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On characterizing the solution for the Fermat–Weber location problem

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On characterizing the solution for the Fermat–Weber location problem

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Abstract

The Fermat–Weber problem consists in finding a point in $\mathbb{R}^n$ that minimizes the weighted sum of distances from m points in $\mathbb{R}^n$ that are not collinear. The Fermat problem is a particular case of the Weber problem. In this article we are interested in characterize the solution of both problems. To do this we show some theoretical results when there exists a symmetry axis. Moreover, we show an approach to generalize that results in absent of symmetry. Finally, we found a family of three–points Weber problem whose solution is the origin of the coordinate system.

Keywords: Fermat–Weber problem, optimal location, symmetry, geometrical center.

1 Introduction

We consider two strongly related location problems. The first one was early proposed by Fermat in the seventeenth century: given three points in the plane, find a fourth point (Fermat point) such that the sum of the distance to the three given points is minimized. The former solution of this problem were given by Torricelli and Simpson by using geometrical ideas. In fact, Simpson generalized the Fermat problem to asking for the minimum weighted sum of distances from three given points in 1750. The economical formulation and a new generalization was given by the economist Alfred Weber in 1909 [14]. This industry location problem consists in finding an optimal point (Weber point) where locate a new industry in order to minimize the weighted

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distances from this point to the raw material manufacturers and to the market to sell the production.

Both problem are strongly related, therefore understanding and having a deep knowledge of the Fermat problem will allow us to understand the Weber problem. Despite of the simplicity of the problem formulation these problems are not easy to solve. Note that a numerical algorithm was proposed by Weiszfeld in 1937 [15] and many articles were recently published about the convergence of the Weiszfeld's algorithm and improvements about it [8, 3, 13, 2]. From a geometric point of view we found many articles about both problems. For instance, historical details and perspectives about those problems in [16, 9]. Sokolowsky [12] and Martelli [10] studied the Fermat problem determining geometrically the solution. Pesamosca [11] obtained some analytic results for the 3–point Weber Problem in 1991. Other results, for arbitrary (positive and negative) weights were established recently in [6, 7, 4, 5]. An interesting revision of historical methods for solving these problems is proposed by Cañavate Bernal and Cobacho Tornal in [1].

In this paper we are interested in characterize exact solutions for the Fermat and Weber problems. First of all, in Section 1, we establish some relationships between geometric center and the Fermat point. Given m points equally weighted, and assuming that exist axial symmetry, we give a new proof to locate the Fermat point. Our approach is based on a Theorem cited in [1]. The formulation of Fermat and Weber problems are given in Section 2. In Section 3, we obtain a characterization of the Fermat point for the symmetric case. When no symmetry is present we also characterize the solution giving a different proof. The Weber problem in the symmetric case is considered in Section 3 and we generalize the results proved for the Fermat problem. Finally, given three points we characterize the solution for the bidimensional Weber problem in Section 4. Conclusions are given in Section 5.

2 The Fermat–Weber problem

The Weber problem can be formulated as an optimization problem in the following way. Let $a_1, \ldots, a_m$ be m distinct points in $\mathbb{R}^n$ and $w_1, \ldots, w_m$ be m positive associated weights. The Fermat–Weber location problem is to find a point $y \in \mathbb{R}^n$, different from $a_k, k = 1, \ldots, m$, which minimizes the cost–function:

$$W(X) = \sum_{i=1}^{m} w_i \, \|X - a_i\| \tag{1}$$

where $\| \cdot \|$ is the euclidean distance.

The Fermat problem is given by the particular case in which $n = 2$, $m = 3$ and weights $w_i = 1$ for all $i = 1, \ldots, m$. In order to generalize our ideas, we consider the extended Fermat problem to m points. That is, we define de Fermat function $F : \mathbb{R}^2 \to \mathbb{R}$, by

$$F(X) = \sum_{i=1}^{m} \|X - a_i\|, \tag{2}$$

for which, we want to minimize $F(X)$ subject to $X \in \mathbb{R}^2$. The solution of this minimization problem is the Fermat point.

3 Characterization of the Fermat point

We will show some theoretical results in order to characterize the Fermat point by using symmetry properties. First of all, we analyze the Fermat function. It is clear that the function F is continuously differentiable in $\mathbb{R}^2 - \{a_1, \ldots, a_m\}$. Since the points $a_1, \ldots, a_m$ are noncollinear it is easy to see that F is a strictly convex function and, consequently, it has a unique mininimizer.

The minimizer of the Fermat function has to satisfy the necessary first order optimality condition. That is

$$\nabla F(X) = \sum_{i=1}^{m} \frac{X - a_i}{\|X - a_i\|} = 0. \tag{3}$$

Moreover, since it is a convex programming problem the first order necessary optimality conditions are sufficient for a minimizer.

Note that since $\nabla F(X)$ is equal to a sum of unit vectors, then this gradient vanishes depending on the relative direction of this vectors. This allows us to enunciate the following theorem.

Theorem 1. *Given $\{a_1, \ldots, a_m\}$ a set of noncollinear points in the plane and let C be the corresponding Fermat point of this set. Assume that C is different from a_i for all i. If the point a_k is replaced by another point a'_k, that lies on the half-line through a_k with initial point C, then the new Fermat problem associated to this set of points has the same solution.*

Proof. Let a'_k given by $a'_k = (a_k - C)t + C$, for $t > 0$ and $H(X)$ the associated Fermat function for the set $\{a_1, \ldots, a'_k, \ldots, a_m\}$, that is,

$$H(X) = \sum_{i=1}^{k-1} \|X - a_i\| + \|X - a'_k\| + \sum_{i=k+1}^{m} \|X - a_i\|$$

Now, is easy to verify that $\nabla H(C) = 0$:

$$\begin{aligned} \nabla H(C) &= \sum_{i=1}^{k-1} \frac{C - a_i}{\|C - a_i\|} + \frac{C - (a_k - C)t - C}{\|C - (a_k - C)t - C\|} + \sum_{i=k+1}^{m} \frac{C - a_i}{\|C - a_i\|} \\ &= \sum_{i=1}^{m} \frac{C - a_i}{\|x - a_i\|} = \nabla F(C) = 0. \end{aligned}$$

Therefore, C is also the Fermat solution for $H(X)$.

□

This important result establishes that given a set of a noncollinear points and the associated Fermat point there exists another set of noncollinear points that has the same Fermat point. It allow us study equivalent problems associated to the same solution.

Let us recall some definitions in order to give our results.

Definition 1. *Given m points $\{a_1, ..., a_m\}$ in the plane. The centroid or geometric center of this set of points is defined by*

$$G = \frac{1}{m}\sum_{i=1}^{m} a_i$$

Given the Fermat point definition and the geometric center, is possible to link them under certain conditions.

We will see that under suitable symmetry conditions the centroid and the Fermat point are strongly related. Before that we will show that the Fermat point lies on some axis of symmetry.

Theorem 2. *Let $\{a_1, ..., a_m\}$ be a finite set of points. If there exists an axis of symmetry for this set, then the Fermat point lies on it.*

Proof. We assume that m is even and there is no point a_i on the axis of symmetry. If C is the Fermat point of the set $\{a_1, ..., a_m\}$ we will show that it lies on the axis of symmetry.

The gradient of the Fermat function is given by

$$\begin{aligned} \nabla F(X) &= \sum_{i=1}^{m} \frac{X - a_i}{\|X - a_i\|} = \frac{X - a_1}{\|X - a_1\|} + ... + \frac{X - a_m}{\|X - a_m\|} \\ &= \frac{(x - a_{1x}, y - a_{1y})}{\|X - a_1\|} + ... + \frac{(x - a_{mx}, y - a_{my})}{\|X - a_m\|} \end{aligned}$$

Without loss of generality we can move the origin of the coordinate system to some arbitrary point onto the axis of symmetry. In addition, if we rotate the coordinate system such that the abscissa axis match with the symmetry axis, we can assure that there are the same amount of points in both sides of this axis.

Consequently, there are pairs of points $a_i = (a_{ix}, a_{iy})$ and $a_j = (a_{jx}, a_{jy})$ that are equidistant from the P and $a_{jx} = a_{ix}$ and $a_{jx} = -a_{ix}$. For simplicity we order the points as we showed in Figure 1.

Given an arbitrary point at the axis of symmetry $P = (p, 0)$, we have

$$\begin{aligned} \|P - a_{2k-1}\| &= \|P - a_{2k}\| && \text{if } k \geq 1 \\ (P - a_{2k-1}) + (P - a_{2k}) &= 2(p - a_{2kx}, 0) && \text{if } k \geq 1 \end{aligned}$$

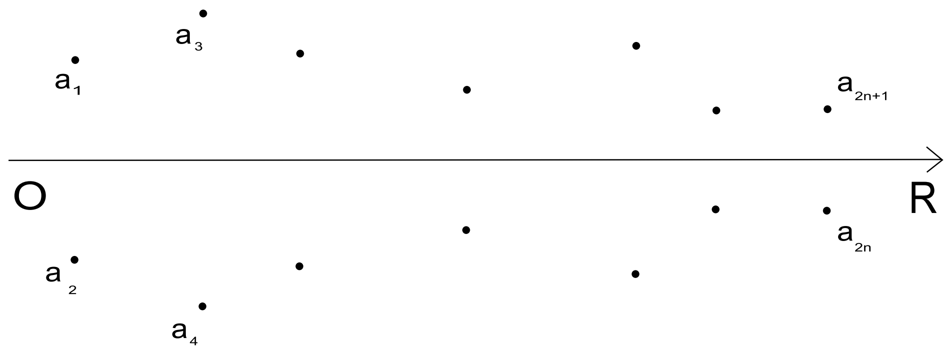

Figure 1: Set of points.

Then

$$\begin{aligned}
\nabla F(P) &= \sum_{i=1}^{m} \frac{P - a_i}{\|P - a_i\|} \\
&= \frac{(p - a_{1x}, -a_{1y})}{\|P - a_1\|} + \frac{(p - a_{2x}, -a_{2y})}{\|P - a_2\|} + \cdots + \frac{(p - a_{mx}, -a_{my})}{\|P - a_m\|} \\
&= \frac{(p - a_{1x}, -a_{1y}) + (p - a_{1x}, a_{1y})}{\|P - a_1\|} + \cdots + \frac{(p - a_{mx}, -a_{my}) + (p - a_{mx}, a_{my})}{\|P - a_m\|} \\
&= \frac{2(p - a_{1x}, 0)}{\|P - a_1\|} + \cdots + \frac{2(p - a_{kx}, 0)}{\|P - a_k\|} \\
&= (2 \sum_{i=1}^{k} \frac{p - a_{ix}}{\|P - a_i\|}, 0)
\end{aligned}$$

where $k = m/2$.

Clearly, $F_y(P) = 0$ for all $P = (p, 0)$ belonging to the axis of symmetry. On the other hand,

$$g(p) = F_x(P) = F_x(p, 0) = \sum_{i=1}^{k} \frac{p - a_{ix}}{\|P - a_k\|} \quad \text{for all } p$$

is a continuous function defined on the variable p. Moreover, if $p = a_{1x}$ then $g(p) < 0$ and if $p = a_{kx}$ then $g(p) > 0$. Therefore, there exists p_0 such that $F_x(p_0, 0) = 0$, that is, the the Fermat point lies on the symmetry axis.

Finally, if m is odd we can assume that there is a point on the symmetry axis and the proof is analogous.

□

Now, we will establish some relationships between the geometric center and the Fermat point.

Theorem 3. *The geometric center lies on the symmetry axis.*

Proof. Without loss of generality, by using the same argument as previous theorem, we can move and rotate the coordinate system in order to match the symmetry axis with the abscissa axis. In consequence, if a_k lies on the symmetry axis for some k then $a_{ky} = 0$. Otherwise, there is a pair of points a_i and a_k such that $a_{iy} = -a_{ky}$. Hence, the coordinates of the geometric center are given by

$$x_{cg} = \frac{a_{1x} + ... + va_{mx}}{m}$$
$$y_{cg} = \frac{a_{1y} + ... + a_{my}}{m} = 0.$$

Since the geometric center exists and it is unique, it follows that it lies on the symmetry axis.

□

Corolary 3.1. *If* $\{a_1, ..., a_m\}$ *determine the vertices of a regular polygon then the Fermat point is equal to the geometric center.*

Proof. Given a regular polygon of m sides we can determine m symmetry axis by considering the lines through the geometric center and a vertex if m is odd and the geometric center and two vertices or any vertices otherwise. By Theorem 2 the Fermat point have to lie on each one of this symmetry axis. On the other hand, the geometric center is the unique point that lies on every symmetry axis. Therefore, the Fermat point is equal to the geometric center.

□

At this point, we observe that it is very easy to obtain analytically the Fermat point when the set of point defining the problem have a symmetry axis. However, by using we Theorem 1, we can extend a characterization of the Fermat solution when no symmetry is present. In that case, we will show that moving adequately some points it is possible to obtaint an equivalent problem, wich has a symmetry axis.

To illustrate our method to characterize the solution of Fermat problem we will consider two triangles with and without axial symmetry.

3.1 Characterization of Fermat solution (axial symmetry case)

For our analysis we are going to consider an arbitrary isosceles triangle as Fig. 2. It is clear that any isosceles triangles has axial symmetry, that is, we are be able to determine a symmetry axis symmetry through the geometrical center and one vertex. Our method is based on Theorem 1 and the transformation of the isosceles triangle in an equilateral triangle. Then we will use Corollary 3.1 to obtain the Fermat point.

Proposition 3.1. *Given three points* $\{a_1, a_2, a_3\}$ *on a plane such that* $\|a_1 - a_3\| = \|a_2 - a_3\|$*, then the Fermat point is given by* $C = (0, \frac{\sqrt{3}}{2}\|a_2 - a_3\|)$*, in an suitable coordinate system.*

Proof. For simplicity, we can suppose that the y-coordinate of a_1 and a_2 are equal to zero and the x-coordinate of a_3 is also equal to zero. That is, $a_1 = (a_{1x}, 0), a_2 = (a_{2x}, 0)$ and $a_3 = (0, y)$. It is clear that there exists an axis of symmetry through a_3 and the middle point between a_1 and a_2. Thus, the points can be rewrite in the form: $a_1 = (a_{1x}, 0), a_2 = (-a_{1x}, 0)$ and $a_3 = (0, y)$.

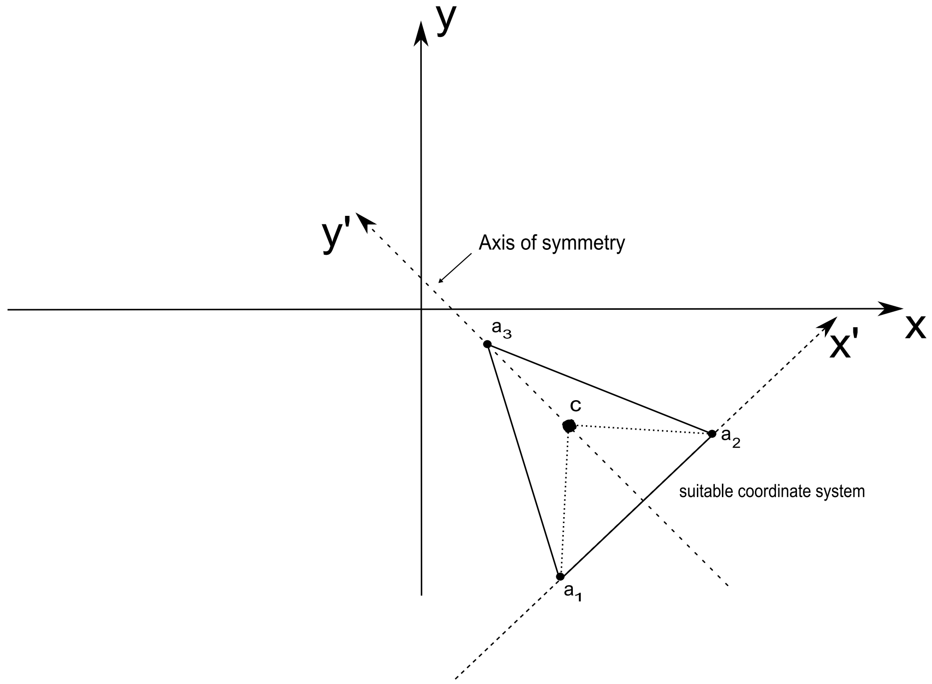

Figure 2: Triangle with axial symmetry.

Now, by using Theorem 1 and Theorem 2, we could move a_3 along this axis without change the solution. In fact, by the Pythagorean Theorem, we can compute y_0 such that an equilateral triangle is obtained. That is

$$(y + y_0)^2 + \left(\frac{\|a_2 - a_1\|}{2}\right)^2 = \|a_2 - a_1\|^2,$$

then,

$$(y + y_0) = \pm\frac{\sqrt{3}}{2}\|a_2 - a_1\|.$$

Since, in our coordinate system, $(y + y_0)$ have to be a positive number we have that

$$(y + y_0) = \frac{\sqrt{3}}{2}\|a_2 - a_1\| = a_{1x}\sqrt{3}.$$

Finally, by Corollary 3.1, we have the characterization of the Fermat point

$$C = (0, \frac{\sqrt{3}}{3} a_{1x}).$$

□

This argument allow us to solve the Fermat problem for any isosceles triangle. Next, we present a new method for solve any Fermat problem (involving three points), even in absence of symmetry.

3.2 Characterization of Fermat solution (without axial symmetry case)

Now, we will consider a scalene right triangle $\{A, B, C\}$ given by the points $A = (0, 0), B = (2, 0)$ and $C = (0, 1)$ for which we would want to find the Fermat point. We assume known the Fermat solution for a model isosceles triangle and we use the Theorem 1 to relate both triangles, after some suitable transformations.

Our method proceeds in the following steps:

1. Consider the isosceles triangle $\{a, b, c\}$ given by $a = (1, 0), b = (0, 1)$ and $c = (-1, 0)$.
2. After fix one vertex, move the others onto the respective half-line connecting them with the Fermat point in order to build a new triangle which has the same side proportions and the same angle between them as the original scalene triangle.
3. Find the Fermat point and scale it properly until it fits to the original problem. Finally, through a suitable rotation and reflection, the solution could be correctly expressed.

3.2.1 The approach

From Proposition 3.1, it follows that the Fermat point for the triangle $\{a, b, c\}$ is given by $F = (0, \frac{\sqrt{3}}{3})$.

Without loss of generality we consider the point c and we would like to determine two points (a' and b') such that the new triangle given by $\{a', b', c\}$ has the same proportions between the edges and one right angle angle as the triangle $\{A, B, C\}$.

We ask that a' and b' are onto the half-lines with the Fermat point F as the origin and passing through a and b, respectively. We denote this two half-lines as $\overline{aF}$ and $\overline{bF}$. Clearly by Theorem 1, the corresponding Fermat point of both triangles will be the same.

The parametric representation of both half-lines is given by

$$\begin{array}{rclcl} \overline{aF} & = & ((1,0) - F)t + F & = & (1,0)t + F(1-t) \\ \overline{bF} & = & ((0,1) - F)s + F & = & (0,1)t + F(1-s) \end{array} \qquad (4)$$

where $t, s > 0$.

The condition of the right angle means that

$$(c - a').(b' - a') = 0,$$

or, equivalently

$$s(\sqrt{3} - 1)(t - 1) + 2t(2t + 1) = 0.$$

See Fig. 3.

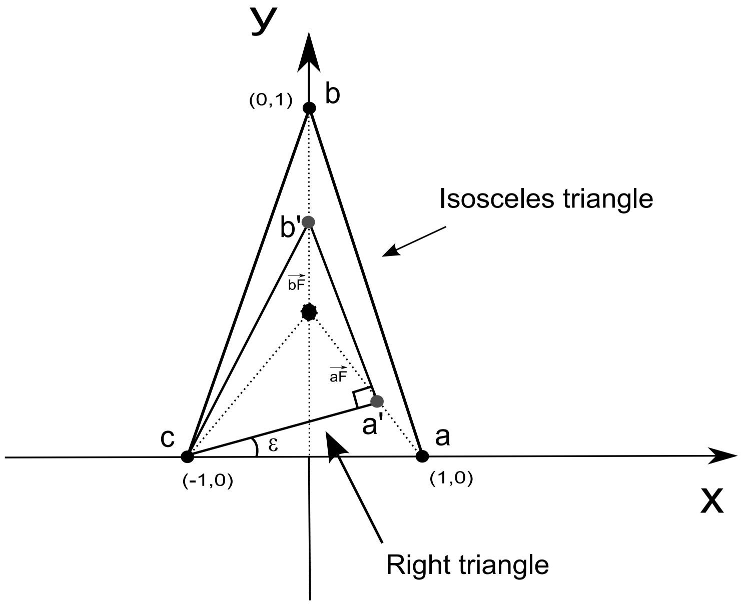

Figure 3: Triangle without axial symmetry.

Thus, by solving this quadratic equation on t we obtain the following two possible solutions:

$$t_{1,2} = \begin{cases} \frac{1}{8}\left(\sqrt{2}\sqrt{s^2(2-\sqrt{3}) + 10s(\sqrt{3}-1) + 2} + s(1-\sqrt{3}) - 2\right) \\ -\frac{1}{8}\left(\sqrt{2}\sqrt{s^2(2-\sqrt{3}) + 10s(\sqrt{3}-1) + 2} - s(1-\sqrt{3}) + 2\right) \end{cases} \tag{5}$$

On the other hand, we will construct the new triangle in a such way the it has the same proportion $(2:1)$ between the edges of the right angle. So, we ask that

$$2|c - a'| = |b' - a'|, \tag{6}$$

or

$$2|b' - a'| = |c - a'|. \tag{7}$$

Both conditions would be satisfied initially.
Now, by using the parametric representation (4), (6) implies that

$$t_{1,2} = \begin{cases} -\frac{1}{12}\left(\sqrt{2}\sqrt{13s^2(2-\sqrt{3})+8s(1-\sqrt{3})-64}+s(1-\sqrt{3})+8\right) \\ \frac{1}{12}\left(\sqrt{2}\sqrt{13s^2(2-\sqrt{3})+8s(1-\sqrt{3})-64}-s(1-\sqrt{3})-8\right) \end{cases} \tag{8}$$

Analogously, (7) implies that

$$t_{1,2} = \begin{cases} \frac{1}{6}\left(\sqrt{16s^2(\sqrt{3}-2)+4s(1-\sqrt{3})+13}+2s(1-\sqrt{3})+1\right) \\ -\frac{1}{6}\left(\sqrt{16s^2(\sqrt{3}-2)+4s(1-\sqrt{3})+13}-2s(1-\sqrt{3})-1\right) \end{cases} \tag{9}$$

Now, by (8)–(9) together with (5), we obtain a nonlinear system of equations. See Figure 4. Since s and t have to be positive, we have only two possible solutions:

- t_1 from (5) with t_2 from (8),
- t_1 from (5) with t_1 from (9).

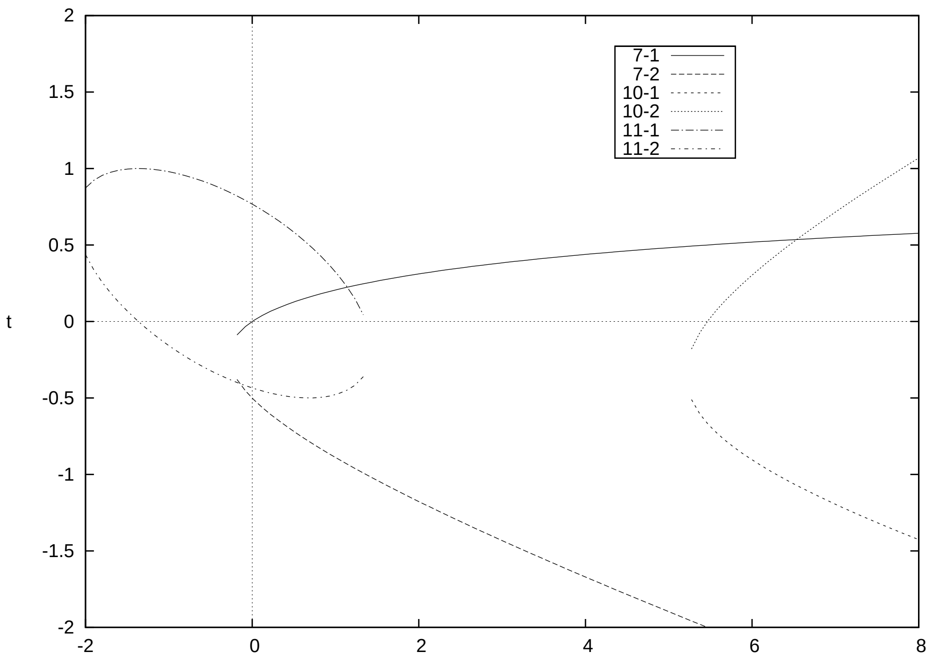

Figure 4: Equations (7), (10) and (11).

Therefore, we can determine two possible pairs for a' and b' such that the Fermat point is preserved after the transformation of the triangle

$$\begin{aligned} a'_1 &= (4-2\sqrt{3}, 2-\sqrt{3}), & b'_1 &= (0, 12-5\sqrt{3}) \\ a'_2 &= (\tfrac{2\sqrt{3}-1}{11}, \tfrac{4\sqrt{3}-2}{11}), & b'_2 &= (0, \tfrac{5\sqrt{3}+3}{11}) \end{aligned}$$

We can choose the pair $a' = a'_2$ and $b' = b'_2$. Now, it is easy to calculate the Fermat point from a', that is, $F_{a'} = F - a'$:

$$F_{a'} = (\frac{1-2\sqrt{2}}{11}, \frac{2}{11} - \frac{\sqrt{3}}{33}) \tag{10}$$

Next, we have to rescale the edges in order to one of the catheti has length equal to 1 and the length of the other one is equal to 2. Remember that in our procedure we adopted t_1 from (5) and t_2 from (8). Hence, we ask for

$$\lambda|b' - a'| = 1 \tag{11}$$

where λ represent the scale factor. Then, by (11), we obtain

$$\lambda = \frac{\sqrt{533 - 78\sqrt{3}}}{13}.$$

Consequently, the relative scaled Fermat point is

$$F'_{a'} = (-\frac{\sqrt{65-26\sqrt{3}}}{13}, \frac{\sqrt{195-78\sqrt{3}}}{39}). \tag{12}$$

Now, to get the final solution we should rotate the coordinate system and translate the origin. First of all, we move the origin to the point c and, consequently, the points a', b' are moved to

$$\begin{aligned} a' &= (\tfrac{2\sqrt{3}-1}{11} + 1, \tfrac{4\sqrt{3}-2}{11}) \\ b' &= (1, \tfrac{5\sqrt{3}+3}{11}) \\ c &= (0, 0). \end{aligned}$$

Let us define the rotation T given by

$$T = \begin{pmatrix} \cos\varepsilon & -\sin\varepsilon \\ \sin\varepsilon & \cos\varepsilon \end{pmatrix}$$

where ε is such that $\tan(\varepsilon) = \frac{4\sqrt{3}-2}{2\sqrt{3}+10}$.

Finally, we apply the rotation T^{-1} to $F'_{a'}$ and reflect it with respect to the vertical axis to obtain the solution

$$(\frac{4\sqrt{3}}{39} + \frac{1}{13}, \frac{8}{13} - \frac{7\sqrt{3}}{39}).$$

4 Results for the (symmetric) Weber problem

The main results proved for the Fermat problem could be easily extended to find the solution of the Weber problem, that is, the Weber point. As in the previous sections, symmetry plays an important role in our idea. In fact, we will assume that we have a set of points in the plane with associated weights and there exists an axial symmetry. Moreover, we assume that if a_{2k} is the reflected point of a_{2k-1} through this symmetry axis then both points have the same weights, that is, $w_{2k} = w_{2k-1}$.

The following result is a generalization of Theorem 2 and the proof is obviously the same.

Theorem 4. *Let $\{a_1, ..., a_m\}$ be a finite set of points such that a_k has an associated weight $w_k > 0$ for all k. If there exists an axis of symmetry of this set and $w_{2k} = w_{2k-1}$, then the Weber point lies on this axis.*

Also is possible to generalize the Theorem 1. The proof is analogous.

Theorem 5. *Given $\{a_1, \ldots, a_m\}$ a set of noncollinear points in the plane with associated positive weights $\{w_1, \ldots, w_m\}$ respectively, and let W be the corresponding Weber point of this set of points. Assume that W is different from a_i for all i. If the point a_k is replaced by another point a'_k, that lies on the half-line through a_k with initial point W, then the new Weber problem associated to this set of points has the same solution.*

This result is clear since the gradient is a linear combination of unit vectors. Also, the gradient vanishes depending on the relative directions of this vectors.

4.1 Characterization of the solution of the Weber problem (bidimensional case)

Now we are interested in the characterization of the solution of the three-points Weber problem, with arbitrary weights. We will use the previous theorem to obtain a solution for a particular two-dimensional Weber problem. Then by using this particular solution and the previous theorema we can solve the three-points Weber problem. Our method is analogous to the proposed for solving the Fermat problem.

To avoid confusion, we will denote the points as follows: $a_i = (a_{ix}, a_{iy}, w_i)$, where w_i is the associated positive weight to the point a_i, for $i = 1, 2, 3$.

Also, for simplicity, we will consider the weighted triangle in the plane given by

$$\{(x_1, y_1, w_1), (-1, y_2, w_2), (0, 1, w_3)\}.$$

We will find the constants x_1, y_1 and y_2 such that the gradient of Weber function in the origin is equal to zero. That is, we will construct a triangle whose Weber point is equal to $(0, 0)$.

Remember that

$$\nabla W(X) = \sum_{i=1}^{m} w_i \frac{X - a_i}{\|X - a_i\|}$$

where $X = (x, y)$. Since we assume that the Weber point will be $(0, 0)$ then we should be to $W_x(0, 0) = 0$ and $W_y(0, 0) = 0$.

We observe that $W_x(0, 0) = 0$ implies that

$$\frac{w_2}{\sqrt{1 + y_2^2}} = \frac{w_1 x_1}{\sqrt{x_1^2 + y_1^2}}. \tag{13}$$

Then $x_1 > 0$ and, moreover, it holds that

$$(\frac{w_1}{w_2})^2 \geq \frac{1}{1 + y_2^2}$$

So, by (13)

$$|y_1| = x_1 \sqrt{(\frac{w_1}{w_2})^2 (1 + y_2^2) - 1}. \tag{14}$$

Similarly, $W_y(0, 0) = 0$ implies that

$$\frac{w_1 y_1}{\sqrt{x_1^2 + y_1^2}} + \frac{w_2 y_2}{\sqrt{1 + y_2^2}} = -w_3.$$

Hence, by (14), we have

$$w_2 y_2 = -\sqrt{w_1^2 y_2^2} - w_3 \sqrt{1 + y_2^2},$$

and, consequently, $y_2 < 0$.

Solving this equation for y_2^2 we have that

$$y_2^2 = \begin{cases} \frac{w_3^2}{(w_1 - w_2)^2 - w_3^2} \\ \frac{w_3^2}{(w_1 + w_2)^2 - w_3^2} \end{cases}$$

Since the condition $(w_1 - w_2) > w_3$ implies $(w_1 + w_2) > w_3$ it is enough to ask that $(w_1 + w_2) > w_3$ in order to $y_2^2 \geq 0$.

Therefore, we could find a particular set of points with theirs corresponding weights such that $(0, 0)$ is the Weber point. This result generalize the solution found for the Fermat problem obtained previously. For instance, applying the results for the Fermat problem we have

$$x_1 = 1, \qquad y_1 = -\sqrt{3}/3, \qquad y_2 = -\sqrt{3}/3$$

and the resulting set of points have $(0, 0)$ as the solution.

Finally, if we apply the same method that in Section 3, we could solve exactly any three-points Weber problem with arbitrary weights.

Note that we can generalize this idea to the Weber problem with m points, where two of them, with their respective weights, are free and the others are fixed in the plane.

5 Conclusions

In this paper we characterize geometrically the exact solution for the Fermat and Weber problems. We would like to emphasize that our results are just based on symmetry and the theorem that the solution remains in the same position when one point is replaced by other one along a ray from the Fermat point. Our ideas can be easily extended for solving the m-points Weber problem when there are two degrees of freedom.

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