i, x_jx_i, $x_k^3x_j$, $x_kx_j^2$ is $x_k^2x_jx_lx_ix_kx_i$. But by hypothesis,

$$\left[\left[x_k^2x_jx_l\right]_c, [x_kx_j]_c\right]_c = \left[x_i, [x_kx_j]_c\right]_c = 0,$$

so $\left[x_k^2x_jx_lx_ix_kx_i\right]_c = \left[\left[x_k^2x_jx_lx_i\right]_c, [x_kx_j]_c\right]_c = 0$, and $x_k^2x_jx_lx_ix_kx_i \notin S_I$.

- $i = k$: $s_i(3\mathbf{e}_i + 2\mathbf{e}_j + \mathbf{e}_l) = \mathbf{e}_i + 2\mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B})$, by Lemma 5.4.
- $i = j$: $s_i(3\mathbf{e}_k + 2\mathbf{e}_i + \mathbf{e}_l) = 3\mathbf{e}_k + 2\mathbf{e}_i + \mathbf{e}_l \notin \Delta^+(\mathfrak{B})$, by Lemma 5.6.
- $i = k$: $s_i(3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_i) = \mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_i \notin \Delta^+(\mathfrak{B})$, as before.

Now, if $\mathbf{w} \in \{\mathbf{e}_k, \mathbf{e}_j, \mathbf{e}_l, \mathbf{e}_k+\mathbf{e}_j, \mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l, 2\mathbf{e}_k+\mathbf{e}_j, 2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l, 2\mathbf{e}_k+2\mathbf{e}_j+\mathbf{e}_l\}$, then $\mathbf{w} \in \Delta(\mathfrak{B}(V_i))$, so $s_i(\mathbf{w}) \in \Delta(\mathfrak{B}(V))$, and then $s_i(\mathbf{w}) \in \Delta(\mathfrak{B})$.

Consider a basis as in Proposition 1.11 for an order such that $x_k < x_j < x_l$. Then, x_jx_k, x_kx_l are elements of this basis. We know x_kx_l, $x_k^3x_j$, $x_kx_j^2$, $x_kx_jx_lx_k$, $x_k^2x_jx_kx_j$ are not elements of such basis, since in $\mathfrak{B}$ hold (5.4), (5.5) and (5.7). By Lemma 2.7, the relations between the Hilbert series in Theorem 2.6 and the fact that $3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l \notin s_i\left(\Delta^+(\mathfrak{B})\right)$, we note that the Lyndon word $x_k^2x_jx_lx_kx_j$ is not an element of such basis. Thus this word is a linear combination of greater words. By Lemma 5.6, this implies that (5.9) holds in $s_i(\mathfrak{B})$. □

As $s_i(\mathfrak{B})$ is of the same type as $\mathfrak{B}$, we conclude the proof. □

Let V of type different of G_2. We define the algebra $\hat{\mathfrak{B}}(V) := T(V)/\mathfrak{I}(V)$, where $\mathfrak{I}(V)$ is the 2-sided ideal of $T(V)$ generated by

- $(\operatorname{ad}_c x_k)^{m_{kj}+1}x_j$, $k \neq j$;
- $\left[(\operatorname{ad}_c x_j)(\operatorname{ad}_c x_k)x_l, x_k\right]_c$, $l \neq k \neq j$, $q_{kk} = -1$, $m_{kj} = m_{kl} = 1$;
- $\left[(\operatorname{ad}_c x_k)^2x_j, (\operatorname{ad}_c x_k)x_j\right]_c$, $k \neq j$, $q_{kk} \in \mathbb{G}_3$ or $q_{jj} = -1$, $m_{kj} = 2, m_{jk} = 1$;
- $\left[(\operatorname{ad}_c x_k)^2(\operatorname{ad}_c x_j)x_l, (\operatorname{ad}_c x_k)x_j\right]_c$, $k \neq j \neq l$, $q_{kk} \in \mathbb{G}_3$ or $q_{jj} = -1$, $m_{kj} = 2, m_{jk} = m_{jl} = 1$.

Compare with the definitions in [AS3, Section 4]. Since V is of Cartan type, $\mathfrak{I}(V)$ is a Hopf ideal, by Lemmata 5.7, 5.8 and 5.9. As also $\mathfrak{I}(V)$ is $\mathbb{Z}^\theta$-homogeneous, we have $\mathfrak{I}(V) \in \mathfrak{S}$.

By Lemmata 5.4, 5.5 and 5.6, the canonical epimorphism $T(V) \to \mathfrak{B}(V)$ induces a epimorphism of braided graded Hopf algebras

$$\pi_V : \hat{\mathfrak{B}}(V) \to \mathfrak{B}(V). \tag{5.39}$$

Also, $\hat{\mathfrak{B}}(V)$ satisfies for each $i \in \{1, \dots, \theta\}$ conditions on Theorem 2.6, so we can transform it.

Lemma 5.24. *With the above notation, $s_i(\hat{\mathfrak{B}}(V)) \cong \hat{\mathfrak{B}}(V_i)$.*

Proof. By Lemma 5.23, the relations defining $\mathfrak{I}(V_i)$ are satisfied in $s_i(\hat{\mathfrak{B}}(V))$. Then, the canonical projections from $T(V_i)$ onto $\hat{\mathfrak{B}}(V_i)$, $s_i\left(\hat{\mathfrak{B}}(V)\right)$ induce a surjective algebra map $\hat{\mathfrak{B}}(V_i) \to s_i\left(\hat{\mathfrak{B}}(V)\right)$. Reciprocally, each relation defining $\mathfrak{I}(V)$ is satisfied in $s_i(\hat{\mathfrak{B}}(V_i))$, so we have the following situation:

$$\begin{tikzcd}
\hat{\mathfrak{B}}(V) \arrow[r, two heads] \arrow[dr, squiggly] & s_i\left(\hat{\mathfrak{B}}(V_i)\right) \\
\hat{\mathfrak{B}}(V_i) \arrow[r, two heads] \arrow[ur, squiggly] & s_i\left(\hat{\mathfrak{B}}(V)\right).
\end{tikzcd}$$

From the relation between the Hilbert series in Theorem 2.6, for each $\mathbf{u} \in \mathbb{N}^\theta$ we have

$$\dim s_i(\hat{\mathfrak{B}}(V))^{\mathbf{u}} = \sum_{k\in\mathbb{N}:\ \mathbf{u}-k\mathbf{e}_i\in\mathbb{N}^\theta,\ s_i(\mathbf{u}-k\mathbf{e}_i)\in\mathbb{N}^\theta} \dim \hat{\mathfrak{B}}(V)^{s_i(\mathbf{u}-k\mathbf{e}_i)},$$

and a analogous relation for $\dim s_i(\hat{\mathfrak{B}}(V_i))^{\mathbf{u}}$. But from the previous surjections we have

$$\dim s_i(\hat{\mathfrak{B}}(V))^{\mathbf{u}} \leq \dim \hat{\mathfrak{B}}(V_i)^{\mathbf{u}}, \quad \dim s_i(\hat{\mathfrak{B}}(V_i))^{\mathbf{u}} \leq \dim \hat{\mathfrak{B}}(V)^{\mathbf{u}},$$

for each $\mathbf{u} \in \mathbb{N}^\theta$. Using that $s_i^2 = \mathrm{id}$, each of above inequalities is in fact an equality, and $s_i(\hat{\mathfrak{B}}(V)) = \hat{\mathfrak{B}}(V_i)$. □

We are now able to prove one of the main results of this paper.

Theorem 5.25. *Let V be a braided vector space of Cartan type, of dimension θ, and $C = (a_{ij})_{i,j\in\{1,\dots,\theta\}}$ the corresponding finite Cartan matrix, where $a_{ij} := -m_{ij}$.*

The Nichols algebra $\mathfrak{B}(V)$ is presented by generators x_i, $1 \leq i \leq \theta$, and relations

$$x_\alpha^{N_\alpha} = 0, \quad \alpha \in \Delta^+; \tag{5.40}$$

$$ad_c(x_k)^{1-a_{kj}}(x_j) = 0, \quad k \neq j; \tag{5.41}$$

if there exist $j \neq k \neq l$ such that $m_{kj} = m_{kl} = 1$, $q_{kk} = -1$, then

$$[(\operatorname{ad} x_k)x_j, (\operatorname{ad} x_k)x_l]_c = 0; \tag{5.42}$$

if there exist $k \neq j$ such that $m_{kj} = 2, m_{jk} = 1$, $q_{kk} \in \mathbb{G}_3$ or $q_{jj} = -1$, then

$$\left[(\operatorname{ad} x_k)^2 x_j, (\operatorname{ad} x_k)x_j\right]_c = 0; \tag{5.43}$$

if there exist $k \neq j \neq l$ such that $m_{kj} = 2, m_{jk} = m_{jl} = 1$, $q_{kk} \in \mathbb{G}_3$ or $q_{jj} = -1$, then

$$\left[(\operatorname{ad} x_k)^2(\operatorname{ad} x_j)x_l, (\operatorname{ad} x_k)x_j\right]_c = 0; \tag{5.44}$$

if $\theta = 2$ and V if of G_2 type, $q_{11} \in \mathbb{G}_4$ or $q_{22} = -1$, then

$$\left[(\operatorname{ad} x_1)^3 x_2, (\operatorname{ad} x_1)^2 x_2\right]_c = 0, \tag{5.45}$$

$$\left[x_1, \left[x_1^2 x_2 x_1 x_2\right]_c\right]_c = 0, \tag{5.46}$$

$$\left[\left[x_1^2 x_2 x_1 x_2\right]_c, \left[x_1 x_2\right]_c\right]_c = 0, \tag{5.47}$$

$$\left[\left[x_1^2 x_2\right]_c, \left[x_1^2 x_2 x_1 x_2\right]_c\right]_c = 0. \tag{5.48}$$

Moreover, the following elements constitute a basis of $\mathfrak{B}(V)$:

$$x_{\beta_1}^{h_1} x_{\beta_2}^{h_2} \dots x_{\beta_P}^{h_P}, \qquad 0 \le h_j \le N_{\beta_j} - 1, \text{ if } \beta_j \in S_I, \quad 1 \le j \le P.$$

Proof. We may assume that C is connected. If V is of type G_2, then the Theorem was proved in Theorem 5.22. So we can assume $m_{kj} \neq 3$, $k \neq j$.

The statement about the PBW basis was proved in Corollary 4.2 – see the definition of the x_α's in Subsection 4.2.

Consider the image of x_α in $\hat{\mathfrak{B}}(V)$; they correspond in $\mathfrak{B}(V)$ with x_α, and are PBW generators for a basis constructed as in Theorem 1.12, considering the same order in the letters. As we observed in (5.39), there exists a surjective morphism of braided Hopf algebras $\hat{\mathfrak{B}}(V) \to \mathfrak{B}(V)$, so

$$\Delta(\mathfrak{B}(V)) \subseteq \Delta(\hat{\mathfrak{B}}(V)).$$

Also, $\hat{\mathfrak{B}}(V)$ verifies the conditions in Theorem 2.6 for each $i \in \{1, \dots, \theta\}$, so we can transform it. By Lemma 5.24, the new algebra is $\hat{\mathfrak{B}}(V_i)$. So, we can continue. Then, consider the sets

$$\begin{aligned} \hat{\Delta} &:= \cup\{\Delta(s_{i_1} \cdots s_{i_k} \hat{\mathfrak{B}}) : k \in \mathbb{N}, 1 \le i_1, \dots, i_k \le \theta\}, \\ \hat{\Delta}^+ &:= \Delta \cap \mathbb{N}^\theta; \end{aligned}$$

and then $\hat{\Delta}$ is invariant by the s_i's. Also, $\Delta(\mathfrak{B}(V)) \subseteq \Delta$, and

$$\Delta(s_{i_1} \cdots s_{i_k} \hat{\mathfrak{B}}(V)) = s_{i_1} \cdots s_{i_k} \Delta(\hat{\mathfrak{B}}(V)).$$

Consider $\alpha \in \hat{\Delta}^+ \setminus \Delta^+(\mathfrak{B}(V))$. Suppose that $\alpha \neq m\alpha_i$, for all $m \in \mathbb{N}$ and $i \in \{1, \dots, \theta\}$, of minimal height among these roots. For each s_i, as α is not a multiple of α_i, we have $s_i(\alpha) \in \Delta^+ \setminus \Delta^+(\mathfrak{B}(V))$, and then $gr(s_i(\alpha)) - gr(\alpha) \ge 0$. But $\alpha = \sum_{i=1}^\theta b_i \mathbf{e}_i$, so $\sum_{i=1}^\theta b_i a_{ij} \le 0$, and as $b_i \ge 0$, we have $\sum_{i,j=1}^\theta b_i a_{ij} b_j \le 0$. This contradicts the fact that (a_{ij}) is definite positive, and $(b_i) \ge 0, (b_i) \neq 0$.

Also, $m\mathbf{e}_i \in \Delta^+(\hat{\mathfrak{B}}) \Leftrightarrow m = N_{\mathbf{e}_i}, 1$, since $x_i^{N_{\mathbf{e}_i}} \neq 0$. Then,

$$\Delta(\hat{\mathfrak{B}}(V)) = \Delta(\mathfrak{B}(V)) \cup \{N_\alpha \alpha : \alpha \in \Delta(\mathfrak{B}(V))\}.$$

It follows since by Corollary 4.2 each $\alpha \in \Delta^+(\mathfrak{B}(V))$ is of the form

$$\alpha = s_{i_1} \cdots s_{i_m}(\mathbf{e}_j), \quad i_1, \dots, i_m, j \in \{1, \dots, \theta\}.$$

Now, $N_{\mathbf{e}_j} \mathbf{e}_j \in \Delta(\hat{\mathfrak{B}}(V))$, so

$$N_\alpha \alpha = N_{\mathbf{e}_j} \alpha = s_{i_1} \cdots s_{i_m}(N_{\mathbf{e}_j} \mathbf{e}_j) \in \Delta(\hat{\mathfrak{B}}(V)).$$

Also, each degree $N_\alpha \alpha$ has multiplicity one too in $\Delta(\hat{\mathfrak{B}}(V))$.

Now, for degrees $N_\alpha \alpha$, suppose that there are some Lyndon words of degree $N_\alpha \alpha$, and consider one of them of minimal height. This word u has a Shirshov decomposition

$$u = vw, \qquad \beta := \deg(v), \ \gamma := \deg(w) \in \Delta^+(\hat{\mathfrak{B}}(V)).$$

From the previous assumption, we have $\beta, \gamma \in \Delta^+(\mathfrak{B}(V))$. Write

$$\alpha = \sum_{k=1}^{\theta} a_k \mathbf{e}_k, \quad \beta = \sum_{k=1}^{\theta} b_k \mathbf{e}_k, \quad \gamma = \sum_{k=1}^{\theta} c_k \mathbf{e}_k,$$

so $N_\alpha a_k = b_k + c_k$, for each $k \in \{1, \dots, \theta\}$. We can consider $a_1, a_\theta \neq 0$ (if not, we look at a smaller subdiagram).

Now, if we consider V of type F_4 and $\beta = 2\mathbf{e}_1 + 3\mathbf{e}_2 + 4\mathbf{e}_3 + 3\mathbf{e}_4$, then $c_1 = 0$, $a_1 = 1$, $N_\alpha = 2$, or $a_1 = c_1 = 1$, $N_\alpha = 3$, since $\alpha, \gamma \neq \beta$.

- If $N_\alpha = 3$, then $3a_2 = 3 + c_2$. Then, $c_2 = 0$, so $c_3 = c_4 = 0$, or $c_2 = 3$, and $c_3 = 4$, $c_4 = 2$. But in both cases we have a contradiction to $\alpha \in \mathbb{N}^4$.
- If $N_\alpha = 2$, $c_1 = 0$, then c_2, c_4 are odd, and c_3 is even, not zero. The unique possibility is $\gamma = \mathbf{e}_2 + 2\mathbf{e}_3 + \mathbf{e}_4$, so $\alpha = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3 + 2\mathbf{e}_4$. But $q_\alpha = q \neq -1$, so $N_\alpha \neq 2$, which is a contradiction.

Thus, we can consider $b_1, c_1 \leq 1$ or $b_\theta, c_\theta \leq 1$, so $a_1 = b_1 = c_1 = 1$ or $a_\theta = b_\theta = c_\theta = 1$, and in both cases, $N_\alpha = 2$. For each possible β with $b_1 \neq 0$ (by assumption of $a_1 \neq 0$, we have $b_1 \neq 0$ or $c_1 \neq 0$, we look for γ such that $\beta + \gamma$ has even coordinates. In types A, D and E there are not such pairs of roots. For the other types,

(1) B_θ: $\beta = \mathbf{v}_{i\theta}$, $\gamma = \mathbf{u}_{i+1,\theta}$. Then, $\alpha = \mathbf{u}_{1\theta}$, but $q_\alpha = q_{11} \neq -1$, which is a contradiction.
(2) C_θ: $\beta = \mathbf{w}_{11}$, $\gamma = \mathbf{e}_\theta$. Then, $\alpha = \mathbf{u}_{1\theta}$, but $q_\alpha = q_{\theta\theta} \neq -1$, which is a contradiction.
(3) F_4: $\beta = \mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_3 + 2\mathbf{e}_4$, $\gamma = \mathbf{e}_1 + \mathbf{e}_2$, or $\beta = \mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3 + 2\mathbf{e}_4$, $\gamma = \mathbf{e}_1$. In both cases, $\alpha = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4$, but $q_\alpha = q \neq -1$, which is a contradiction.

Then, each root $N_\alpha \alpha$ corresponds to $x_\alpha^{N_\alpha}$, and each x_α as before has infinite height. The elements

$$x_{\beta_1}^{h_1} x_{\beta_2}^{h_2} \dots x_{\beta_P}^{h_P}, \qquad 0 \leq h_j < \infty, \text{ if } \beta_j \in S_I, \quad 1 \leq j \leq P,$$

constitute a basis of $\hat{\mathfrak{B}}(V)$ as vector space.

Now, let $\bar{I}(V)$ be the ideal of $T(V)$ generated by relations (5.41), (5.42), (5.43), (5.44), and also (5.40). Then we have $\mathfrak{I}(V) \subseteq \bar{I}(V) \subseteq I(V)$, so the corresponding projections induce a surjective morphisms of algebras $\phi : \mathfrak{B} \to \mathfrak{B}(V)$, where $\mathfrak{B} := T(V)/\bar{I}(V)$.

$$\begin{tikzcd} T(V) \arrow[r, two heads] \arrow[d, two heads] & \hat{\mathfrak{B}}(V) \arrow[dl, two heads] \arrow[d, two heads] \\ \mathfrak{B}(V) & \mathfrak{B} \arrow[l, two heads, "\phi"'] \end{tikzcd}$$

Also, the elements

$$x_{\beta_1}^{h_1} x_{\beta_2}^{h_2} \dots x_{\beta_P}^{h_P}, \qquad 0 \leq h_j < N_{\beta_j}, \text{ if } \beta_j \in S_I, \quad 1 \leq j \leq P,$$

generate $\mathfrak{B}$ as vector space, because they correspond to the image of elements that generate $\hat{\mathfrak{B}}(V)$, and are not zero (each non increasing product of hyperwords as before such that $h_j \geq N_{\beta_j}$ is zero in $\mathfrak{B}$). But also ϕ is surjective, and the corresponding images of these elements constitute a basis of $\mathfrak{B}(V)$, so ϕ is an isomorphism.

□

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References

[A] N. Andruskiewitsch, *About finite dimensional Hopf algebras.* Notes of a course given at the CIMPA School "Quantum symmetries in theoretical physics and mathematics", Bariloche 2000. Contemp. Math 294, 1–57 (2002).

[AA] N. Andruskiewitsch and I. Angiono, *On Nichols algebras with generic braiding.* Abstract and file, arXiv:math/0703924v2. Accepted in "Modules and Comodules", Proceedings of a conference dedicated to Robert Wisbauer, 17 pp.

[AD] N. Andruskiewitsch and S. Dascalescu, *On finite quantum groups at -1.* Algebr. Represent. Theory 8 (2005), no. 1, 11–34.

[AG] N. Andruskiewitsch and M. Graña, *Braided Hopf algebras over non-abelian groups.* Bol. Acad. Nac Cienc. (Córdoba) 63, 45-78 (1999).

[AHS] N. Andruskiewitsch, I. Heckenberger and H.-J. Schneider, *The Nichols algebra of a semisimple Yetter-Drinfeld module.* Abstract and file, arXiv:0803.2430v1.

[AJS] H.H. Andersen, J. Jantzen and W. Soergel, *Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: Independence of p.* Astérisque, Vol. 220, 1994.

[AS1] N. Andruskiewitsch and H.-J. Schneider, *Lifting of quantum linear spaces and pointed Hopf algebras of order p^3.* J. Algebra 209, pp. 658-691 (1998). (MR 99k:16075).

[AS2] N. Andruskiewitsch and H.-J. Schneider, *Finite quantum groups and Cartan matrices.* Adv. Math 54, 1–45 (2000). (MR 2001g: 16070).

[AS3] N. Andruskiewitsch and H.-J. Schneider. *Finite quantum groups over abelian groups of prime exponent.* Ann. Sci. Ec. Norm. Super. 35 (2002), 1-26. (MR 2003a:16055).

[AS4] N. Andruskiewitsch and H.-J. Schneider, *A characterization of quantum groups.* J. Reine Angew. Math. 577 (2004), 81-104.

[AS5] N. Andruskiewitsch and H.-J. Schneider, *Pointed Hopf algebras.* En *Recent developments in Hopf algebra Theory*, MSRI Publications 43 (2002), 1-68, Cambridge Univ. Press. (MR 2003a:16002).

[AS6] N. Andruskiewitsch and H.-J. Schneider, *On the classification of finite-dimensional pointed Hopf algebras.* `math.QA/0502157`. Ann. Math., accepted. 43 pp.

[B] N. Bourbaki, *Groupes et alg'ebres de Lie, Ch. 4, 5 et 6.* Éléments de mathematique. Hermann, Paris, 1968.

[dCP] C. de Concini and C. Procesi *Quantum Groups*, in "D-modules, Representation theory and Quantum Groups", 31–140, Lecture Notes in Maths. **1565** (1993), Springer-Verlag.

[Dr] V. Drinfeld, *Quantum groups*, Proceedings of the ICM Berkeley 1986, A.M.S.

[Gr] M. Graña, *On Nichols algebras of low dimension*, New trends in Hopf algebra theory (La Falda, 1999), Contemp. Math., vol. 267, Amer. Math. Soc. 2000, 111-134.

[H1] I. Heckenberger, *Rank 2 Nichols algebras with finite arithmetic root system.* Preprint math.QA/0412458. Accepted in Algebr. and Represent. Theory.

[H2] I. Heckenberger, *The Weyl groupoid of a Nichols algebra of diagonal type*, Inventiones Math. 164, 175–188 (2006).

[H3] I. Heckenberger, *Classification of arithmetic root systems*, `math.QA/0605795`.

[H4] I. Heckenberger, *Examples of finite dimensional rank 2 Nichols algebras of diagonal type.* Compositio Math. 143 (2007) 165-190

[Ji] M. Jimbo, *A q-difference analogue of $U(g)$ and the Yang Baxter equation*, Lett. Math. Phys. 10 (1985), pp. 63–69.

[K] V. Kac, *Infinite-dimensional Lie algebras*, 3rd Edition, Cambridge University Press, Cambridge, 1990.

[Kh] V. Kharchenko, *A quantum analog of the Poincare-Birkhoff-Witt theorem*, Algebra and Logic, 38, (1999), 259-276.

[LR] P. Lalonde and A. Ram, *Standard Lyndon bases of Lie algebras and enveloping algebras*, Trans. Amer. Math. Soc. 137, (1995), 1821-1830.

[L1] G. Lusztig, *Finite dimensional Hopf algebras arising from quantized universal enveloping algebras*, J. of Amer. Math. Soc. **3** 257–296.

[L2] G. Lusztig, *Quantum groups at roots of 1*, Geom. Dedicata **35** (1990), 89–114.

[L3] G. Lusztig, *Introduction to quantum groups*, Birkhäuser (1993).

[M] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Lecture Notes 82, Amer. Math. Soc., 1993.

[R] M. Rosso, *Quantum groups and quantum shuffles*, Inventiones Math. 133 (1998), 399–416.

[R2] M. Rosso, *Lyndon words and Universal R-matrices*, talk at MSRI, October 26, 1999, available at `http://www.msri.org`; *Lyndon basis and the multiplicative formula for R-matrices*, preprint (2003).

[Sw] M.E. Sweedler, *Hopf algebras*, Benjamin, New York, 1969.

[U] S. Ufer, *PBW bases for a class of braided Hopf algebras*, J. of Alg. 280 (2004) 84-119.

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