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An Introduction to Finite-Dimensional Representations of Classical and Quantum Affine Algebras

Adriano Moura

Editores: Jorge R. Lauret–Jorge G. Adrover

CIUDAD UNIVERSITARIA – 5000 CÓRDOBA

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An Introduction to Finite-Dimensional Representations of Classical and Quantum Affine Algebras

Adriano Moura

Universidade Estadual de Campinas

Abstract

This is a preliminary version of expanded lecture notes of a mini course to be given by the author at the XV Latin American School of Mathematics. The aim is to give an introduction to the finite-dimensional representation theory of affine Kac-Moody algebras and their quantum groups covering topics such as the classification of the simple modules and the notions of Weyl modules, ℓ-weight modules, and qcharacters. We also present a few results regarding tensor products and extensions which will be further expanded in a revised version.

Introduction

These notes are intended as supporting material for the mini-course I was invited to give at the XV Latin American School of Mathematics (ELAM) to take place in Cordoba, Argentina, from May 16 to May 27, 2011. The courses at the ELAM are intended to survey specific areas or problems thus providing motivation for potential directions towards which graduate students may incline their careers. I know two styles of providing motivation: the "fancy" one and the "down-to-earth" one.

Let me begin with the fancy one which, in the case of finite-dimensional representation theory of affine Kac-Moody algebras and their quantum groups, is two-fold. From one side we have its connection to areas closer to the "real world". Namely, it is by now well-known that affine Kac-Moody algebras are intimately related with areas of mathematical physics, especially conformal field theory and related areas. Also, the introduction of the concept of quantum groups was motivated by the study of the Yang-Baxter equation in statistical mechanics. Therefore, the connection of quantum affine algebras with mathematical physics comes from both classical and quantum perspectives. Their finite-dimensional representations play a role in the study of integrability of certain lattice models. As I do not work on the mathematical physics side of the story, I will not give further insights in this direction. The interested reader will have no difficulty finding papers and books with very insightful introductions giving a very broad overview of this side of the story in a manner accessible to a non-expert reader. The other side of the fancy motivation comes from a purely representation theoretical point of view. Namely, the finite-dimensional representations of these algebras form a Jordan-Hölder tensor category which is not semi-simple (by a Jordan-Hölder category we mean an abelian category such that all objects have finite-length and the Jordan-Hölder theorem holds). Given a category with such properties, there is a plethora of natural questions such as understanding composition series and extensions as well as the tensor structure. Moreover, these representations breakup as a direct sum of generalized eigenspaces for the action of a commutative subalgebra called the ℓ-weight spaces. This decomposition gives rise to a character theory which is especially interesting in the quantum setting (and has a high level of combinatorial flavor). Although several important and profound results regarding these characters have been proved, the theory (which is also very relevant in mathematical physics) is far from being settled. The systematic study of extensions is still in its infancy and the first Ext groups have been described, in the classical context only, two years ago. In the quantum setting, the only result so far is the description of the block decomposition of the category. The tensor structure of the category in the classical setting is somewhat simple: tensor products are "symmetric" (the category has trivial braiding) and tensor products of simple modules are completely reducible. The story is very different in the quantum setting (the category is not braided!). Very recent results show that the understanding of the tensor structure of this category interacts with the study of one of the most fashionable topics of the moment – the theory of cluster algebras. In a nutshell, the finite-dimensional representation theory of quantum and classical affine algebras is very rich with many open interesting questions which are relevant in mathematical physics.

The remainder of the text and the course are intended to provide the "down-to-earth" motivation. When one is trying to decide what to do for a living, one should rather be as sure as possible that doing that thing on a daily basis will be enjoyable than if it is sounding "fancy". In mathematics, each area or subarea has its own flavor essentially determined by the style of computations one ends up doing on a daily basis. The selected topics covered in these notes should give the students a glimpse of a few possible flavors of the theory. In other words, the text brings computations which should give an idea of the typical computations they would be "forced to enjoy" every day should they choose to work on the area. Hopefully, several of the students will agree that the computations are indeed enjoyable and will feel motivated to join the team of mathematicians trying to unravel the structure of these representations.

As this is one of the "advanced" courses of the XV ELAM, the text is written assuming that the students are familiar with the basics of ring and module theory (including the concept of composition series and the Jordan-Hölder Theorem) as well as with the finite-dimensional representation theory of finite-dimensional semi-simple Lie algebras over an algebraically closed field of characteristic zero (the material of Humphreys' book [52], for instance). The knowledge of what a Kac-Moody algebra is should be helpful but not necessary. Section 1 gives a review of the basic aspects of Kac-Moody algebras and the class of representations that resembles that of finite-dimensional representations of simple Lie algebras – the class of integrable representations in the famous BGG category $\mathscr{O}$. Since there are several good books on this matter, we only present the basic constructions and state the main results which are needed so that the readers will be able to understand the main part of the course even if this is the first time they are exposed to Kac-Moody algebras. In fact, the students who never studied the finite-dimensional simple Lie algebras should also be able to follow the main part of the course after reading the review in Section 1. We also review the definition of the Drinfeld-Jimbo quantum groups associated to Kac-Moody algebras in Section 1. We follow the same approach we had used for Kac-Moody algebras since there are several good books on quantum groups as well. A list of books covering this material is given in the introduction to Section 1.

The material of the course properly starts in Section 2. The main goal of the section is to answer the most basic question in representation theory: given an algebra, classify its simple modules. We shall see that, already in such a basic level, there are some differences among the quantum and the classical contexts. We also introduce the concepts of global and local Weyl modules and the notions of ℓ-weight modules and their qcharacters. The main goal of Section 3 is to present results about tensor products of simple modules and prepare ground for the study of some combinatorial aspects of the theory of qcharacters to be done in Section 4. In particular, in Section 3 we also study results on duality and present the description of the block decomposition of our category of modules. However, in this section we prove the result on the block decomposition only in the case that the underlying simple Lie algebra is $\mathfrak{sl}_2$ (which is what we will need for Section 4). Section 4 is then entirely dedicated to the study of algorithms designed to compute the qcharacter of certain classes of simple modules. All sections, except for Section 1, end with a subsection named "bibliographical notes" where the due credit for the original proofs are given as well as a few comments regarding the present and future perspectives. This is the reason no citation is made in this introduction nor in the main body of each section.

The material contained in this preliminary version is already larger than what can be covered during the mini-course. However, there are certain additional topics which will most likely appear in the revised version such as a new section on extensions where, in particular, we will finish the proof of the block decomposition theorem. An extra subsection may be added to Section 3 concerning simple tensor products and prime representations and another one to Section 4 containing a description of Nakajima's algorithm for qcharatacters. There are several other interesting topics which may eventually be included in future versions such as: the proof of the Kirillov-Reshetikhin conjecture and fermionic formulas (passing through the theory of T-systems), minimal affinizations, fusion products, Demazure modules, crystal bases, and monomial categorification of cluster algebras. Since there will certainly be a revised version, corrections and suggestions about the already included material are more than well come.

Acknowledgements

I thank the organizing committee of the XV ELAM, especially Vyacheslav Futorny and Nicolás Andruskiewitsch, for inviting me to give this mini-course. I also thank the mathematicians with whom I worked on these topics for all the discussions that helped me learn many things – my advisor Pavel Etingof, my post-doc supervisor Vyjayanthi Chari, and my co-author Dijana Jakelić who also helped me to find a few mistakes in the first draft of these notes. The support of the Brazilian foundations for promotion of sciences CAPES, CNPq, and FAPESP at various times is gratefully acknowledged. About half of the material covered here was part of a course I gave at Unicamp during the second half of 2009. The students of that course typed the lecture notes which served as basis for the present text. Their first draft made my life much easier now, so they deserve the credit for helping me bring these notes to existence. Here are their names: Angelo Bianchi, Matheus Brito, Tiago Macedo, Fernanda Pereira (who also helped me to correct a few mistakes in the first draft of the present text), Vitor da Silva, and Manuela Souza.

Contents

Basic Notation

$\mathbb{Z}$ and $\mathbb{Q}$ denote the sets of integer and rational numbers, respectively. The notation $\mathbb{Z}_{\geq m}$ will be used for the set of integers bigger or equal to m and similarly for $\mathbb{Z}_{>m}$ and son on. Given a ring R, the underlying multiplicative group of units is denoted by $R^\times$ and R^{op} denotes the ring whose underlying additive group is R but equipped with the opposite multiplication. The identity element of a multiplicative group will be denoted generically by 1. The cardinality of a set S will be denoted by $|S|$. The symmetric group of a set with cardinality m is denoted by S_m. The symbol $\cong$ means "isomorphic to".

Throughout the text, $\mathbb{F}$ denotes an algebraically closed field of characteristic zero and, unless otherwise stated, all vector spaces and algebras considered are $\mathbb{F}$-vector spaces. The dual of a vector space V is denoted by V^*. Given a subset α of a vector space, we denote by $[\alpha]$ the span of α. If α and β are bases of finite-dimensional vector spaces V and W, respectively, and $T : V \to W$ is a linear transformation, the matrix of T with respect to α and β is denoted by $[T]^\alpha_\beta$. The transpose of a matrix A is denoted by A^t. Tensor products without a subscript are assumed to be over $\mathbb{F}$. The r-th graded piece of a graded vector space V will be denoted by $V[r]$.

1. Kac-Moody Algebras and Quantum Groups

This is a review section so all the proofs are omitted. Unless otherwise noted, the proofs concerning the classical context can be found in the books [6, 8, 39, 52, 53, 54, 61, 77, 82] while the ones concerning the quantum setting can be found in [25, 29, 43, 59, 64, 67].

1.1. Basic concepts on Kac-Moody algebras

Definition 1.1.1. Let I be a finite set and $C = (c_{ij})$, $i, j \in I$, be a matrix. The matrix C is said to be indecomposable if for any choice of nonempty disjoint subsets I_1 and I_2 of I such that $I = I_1 \cup I_2$, there exist $i \in I_1$ and $j \in I_2$ satisfying $c_{ij} \neq 0$. Otherwise, C is said to be decomposable. The matrix C is said to be a generalized Cartan matrix if:

(a) $c_{ij} \in \mathbb{Z}$ and $c_{ii} = 2$ for all $i, j \in I$,

(b) $c_{ij} \leq 0$ for all $i \neq j$,

(c) $c_{ij} = 0 \Leftrightarrow c_{ji} = 0$ for all $i, j \in I$.

A generalized Cartan matrix is said to be symmetrizable if it satisfies:

(d) there exist $s_i \in \mathbb{Z}_{>0}, i \in I$, such that SC is symmetric where $S = \operatorname{diag}(s_i : i \in I)$. ◊

Notice that, if C is symmetrizable, we can choose the numbers $s_i \in \mathbb{Z}, i \in I$, to be relatively prime. We shall always assume that C is an indecomposable symmetrizable generalized Cartan matrix and that s_i are chosen in this way.

Proposition 1.1.2. One, and only one, of the following options holds for C.

(a) SC is positive definite.

(b) SC is semi positive definite of corank one.

(c) SC is indefinite. □

Definition 1.1.3. A generalized Cartan matrix C is said to be of finite, affine, or indefinite type if C satisfies condition (i), (ii), or (iii) of the above proposition, respectively. ◊

The entries of a generalized Cartan matrix C can be encoded in a picture called the Dynkin diagram of C. We will describe how to construct the diagram for finite and affine types only since this is all we will need. In this case we have $c_{ij}c_{ji} \leq 4$ for all $i, j \in I$. If $n = |I|$, the Dynkin diagram is a graph with n vertices and $c_{ij}c_{ji}$ edges joining the distinct vertices i and j. If $c_{ij} < -1$, we adorn the set of edges joining i and j with a > pointing towards i. One easily checks that this picture indeed determines completely the matrix C. The following tables is a summary of are the theorem classifying generalized Cartan matrices of finite and affine type. In Table 1.1.1, the number of vertices in the diagram of type X_n is n while in Tables 1.1.2 and 1.1.3, the number of vertices of the diagram of type $X^{(k)}_{f(n)}$ is $n+1$. The diagrams on the right-side column of Table 1.1.1 are known as diagrams of exceptional type. We did not label the vertices in Table 1.1.3 since we will not work with them here.

Table 1.1.1: Dynkin Diagrams of Finite Type

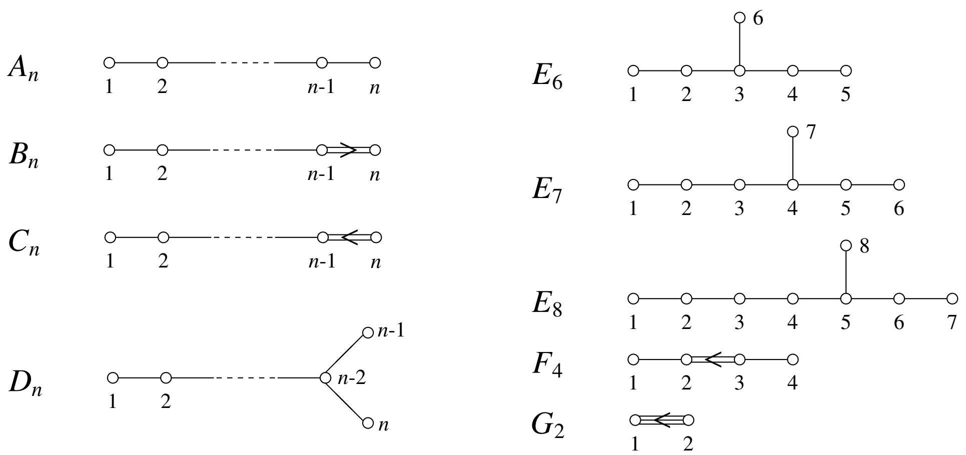


Table 1.1.2: Dynkin Diagrams of Non Twisted Affine Type

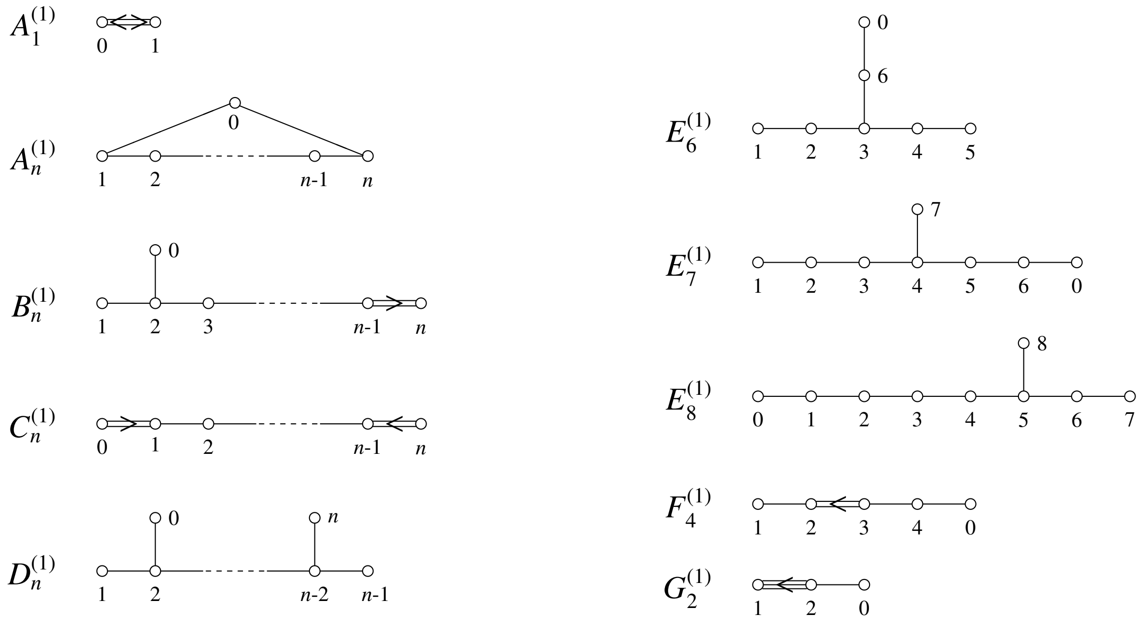
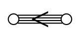
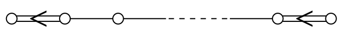
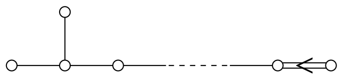
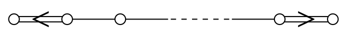


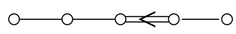
Table 1.1.3: Dynkin Diagrams of Twisted Affine Type

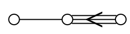
$A_2^{(2)}$ 

$A_{2n}^{(2)}$ 

$A_{2n-1}^{(2)}$ 

$D_{n+1}^{(2)}$ 

$E_6^{(2)}$ 

$D_4^{(3)}$ 

Definition 1.1.4. Let r be the rank of C and choose $I' \subseteq I$ such that $|I'| = r$ and (c_{ij}), $i, j \in I'$, is invertible. The *Kac-Moody algebra* $\mathfrak{g} = \mathfrak{g}(C)$ is the Lie algebra given by generators $x_i^{\pm}, h_i, d_j, i, j \in I, j \notin I'$, satisfying the defining relations:

$$[h_i, h_j] = 0, \qquad [x_i^+, x_j^-] = \delta_{ij} h_i, \qquad [h_i, x_j^{\pm}] = \pm c_{ij} x_j^{\pm} \qquad \text{for all} \qquad i, j \in I,$$
$$\mathrm{ad}(x_i^{\pm})^{1-c_{ij}}(x_j^{\pm}) = 0 \qquad \text{for all} \qquad i, j \in I, i \neq j,$$
$$[d_j, d_k] = 0, \qquad [h_i, d_j] = 0, \qquad [d_j, x_i^{\pm}] = \pm\delta_{ij} x_i^{\pm} \qquad \text{for all} \qquad i, j, k \in I, j, k \notin I'.$$

The generators $x_i^{\pm}$, h_i, $i \in I$, are called *Chevalley* or *Chevalley-Kac generators*, and the relations in the second line are called *Serre's relations*. Denote by $\mathfrak{d}$ the subalgebra of $\mathfrak{g}$ generated by $d_j, j \in I \setminus I'$, by $\mathfrak{h}'$ the subalgebra generated by $h_i, i \in I$, and by $\mathfrak{n}^{\pm}$ the ones generated by $x_i^{\pm}, i \in I$, respectively. Set also $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{d}$. $\Diamond$

In the remainder of the section, unless stated otherwise, $\mathfrak{g}$ denotes a given Kac-Moody algebra. The universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ can be similarly presented by generators and relations and the Serre's relations can then be rewritten as

$$\sum_{k=0}^{1-c_{ij}} (-1)^k \binom{1-c_{ij}}{k} (x_i^{\pm})^{1-c_{ij}-k} x_j^{\pm} (x_i^{\pm})^k = 0.$$

Notice that the derived algebra $\mathfrak{g}'$ of $\mathfrak{g}$ is the subalgebra of $\mathfrak{g}$ generated by $x_i^{\pm}, i \in I$, and that $\mathfrak{g}/\mathfrak{g}' = \mathfrak{d}$. In particular, if C is invertible, $\mathfrak{g} = \mathfrak{g}'$.

Proposition 1.1.5. The Serre's relations are defining relations for the subalgebras $\mathfrak{n}^{\pm}$. In particular, $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ as a vector space. $\square$

Given $i \in I$, the subalgebra of $\mathfrak{g}$ generated by $x_i^{\pm}$ is isomorphic to the Lie algebra $\mathfrak{sl}_2$ of traceless 2-by-2 matrices. We shall denote this subalgebra by $\mathfrak{g}_i$. In general, given any subset J of I, we denote by $\mathfrak{g}_J$ the subalgebra of $\mathfrak{g}$ generated by $x_j^{\pm}, j \in J$. The subalgebras $\mathfrak{h}_J, \mathfrak{n}_J^{\pm}$ are defined in the obvious way. Then $\mathfrak{g}_J$ is isomorphic to the derived algebra of the Kac-Moody algebra associated to the matrix $C_J = (c_{ij}), i, j \in J$.

Proposition 1.1.6. Let C be a generalized Cartan matrix and $\mathfrak{g} = \mathfrak{g}(C)$.

(i) C is of finite type if and only if $\mathfrak{g}$ is finite-dimensional. Moreover, in that case, $\mathfrak{g}$ is simple.

(ii) C is of affine type if and only $\mathfrak{g}_J$ is finite-dimensional for all proper subsets J of I.

(iii) C is of indefinite type if and only if C has a negative principal minor. □

Remark 1.1.7. It follows that C is of finite-type if and only if C is a Cartan matrix in the usual sense and, hence, all principal minors of C are positive. Thus, C is of affine type if and only if C is singular and all proper principal minors of C are positive. The algebras associated with the diagrams of type A_n, B_n, C_n, and D_n are the classical matrix algebras $\mathfrak{sl}_{n+1}, \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}$, and $\mathfrak{so}_{2n}$, respectively. ◊

Proposition 1.1.8. There exists a unique symmetric invariant bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$ satisfying:

$$(h_i, h_j) = \frac{c_{ij}}{s_j}, \qquad (x_i^+, x_j^-) = \frac{\delta_{ij}}{s_j}, \qquad (x_i^\pm, x_j^\pm) = 0, \qquad (h_i, x_j^\pm) = 0, \qquad \text{for all} \qquad i, j \in I,$$

$$(d_j, d_k) = 0, \qquad (d_j, x_i^\pm) = 0, \qquad (h_i, d_j) = \frac{\delta_{ij}}{s_j}, \qquad \text{for all} \qquad i, j, k \in I, j, k \notin I'.$$

Moreover, $(\cdot,\cdot)$ is nondegenerate and its restriction to $\mathfrak{h}$ is also nondegenerate. □

Remark 1.1.9. If $I' \neq I$, the restriction of $(\cdot,\cdot)$ to $\mathfrak{g}'$ is degenerate. In fact, notice that the matrix of the restriction of $(\cdot,\cdot)$ to $\mathfrak{h}'$ with respect to the basis $\{h_i,\ i \in I\}$ is CS^{-1}. In particular, the restriction of $(\cdot,\cdot)$ to $\mathfrak{h}'$ is nondegenerate if and only if C is invertible. In the case that $\mathfrak{g}$ is finite-dimensional, $(\cdot,\cdot)$ is a scalar multiple of the Killing form (several authors call it Cartan-Killing form). Thus, by abuse of terminology, we shall refer to $(\cdot,\cdot)$ as the Killing form of $\mathfrak{g}$ in general. ◊

Proposition 1.1.10. Given $i \in I$, there exists unique $\alpha_i \in \mathfrak{h}^*$ such that $\alpha_i(h_j) = c_{ji}$ an $\alpha_i(d_j) = \delta_{ij}$. Moreover, $\{\alpha_i : i \in I\}$ is linearly independent and $[h, x_i^\pm] = \pm\alpha_i(h)x_i^\pm$ for all $i \in I, h \in \mathfrak{h}$.

Remark 1.1.11. By abuse of notation, we shall denote the restriction of α_i to $\mathfrak{h}'$ by α_i as well. If C is singular, the set $\{\alpha_i : i \in I\}$ is linearly dependent when regarded as a subset of $\mathfrak{h}'^*$. ◊

Definition 1.1.12. The root lattice Q of $\mathfrak{g}$ is the subgroup of $\mathfrak{h}^*$ generated by $\{\alpha_i : i \in I\}$. Let also Q^+ be the corresponding submonoid. Given, $\eta = \sum_{i\in I} a_i\alpha_i \in Q$, the number $|\eta| = \sum_{i\in I} a_i$ is called the height of η. Define a partial order on $\mathfrak{h}^*$ by setting $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$. Given $i \in I$, the unique element $\omega_i \in \mathfrak{h}^*$ satisfying $\omega_i(h_j) = \delta_{ij}$ and $\omega_i(d_k) = 0$ for all $j, j \in I \setminus I'$ is called the i-th fundamental weight of $\mathfrak{g}$. The weight lattice of $\mathfrak{g}$ is the subgroup of $\mathfrak{h}^*$ generated by the fundamental weights. Let also P^+ denote the corresponding submonoid of $\mathfrak{h}^*$. The elements of P are called integral weights while the elements of P^+ are called dominant integral weights. ◊

Remark 1.1.13. Notice that $Q \subseteq P$. Also, $\sum_{i\in I} \mathbb{Z}\omega_i \subseteq P$ and equality holds if and only if C is invertible. We shall refer to elements of P simply by the weights of $\mathfrak{g}$. ◊

Given $\alpha \in \mathfrak{h}^*$, set

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \ \text{ for all } \ h \in \mathfrak{h}\}.$$

One quickly checks that

$$[\mathfrak{g}_\alpha,\ \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} \qquad \text{for all} \qquad \alpha, \beta \in \mathfrak{h}^* \tag{1.1.1}$$

and that $\mathfrak{g}_{\pm\alpha_i}$ is spanned by $x_i{}^+$ for all $i \in I$.

Definition 1.1.14. An nonzero element $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \neq 0$ is said to be a root of $\mathfrak{g}$ while $\mathfrak{g}_\alpha$ is the associated root space. A nonzero vector in a root space is called a root vector. The set R of all roots of $\mathfrak{g}$ is called the root system of $\mathfrak{g}$. The elements of $R^+ = R \cap Q^+$ are called positive roots while the ones in $-R^+$ are called negative roots. The elements $\alpha_i, i \in I$, are called simple roots. ◊

Proposition 1.1.15. In any Kac-Moody algebra we have $R \subseteq Q$ and $R = R^+ \cup -R^+$. Moreover, $\mathfrak{g}_0 = \mathfrak{h}$, $\mathfrak{n}^\pm = \bigoplus\limits_{\alpha \in R_+} \mathfrak{g}_{\pm\alpha}$, and $\mathfrak{g}_\alpha$ is finite-dimensional for all $\alpha \in R$. □

Given $\eta \in Q$, set

$$U(\mathfrak{g})_\eta = \{x \in U(\mathfrak{g}) : [h, x] = \eta(h)x \quad \text{for all} \quad h \in \mathfrak{h}\}$$

and $U(\mathfrak{n}^\pm)_\eta = U(\mathfrak{n}^\pm) \cap U(\mathfrak{g})_\eta$. One quickly checks that

$$U(\mathfrak{g}) = \bigoplus_{\eta \in Q} U(\mathfrak{g})_\eta \tag{1.1.2}$$

is a Q-gradation on $U(\mathfrak{g})$, $U(\mathfrak{h}) \subseteq U(\mathfrak{g})_0$, and $U(\mathfrak{n}^\pm)_\eta \neq 0$ only if $\eta \in \pm Q^+$.

Since the restriction of $(\cdot,\cdot)$ to $\mathfrak{h}$ is nondegenerate, there exists a unique linear isomorphism $\mathfrak{h}^* \to \mathfrak{h}, \lambda \mapsto t_\lambda$, where t_λ is the unique element of $\mathfrak{h}$ satisfying $(t_\lambda, h) = \lambda(h)$ for all $h \in \mathfrak{h}$. Define a symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}^*$ by requiring that this isomorphism be orthogonal, i.e., $(\lambda, \mu) = (t_\lambda, t_\mu)$ for all $\lambda, \mu \in \mathfrak{h}^*$. Notice that

$$(\lambda, \mu) = \lambda(t_\mu) = \mu(t_\lambda) \quad \text{for all} \quad \lambda, \mu \in \mathfrak{h}^*$$

and that $(\lambda, \lambda) \neq 0$ for all $\lambda \in \mathfrak{h}^* \setminus \{0\}$. Moreover, the matrix of the restriction of $(\cdot,\cdot)$ to the subspace generated by Q with respect to the basis $\{\alpha_i : i \in I\}$ is SC, i.e.,

$$(\alpha_i, \alpha_j) = s_i c_{ij} \qquad \text{for all} \qquad i, j \in I.$$

For $i \in I$, let $r_i \in \mathrm{End}_{\mathbb{F}}(\mathfrak{h}^*)$ be given by

$$r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i = \lambda - 2\frac{(\lambda,\, \alpha_i)}{(\alpha_i,\, \alpha_i)}\alpha_i.$$

Observe that $r_i^2 = 1, r_i(\lambda) = \lambda$ if $(\lambda, \alpha_i) = 0$, and $r_i(\alpha_i) = -\alpha_i$. Because of this, r_i is called a simple reflection. Notice that

$$\alpha_i = \omega_i - r_i(\omega_i). \tag{1.1.3}$$

Definition 1.1.16. The *Weyl group* $\mathscr{W}$ of $\mathfrak{g}$ is the subgroup of $\mathrm{Aut}_{\mathbb{F}}(\mathfrak{h}^*)$ generated by the simple reflections. Given $w \in \mathscr{W}$, an expression $w = r_{i_1} r_{i_2} \cdots r_{i_l}$ is a said to be a *reduced expression for* w if l is minimal. In that case, l is called the *length of* w. The length of $w \in \mathscr{W}$ will be denoted by $\ell(w)$. Two elements $\mu, \nu \in \mathfrak{h}^*$ in the same $\mathscr{W}$-orbit are said to be $\mathscr{W}$-conjugate. A root α of $\mathfrak{g}$ is said to be a *real root* if it is $\mathscr{W}$-conjugate to a simple root. Otherwise, α is said to be an *imaginary root*. ◊

Proposition 1.1.17. For all $\alpha \in R$ and $w \in \mathscr{W}$ we have $\dim(\mathfrak{g}_\alpha) = \dim(\mathfrak{g}_{w\alpha})$. In particular, if α is a real root, then $\dim \mathfrak{g}_\alpha = 1$. In that case, the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_{\pm\alpha}$ is isomorphic to $\mathfrak{sl}_2$. □

Proposition 1.1.18. The following conditions are equivalent:

(a) $\mathscr{W}$ is finite.

(b) R is finite.

(c) $\mathfrak{g}$ is finite-dimensional.

(d) Every root is real. □

We will also make use the of braid group associated to C.

Definition 1.1.19. The braid group $\mathscr{B}$ associate to $\mathfrak{g}$ is the group generated by elements $T_i, i \in I$, subject to the following defining relations:

$$\begin{array}{lcl} T_iT_j = T_jT_i, & \text{if} & c_{ij} = 0, \\ T_iT_jT_i = T_jT_iT_j, & \text{if} & c_{ij}c_{ji} = 1, \\ (T_iT_j)^2 = (T_jT_i)^2, & \text{if} & c_{ij}c_{ji} = 2, \\ (T_iT_j)^3 = (T_jT_i)^3, & \text{if} & c_{ij}c_{ji} = 3. \end{array}$$

◊

Proposition 1.1.20.

(i) The Weyl group $\mathscr{W}$ of $\mathfrak{g}$ is the quotient of $\mathscr{B}$ by the normal subgroup generated by $T_i^2 - 1$.

(ii) If $w = r_{i_1} \cdots r_{i_l}$ and $w = r_{i'_1} \cdots r_{i_l}$ are two reduced expressions for $w \in \mathscr{W}$, then $T_{i_1} \cdots T_{i_l} = T_{i'_1} \cdots T_{i'_l}$. □

The second statement of the above proposition can be rephrased by saying that we have a function $\mathscr{B} \to \mathscr{W}$ given by

(1.1.4) $$w \mapsto T_w := T_{i_1} \cdots T_{i_l}$$

where $w = r_{i_1} \cdots r_{i_l}$ is any reduce expression for w (this is not a group homomorphism).

We end this subsection recalling some extra facts that hold in case $\mathfrak{g}$ is finite-dimensional as well as some extra terminology. In that case, there exists a unique root which is maximal in R with respect to the partial order on $\mathfrak{h}^*$. This is also the highest root of $\mathfrak{g}$ and it will be denoted by θ. Similarly, there exists a unique element of maximum length in $\mathscr{W}$ which will be denoted by w_0. The order of w_0 is two and, if $w_0 = r_{i_1} \cdots r_{i_l}$ is a reduced expression for w_0, then

$$R^+ = \{\alpha_{i_1},\ r_{i_1}\alpha_{i_2},\ r_{i_1}r_{i_2}\alpha_{i_3}, \cdots,\ r_{i_1} \cdots r_{i_{l-1}}\alpha_{i_l}\}.$$

The number $r^\vee := \max\{s_j : i \in I\} = \max\{c_{ij}c_{ji} : i \neq j\}$ is called the lacing number of $\mathfrak{g}$. The set $\{(\alpha, \alpha) : \alpha \in R\}$ has at most two elements and the cardinality is one if and only if $\mathfrak{g}$ is simply laced, i.e., if $r^\vee = 1$. If $r^\vee > 1$, a positive root α is said to be a short root if $(\alpha, \alpha) < (\theta, \theta)$. Otherwise, α is said to be a long root (in the simply laced case we shall use the convention that all roots are short and long). We have $(\alpha, \alpha) = 2$ if α is a short root and $(\alpha, \alpha) = 2r^\vee$ if α is a long root. The Coxeter number of $\mathfrak{g}$ is $h = 1 + |\theta|$ while the dual Coxeter number is $h^\vee = 1 + |\vartheta|$ where ϑ is the highest short root.

Given a nonzero $\lambda \in \mathfrak{h}^*$, define

$$\lambda^\vee = \frac{2t_\lambda}{(\lambda, \lambda)}.$$

We have $\alpha_i^\vee = h_i$ and, if $\lambda = \sum_{i \in I} m_i \alpha_i \neq 0$, then $\lambda^\vee = \sum_{i \in I} \frac{s_i}{r^\vee} m_i h_i$. Moreover, if $\lambda \in R$, then $\frac{s_i}{r^\vee} m_i \in \mathbb{Z}$ for all $i \in I$ and $R^\vee := \{\alpha^\vee : \alpha \in R\}$ is isomorphic to the root system associated to the matrix C^t. Every element of P is $\mathscr{W}$-conjugate to a unique element of P^+. We also have:

Proposition 1.1.21. Let $\lambda \in P^+$ and set $\mathrm{wt}(\lambda) = \{w(\mu) : w \in \mathscr{W}, \mu \in P^+, \mu \leq \lambda\}$.

(i) $w(\lambda) \leq \lambda$ for all $w \in \mathscr{W}$. In particular, $\nu \leq \lambda$ for all $\nu \in \mathrm{wt}(\lambda)$.

(ii) For all $\lambda \in P^+$, $\mathrm{wt}(\lambda)$ is a finite set, $w_0(\lambda) \leq \mu$ for all $\mu \in \mathrm{wt}(\lambda)$, and $w_0(\lambda) \in -P^+$.

(iii) If $i \in I$ and $w \in \mathscr{W}$ are such that $\ell(r_i w) = \ell(w) + 1$, then $w^{-1}(\alpha_i) \in R^+$. In particular, $w(\lambda) + \alpha_i \notin \mathrm{wt}(\lambda)$. □

Remark 1.1.22. For each $i \in I$, $-w_0(\omega_i) = \omega_j$ for some $j \in I$. Thus, w_0 defines an involution on $I, i \mapsto w_0(i) = j$. ◊

An element $\lambda \in P^+$ is said to be minuscule $\mathrm{wt}(\lambda) \cap P^+ = \{\lambda\}$, i.e., if $\mathrm{wt}(\lambda)$ is the $\mathscr{W}$-orbit of λ. It turns out that a nonzero minuscule weight is necessarily a fundamental weight (but not all fundamental weights are minuscule). Given $\lambda \in P^+$, let $\mathrm{supp}(\lambda) = \{i \in I : \lambda(h_i) \neq 0\}$ and let $\mathscr{W}^\lambda$ be the subgroup of $\mathscr{W}$ generated by $\{r_i : i \in I \setminus \mathrm{supp}(\lambda)\}$.

Lemma 1.1.23. Let $\lambda \in P^+$.

(i) $\mathscr{W}^\lambda = \{w \in W : w(\lambda) = \lambda\}$.

(ii) Each left coset of $\mathscr{W}^\lambda$ in $\mathscr{W}$ contains a unique element of minimal length.

(iii) Let $\mathscr{W}_\lambda$ the set of all left coset representatives of minimal length. If $w \in \mathscr{W}_\lambda$ and $i \in I$ are such that $\ell(r_i w) = \ell(w) - 1$, then $r_i w \in \mathscr{W}_\lambda$. □

In the next lemma we record some special properties of the fundamental weights in the case that $\mathfrak{g}$ is not of exceptional type. For this lemma, we identify I with the finite set $\{1, \ldots, n\}$ as in Table 1.1.1.

Lemma 1.1.24. Suppose $\mathfrak{g}$ is not of exceptional type and let $i \in I$.

(i) If $\lambda \in P^+$ is such that $\lambda < \omega_i$, then either $\lambda = 0$ or $\lambda = \omega_j$ for some $j < i$.

(ii) If $j \in I$ is such that $i > j$, then $\omega_i - (\omega_j + \alpha_j) \notin Q^+$.

(iii) If $\lambda \in P^+$ is such that $\lambda < \omega_i$, then $\omega_i - (\lambda + 2\alpha_j) \notin \mathrm{wt}(\omega_i)$ for all $j \in I$. □

1.2. Quantum groups

In this subsection we overview (one version of) the definition of Drinfeld-Jimbo's quantized universal enveloping algebra $U_q(\mathfrak{g})$ most commonly known as the quantum group over $\mathfrak{g}$.

Given $q \in \mathbb{F}$ and $m \in \mathbb{Z}$, define

$$[m]_q = q^{m-1} + q^{m-3} + \cdots + q^{3-m} + q^{1-m}.$$

Notice that $[m]_1 = m$ and that, if $q^2 \neq 1$, then $[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$. The numbers $[m]_p$ are often referred to as quantum numbers. One can then define quantum factorial numbers in the obvious way:

$$[0]_q! = 1 \qquad \text{and} \qquad [m]_q! = [m]_q[m-1]_q \ldots [1]_q \qquad \text{for} \qquad m > 0.$$

Also, given $m, n \in \mathbb{Z}$, $m \geq 0$ such that $q^{2m} \neq 1$, define the quantum binomial

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q[n-1]_q \cdots [n-m+1]_q}{[m]_q!}.$$

Notice that, if $n \geq m$, then $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]_q = \frac{[n]_q!}{[n-m]_q![m]_q!}$. One easily checks that there exists $f \in \mathbb{Z}[u, u^{-1}]$ (depending only on m and n but not on q) such that $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]_q = f(q)$ and, therefore, one can remove the hypothesis $q^{2m} \neq 1$. Moreover, if $n \geq m$, we have $f(1) = \binom{n}{m}$.

Henceforth we fix a nonzero $q \in \mathbb{F}$ which is not a root of unity and set $q_i := q^{s_i}$.

Definition 1.2.1. The *quantized universal enveloping algebra* $U_q(\mathfrak{g})$ is the associative algebra (with 1) with generators $x_i^\pm, k_i^{\pm 1}, g_j^{\pm 1}, i \in I, j \in I \setminus I'$, satisfying the following defining relations:

$$k_i k_i^{-1} = 1, \quad k_i k_j = k_j k_i, \quad k_i x_j^\pm k_i^{-1} = q_i^{\pm c_{ij}} x_j^\pm, \quad [x_i^+, x_j^-] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \quad \text{for all} \quad i, j \in I,$$

$$\sum_{m=0}^{1-c_{ij}} (-1)^m \left[\begin{smallmatrix} 1-c_{ij} \\ m \end{smallmatrix}\right]_{q_i} (x_i^\pm)^{1-c_{ij}-m} x_j^\pm (x_i^\pm)^m = 0 \qquad \text{for all} \qquad i, j \in I, i \neq j,$$

$$g_j g_j^{-1} = 1, \quad g_j g_k = g_k g_j, \quad k_i g_j = g_j k_i, \quad g_j x_i^\pm g_j^{-1} = q^{\pm \delta_{ij}} x_i^\pm, \quad \text{for all} \quad i, j, k \in I, j, k \notin I'.$$

The relations in the second line are called *quantum Serre's relations*. Denote by $U_q(\mathfrak{d})$ the subalgebra of $U_q(\mathfrak{g})$ generated by $g_j^{\pm 1}, j \in I \setminus I'$, by $U_q(\mathfrak{h}')$ the subalgebra generated by $k_i^{\pm 1}, i \in I$, by $U_q(\mathfrak{h})$ the one generated by $U_q(\mathfrak{h}')$ and $U_q(\mathfrak{d})$, by $U_q(\mathfrak{n}^\pm)$ the ones generated by $x_i^\pm, i \in I$, respectively, and by $U_q(\mathfrak{g}')$ the one generated by $U_q(\mathfrak{n}^\pm)$ and $U_q(\mathfrak{h}')$. ◊

In the case that $\mathfrak{g}$ is of affine type, the quantum group $U_q(\mathfrak{g})$ is frequently called a quantum affine algebra.

Remark 1.2.2. Notice that the above definition makes sense as long as $q^{s_i} \neq \pm 1$. We will see below that one can make sense of quantum groups even when $q^{s_i} \neq \pm 1$ and, also, for $q = 1$ in which case the resulting algebra is essentially $U(\mathfrak{g})$. The elements k_i and g_j are, roughly speaking, "q-exponentials" of the elements h_i and d_j, respectively. This is made more precise if one works with a larger version of $U_q(\mathfrak{g})$ which, in some instances, is more convenient than the above. Namely, one replaces the generators k_i and g_j by generators q^h with h running in the $\mathbb{Z}$-span of h_i and d_j. One then imposes the relations $q^0 = 1, q^{h+h'} = q^h q^{h'}$, and $q^h x_j^\pm q^{-h} = q^{\alpha_j(h)} x_j^\pm$. The element k_i then corresponds to $q^{s_i h_i}$ while g_j corresponds to q^{d_j}. ◊

The next proposition establishes a quantum analogue of the triangular decomposition.

Proposition 1.2.3. The multiplication of $U_q(\mathfrak{g})$ induces isomorphism of vector spaces:

$$U_q(\mathfrak{n}^-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}^+) \to U_q(\mathfrak{g}) \quad \text{and} \quad U_q(\mathfrak{h}') \otimes U_q(\mathfrak{d}) \to U_q(\mathfrak{h}). \qquad \square$$

Given $J \subseteq I$, we denote by $U_q(\mathfrak{g}_J)$ the subalgebra of $U_q(\mathfrak{g})$ generated by $k_j, x_j^\pm, j \in J$, and set $U_q(\mathfrak{h}_J) = U_q(\mathfrak{h}) \cap U_q(\mathfrak{g}_J)$ and so on. If $J = \{i\}$ for some $i \in I$ we simplify notation and write $U_q(\mathfrak{g}_i)$. It is not difficult to see that $U_q(\mathfrak{g}_i)$ is isomorphic to $U_{q_i}(\mathfrak{sl}_2)$ and similarly for general J.

There exists a unique Q-gradation on $U_q(\mathfrak{g})$ such that the degree of $x_i^\pm$ is $\pm\alpha_i$, respectively, and $U_q(\mathfrak{h})$ is contained in the graded piece of degree 0. For $\eta \in Q$ we let $U_q(\mathfrak{g})_\eta$ denote the graded piece of degree η. Notice that if x has degree η, then $k_i x k_i^{-1} = q^{(\alpha_i,\eta)}x$ for all $i \in I$. As before, we set $U_q(\mathfrak{n}^\pm)_\eta = U_q(\mathfrak{n}^\pm) \cap U_q(\mathfrak{g})_\eta$.

Proposition 1.2.4. There exist unique algebra homomorphisms

$$\varepsilon : U_q(\mathfrak{g}) \longrightarrow \mathbb{F}, \quad \Delta : U_q(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g}) \otimes_{\mathbb{F}} U_q(\mathfrak{g}), \quad \text{and} \quad S : U_q(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g})^{op}$$

such that

$$\begin{gathered} \varepsilon(k_i) = \varepsilon(g_j) = 1, \quad \varepsilon(x_i^\pm) = 0, \\ \Delta(k_i) = k_i \otimes k_i, \quad \Delta(g_j) = g_j \otimes g_j, \quad \Delta(x_i^+) = x_i^+ \otimes 1 + k_i \otimes x_i^+, \quad \Delta(x_i^-) = x_i^- \otimes k_i^{-1} + 1 \otimes x_i^-, \\ S(k_i) = k_i^{-1}, \quad S(g_j) = g_j^{-1}, \quad S(x_i^+) = -k_i^{-1}x_i^+, \quad S(x_i^-) = -x_i^- k_i \end{gathered}$$

for all $i, j \in I, j \notin I'$. $\square$

It follows from the above proposition that $U_q(\mathfrak{g})$ has a structure of a non cocomutative Hopf algebra with counit ε, comultiplication Δ, and (invertible) antipode S. Notice that $U_q(\mathfrak{g}')$, $U_q(\mathfrak{n}^\pm)$, and $U_q(\mathfrak{h})$ are Hopf subalgebras of $U_q(\mathfrak{g})$. We shall denote by H^0 the augmentation ideal of a Hopf algebra H, i.e., the kernel of ε.

We end this subsection with an overview of the constructions and results establishing the relation between $U_q(\mathfrak{g})$ and $U(\mathfrak{g})$. For that purpose, assume that q is transcendental over $\mathbb{Q}$ so that the field $\mathbb{Q}(q)$ is a subfield of $\mathbb{F}$. Let also $\mathbb{A} = \mathbb{Q}[q, q^{-1}]$ and notice that $[m]_{q_i}, [m]_{q_i}!, \left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_{q_i} \in \mathbb{A}$, for all $m, n \in \mathbb{Z}, n \geq 0$ (in fact, they lie in $\mathbb{Z}[q, q^{-1}]$ as remarked above).

Definition 1.2.5. If V is an $\mathbb{F}$-vector space, an $\mathbb{A}$-*lattice* of V is the $\mathbb{A}$-span of an $\mathbb{F}$-basis of V. An $\mathbb{A}$-*form* of $U_q(\mathfrak{g})$ is an $\mathbb{A}$-subalgebra of $U_q(\mathfrak{g})$ which is also an $\mathbb{A}$-lattice of $U_q(\mathfrak{g})$. $\lozenge$

Given $\xi \in \mathbb{F}$, consider the ring homomorphism

$$\mathrm{ev}_\xi : \mathbb{A} \to \mathbb{F}, \quad f(q) \mapsto f(\xi).$$

Denote by $\mathbb{F}_\xi$ the $\mathbb{A}$-module induced by ev_ξ, i.e., $\mathbb{F}_\xi$ is the $\mathbb{Q}$-vector space $\mathbb{F}$ where q acts as multiplication by ξ. Then, for an $\mathbb{A}$-form U of $U_q(\mathfrak{g})$, set

$$U_\xi = \mathbb{F}_\xi \otimes_{\mathbb{A}} U$$

which is naturally an $\mathbb{F}$-algebra. Moreover, if U is an $\mathbb{A}$-Hopf subalgebra of $U_q(\mathfrak{g})$, then U_ξ has a natural Hopf algebra structure. The algebra U_ξ is called the *specialization* of $U_q(\mathfrak{g})$ at $q = \xi$ via U. The elements of the form $\lambda \otimes x$ of U_ξ, with $x \in U$ and $\lambda \in \mathbb{F}$, will be denoted simply by λx. Notice that if $\xi = q$, then U_ξ is naturally isomorphic to $U_q(\mathfrak{g})$.

There are two $\mathbb{A}$-forms of $U_q(\mathfrak{g})$ used in the literature. One was first studied by Kac and De Concini and is often referred to as the nonrestricted integral form of $U_q(\mathfrak{g})$. The other was first considered by Lusztig in connection with the theory of algebraic groups and is often referred to as the restricted integral form of $U_q(\mathfrak{g})$ as it is also related to restricted Lie algebras in positive characteristic. It turns out that if ξ is not a root of unity, then the specialized algebras obtained from them are isomorphic to the algebra given by generators and relations as above with ξ in place of q. However, the situation is very different if ξ is a nontrivial root of unity. We will focus only in the case $\xi = 1$ here in which case both specializations are again isomorphic. Thus, for simplicity, we introduce the nonrestricted form only. Namely, we consider the $\mathbb{A}$-subalgebra U of $U_q(\mathfrak{g})$ generated by $x_i^\pm, k_i, g_j, \frac{g_j - g_j^{-1}}{q-q^{-1}}, i, j \in I, j \notin I'$.

Theorem 1.2.6. U is an $\mathbb{A}$-form of $U(\mathfrak{g})$ and $U(\mathfrak{g})$ is isomorphic to the quotient of U_1 by the ideal generated by $k_i - 1, g_j - 1, i, j \in I, j \notin I'$. □

Remark 1.2.7. Notice that, under the isomorphism of the above theorem, h_i is the image of the element $\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}$ which is an $\mathbb{A}$-basis element of U (similar remark is valid for the elements d_j). It turns out that the restricted integral form is an $\mathbb{A}$-Hopf subalgebra of $U_q(\mathfrak{g})$ and the induced Hopf algebra structure on $U(\mathfrak{g})$ coincides with the usual one: $\varepsilon(x) = 0, \Delta(x) = x \otimes 1 + 1 \otimes x, S(x) = -x$, for all $x \in \mathfrak{g}$. ◊

1.3. Integrable representations in category $\mathscr{O}$

We now give a review on a certain category of representations of Kac-Moody algebras (and their quantum groups) which runs parallel to that of finite-dimensional representations of finite-dimensional simple Lie algebras. This is the category $\mathscr{O}^{\rm int}$ of integrable modules in Bernstein-Gelfand-Gelfand's category $\mathscr{O}$. The classical and quantum are developed essentially in the same way. For simplicity we will consider the classical case and then point out the small modifications needed to treat the quantum case.

Let V be a $\mathfrak{g}$-module. Given $\mu \in \mathfrak{h}^*$ define

$$V_\mu = \{v \in V : hv = \mu(h)v \text{ for all } h \in \mathfrak{h}\}. \tag{1.3.1}$$

The space V_μ is said to be the weight space of V of weight μ and the nonzero vectors of V_μ are called weight vectors of weight μ. Let $\mathrm{wt}(V) = \{\mu \in P : V_\mu \neq 0\}$ be the set of weights of V. The module V is said to be a weight-module if

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu.$$

Notice that the relations $[h, x_i^\pm] = \pm\alpha_i(h)x_i^\pm, i \in I, h \in \mathfrak{h}$, imply

$$x_i^\pm V_\mu \subseteq V_{\mu \pm \alpha_i} \qquad \text{for all} \qquad i \in I. \tag{1.3.2}$$

In particular, the sum of the weight spaces of a given representations is a subrepresentation. A weight vector v is said to be a highest-weight vector if $x_i^+ v = 0$ for all $i \in I$ (equivalently, if $\mathfrak{n}^+ v = \{0\}$). Equation (1.3.2) implies that the weights of a subrepresentation generated by a highest-weight vector are smaller than the weight of its generating vector. This explains the term "highest-weight vector". A representation which is generated by a highest-weight vector is said to be a highest-weight module. Using (1.3.2) once more one easily sees that highest-weight modules are weight-modules. Notice that the sum of the proper submodules of a highest-weight module is again a proper submodule. This proves the following proposition.

Proposition 1.3.1. Let V be a highest-weight module. Then, V has a unique maximal proper submodule and, hence, a unique irreducible quotient. In particular, V is indecomposable. □

Given $\lambda \in \mathfrak{h}^*$, let $M(\lambda)$ be the $\mathfrak{g}$-module generated by a vector satisfying the defining relation of being a highest-weight vector of weight λ. In other words, $M(\lambda)$ is the quotient of $U(\mathfrak{g})$ by the left ideal generated by $h - \lambda(h), x_i^+$ for all $h \in \mathfrak{h}$ and $i \in I$. One easily sees that, as an $\mathfrak{n}^-$-module, $M(\lambda)$ is isomorphic to $U(\mathfrak{n}^-)$. By definition, any other highest-weight module of highest weight λ is a quotient of $M(\lambda)$, i.e., $M(\lambda)$ is the universal highest-weight module of highest weight λ. It is also called the Verma module of highest-weight λ. We shall denote by $V(\lambda)$ the unique irreducible quotient of $M(\lambda)$. Evidently that $M(\lambda)$ is isomorphic to $M(\mu)$ if and only if $\lambda = \mu$ and similarly for their irreducible quotients.

A $\mathfrak{g}'$-module V is said to be integrable if the elements $x_i^\pm, i \in I$, act locally nilpotently, i.e., if for every $v \in V$ and $i \in I$, there exists $m \in \mathbb{Z}_{\geq 0}$ such that $(x_i^\mp)^m v = 0$. A $\mathfrak{g}$-module is said to be integrable if it is integrable when regarded as a $\mathfrak{g}'$-module.

Lemma 1.3.2. Let V be a $\mathfrak{g}'$-module generated by a vector v which is an eigenvector for the action of $\mathfrak{h}'$. Suppose there exist $m_i \in \mathbb{Z}_{>0}, i \in I$, such that $(x_i^\pm)^{m_i} v = 0$. Then, V is integrable. □

Proposition 1.3.3. If V is an integrable $\mathfrak{g}'$-module, then $\mathfrak{h}'$ act semisimply on V and the eigenvalues of h_i are in $\mathbb{Z}$ for all $i \in I$. In particular, if V is a $\mathfrak{g}$-module, then it is a weight-module and $V_\mu \neq 0$ only if $\mu \in P$. Also, $U(\mathfrak{g}_i)v$ is a finite-dimensional $\mathfrak{g}_i$-submodule of V for all $i \in I, v \in V$. □

Theorem 1.3.4. Let $\lambda \in \mathfrak{h}^*$ and v be a highest-weight vector of $M(\lambda)$. The module $V(\lambda)$ is integrable if and only if $\lambda \in P^+$. In that case, it is the quotient of $M(\lambda)$ by the submodule generated by $(x_i^-)^{\lambda(h_i)+1} v$ for all $i \in I$. □

Remark 1.3.5. The image of the vectors $(x_i^-)^{\lambda(h_i)} v, i \in I$, in $V(\lambda)$ are all nonzero as a consequence of the study of the $\mathfrak{sl}_2$ case. In particular, $V(0)$ is the one-dimensional trivial representation. The notions of lowest-weight vector and module is defined by interchanging the role of the generators $x_i^\pm$. Evidently, all results proved for highest-weight modules are true for lowest-weight ones mutatis-mutandis. ◊

Given $\lambda \in \mathfrak{h}^*$, set

$$D(\lambda) = \lambda - Q^+ = \{\mu \in \mathfrak{h}^* : \lambda - \mu \in Q^+\}.$$

Let $\mathscr{O}$ be the category of weight-modules with finite dimensional weight spaces whose weights lie in the union of finitely many sets of the form $D(\lambda), \lambda \in P^+$. Let also $\mathscr{O}^{\rm int}$ be the full subcategory of $\mathscr{O}$ consisting of integrable modules. Notice that the Verma modules $M(\lambda), \lambda \in P^+$, are in $\mathscr{O}$ and, hence, so are their irreducible quotients.

Theorem 1.3.6. If V is a simple module in category $\mathcal{O}^{\rm int}$ it is isomorphic to $V(\lambda)$ for some $\lambda \in P^+$. Moreover, every object in $\mathcal{O}^{\rm int}$ is completely reducible. □

If V is a weight-module with finite-dimensional weight spaces, the character of V is the function $\mathrm{ch}(V) : P \to \mathbb{Z}$ given by

$$\mathrm{ch}(V)(\mu) = \dim(V_\mu).$$

The action of $\mathscr{W}$ on P can be naturally extended to an action of $\mathscr{W}$ on $\mathbb{Z}^P$ by

$$(wf)(\mu) = f(w(\mu)) \qquad \text{for all} \qquad w \in \mathscr{W}, f \in \mathbb{Z}^P, \mu \in P.$$

Proposition 1.3.7. Let V be an integrable weight-module. Then, $V_\mu \cong V_{w(\mu)}$ as vector spaces for all $\mu \in P, w \in \mathscr{W}$. In particular, if V has finite-dimensional-weight spaces, $\mathrm{ch}(V)$ is $\mathscr{W}$-invariant. □

For $\mu \in P$, let $e^\mu \in \mathbb{Z}^P$ be the characteristic function of $\{\mu\}$, i.e., $e^\mu(\lambda) = \delta_{\lambda,\mu}$. Then, every element $f \in \mathbb{Z}^P$ can be reinterpreted as a formal sum of the form $f = \sum_{\mu\in P} f(\mu)e^\mu$.

Corollary 1.3.8. If $\mathfrak{g}$ is finite-dimensional and $\lambda \in P^+$ is minuscule, then $\mathrm{ch}(V(\lambda)) = \sum_{w\in\mathscr{W}_\lambda} e^{w(\lambda)}$. □

More generally, we have:

Theorem 1.3.9. For all $\lambda \in P^+$, $\mathrm{ch}(V(\lambda))$ is given by the Weyl-Kac character formula. □

Corollary 1.3.10. If $\mathfrak{g}$ is finite-dimensional and $\lambda \in P^+$, then $\mathrm{wt}(V(\lambda)) = \mathrm{wt}(\lambda)$. In particular, $\mathcal{O}^{\rm int}$ is the category of finite-dimensional $\mathfrak{g}$-modules. □

Corollary 1.3.11. Suppose $\mathfrak{g}$ is finite-dimensional and V is an integrable weight-module such that $\mathrm{wt}(V)$ has a unique maximal element, say λ. Then, $\mathrm{wt}(V) = \mathrm{wt}(\lambda)$. □

We will need an expression for vectors spanning the weight spaces $V(\lambda)_{w(\lambda)}$, $\lambda \in P^+, w \in \mathscr{W}$. Fix a reduced expression $w = r_{i_l}\cdots r_{i_1}$ for w and consider

$$m_j = (r_{i_{j-1}}\cdots r_{i_1}\lambda)(h_{i_j}). \tag{1.3.3}$$

Proceeding inductively on j, it follows from Proposition 1.1.21(iii) that $m_j \in \mathbb{Z}_{\geq 0}$. Let v be a highest-weight vector for $V = V(\lambda)$ and set

$$v_w = (x_{i_l}^-)^{m_l}\cdots(x_{i_1}^-)^{m_1}v \in V_{w(\lambda)}. \tag{1.3.4}$$

Since $\dim(V_{w(\lambda)}) = 1$ by the previous proposition and $v_w \neq 0$ by the first comment on Remark 1.3.5, v_w spans $V_{w(\lambda)}$. Moreover, by reverting the process, we see that v is a nonzero multiple of $(x_{i_1}^+)^{m_1}\cdots(x_{i_l}^+)^{m_l}(x_{i_l}^-)^{m_l}\cdots(x_{i_1}^-)^{m_1}v$.

Given $V \in \mathcal{O}^{\rm int}$, we can write $V = \bigoplus\limits_{\lambda\in P^+} V(\lambda)^{\oplus[V:\lambda]}$ for unique nonnegative integers $[V:\lambda]$. The number $[V:\lambda]$ is called the multiplicity of $V(\lambda)$ in V. In particular,

$$\mathrm{ch}(V) = \sum_{\lambda\in P^+}[V:\lambda]\,\mathrm{ch}(V(\lambda)).$$

Conversely, one can algorithmically recover the numbers $[V : \lambda]$ from $\mathrm{ch}(V)$ (notice this is a very laborious task in practice). In other words, from a theoretical point of view, the collection of numbers $[V : \lambda], \lambda \in P^+$, and the collection of numbers $\dim(V_\mu), \mu \in P$, provide the same information about V.

Recall that if V and W are representations of a Hopf algebra A, then $V \otimes W$ can be turned into a representation of A by setting

$$(1.3.5) \qquad x(v \otimes w) = \Delta(x)(v \otimes w) \qquad \text{for all} \qquad v \in V, w \in W, x \in A$$

where $V \otimes W$ is naturally interpreted as a representation for the algebra $A \otimes A$. In particular, since $U(\mathfrak{g})$ is a Hopf algebra, the tensor product of two given $\mathfrak{g}$-modules is defined in this way. Notice that, since the formula for $\Delta(x)$ is symmetric (the comultiplication of $U(\mathfrak{g})$ is cocommutative), we have

$$(1.3.6) \qquad V \otimes W \cong W \otimes V.$$

One can define the multiplication of the elements e^μ by setting

$$e^\mu e^\nu = e^{\mu+\nu}.$$

This multiplication extends naturally to the subset of $\mathbb{Z}^P$ of functions with finite support turning it into a ring which is nothing else but the group ring $\mathbb{Z}[P]$. Evidently, this does not extend to a multiplication on all of $\mathbb{Z}^P$. However, one can multiply characters of objects in $\mathscr{O}^{\mathrm{int}}$ in this way. In fact, $\mathscr{O}^{\mathrm{int}}$ is a tensor category and, for every two modules V, W in $\mathscr{O}^{\mathrm{int}}$, we have

$$(1.3.7) \qquad \mathrm{ch}(V \otimes W) = \mathrm{ch}(V)\mathrm{ch}(W).$$

The above essentially follows from the following which is easily checked using (1.3.5) and the formula for $\Delta(x)$ given in Remark 1.2.7.

$$(1.3.8) \qquad V_\mu \otimes W_\nu \subseteq (V \otimes W)_{\mu+\nu}.$$

Proposition 1.3.12. Let $\lambda, \mu \in P^+ \setminus \{0\}$. Then, $[V(\lambda) \otimes V(\mu) : \nu] \neq 0$ for some $\nu \in P^+, \nu \neq \lambda + \mu$. □

The dual space V^* of a representation for a Hopf algebra A can be turned into a representation as well by setting

$$(1.3.9) \qquad (xf)(v) = f(S(x)v) \qquad \text{for all} \qquad f \in V^*, v \in V, x \in A.$$

If V is a weight-module for $\mathfrak{g}$, then $V^r := \bigoplus_{\mu \in P} (V_\mu)^*$ is a submodule of V^* usually called the restricted dual of V (if V is finite-dimensional, then $V^r = V^*$). We will only consider restricted duals, so we abuse of notation and write V^* instead of V^r. Since $S(x) = -x$ for all $x \in \mathfrak{g}$ (Remark 1.2.7), it easily follows that

$$(1.3.10) \qquad (V_\mu)^* = (V^*)_{-\mu} \qquad \text{for all} \qquad \mu \in P.$$

Notice also that, if all the weight-spaces are finite-dimensional, then $(V^*)^* \cong V$ as a $\mathfrak{g}$-module.

Proposition 1.3.13. Let U, V, W be $\mathfrak{g}$-weight-modules.

(i) If V is integrable, so is V^*.

(ii) If $0 \to U \to V \to W \to 0$ is a short exact sequence, so is $0 \to W^* \to V^* \to U^* \to 0$.

(iii) Suppose $\dim(V_\mu) < \infty$ for all $\mu \in P$. Then, V is irreducible if and only if V^* is irreducible. □

Remark 1.3.14. Notice that if V is a simple highest-weight module of highest weight λ, then V^* is a simple lowest-weight module of lowest weight $-\lambda$. However, the dual of the Verma module is not a lowest-weight module in general as it is easily seen from part (ii) of the above proposition (although $-\lambda$ is its unique minimal weight). ◊

The next corollary follows from Proposition 1.1.21(ii) and the above remark.

Corollary 1.3.15. Suppose $\mathfrak{g}$ is finite-dimensional and let $\lambda \in P^+$. Then, $V(\lambda)$ is a lowest-weight module of lowest weight $w_0(\lambda)$ and, if v is a highest-weight vector, v_{w_0} is a lowest-weight vector. In particular, $V(\lambda)^* \cong V(-w_0(\lambda))$. □

Remark 1.3.16. Before turning to the quantum setting, we find interesting to remark the following. Let $V = V(\lambda)$ for some $\lambda \in \mathfrak{h}^*$ and notice that V can be naturally considered as $\mathbb{Z}_{\geq 0}^{I\setminus I'}$-graded $\mathfrak{g}'$-module. Namely, given $\boldsymbol{r} = (r_j)_{j\in I\setminus I'}$, the $\boldsymbol{r}$-th graded piece of V is set to be

$$V[\boldsymbol{r}] = \{v \in V : d_j v = (\lambda(d_j) - r_j)v \text{ for all } j \in I \setminus I'\} = \bigoplus_{\mu\,:\,\mu(d_j) = \lambda(h_j) - r_j} V_\mu.$$

Thus, if we set $\boldsymbol{\delta}_i = (\delta_{ij})_{j\in I\setminus I'}, i \in I$, we have $x_i^\pm V[\boldsymbol{r}] \subseteq V[\boldsymbol{r} \mp \boldsymbol{\delta}_i]$ for all $i \in I, \boldsymbol{r} \in \mathbb{Z}_{\geq 0}^{I\setminus I'}$. It is not difficult to see that, if $\lambda, \mu \in \mathfrak{h}^*$ are such that $(\lambda - \mu)(d_j) \in \mathbb{Z}$ for all $j \in I \setminus I'$, then $V(\lambda) \cong V(\mu)$ as a $\mathbb{Z}_{\geq 0}^{I\setminus I'}$-graded $\mathfrak{g}'$-module. Moreover, one can easily recover the $\mathfrak{g}$-module structure of $V(\lambda)$ from the graded $\mathfrak{g}'$-module structure together with the values $\lambda(h_j), j \in I \setminus I'$. Thus, studying modules in $\mathscr{O}^{\rm int}$ is essentially equivalent to studying $\mathbb{Z}_{\geq 0}^{I\setminus I'}$-graded $\mathfrak{g}'$-modules. Also, without loss of generality, we restrict our attention to modules whose weights lie in $P_d = \{\mu \in P : \mu(d_j) \in \mathbb{Z} \text{ for all } j \in I \setminus I'\}$ from now on (evidently, $P = P_d$ if C is invertible). ◊

We now turn to the quantum case. One easily checks that, given a function $\sigma : I \longrightarrow \{-1, 1\}$, there exists a unique $\mathbb{F}$-algebra automorphism $U_q(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g})$, also denoted by σ, such that

$$\sigma(x_i^\pm) = \sigma(i)x_i^\pm, \qquad \sigma(k_i) = \sigma(i)k_i, \qquad \sigma(g_j) = g_j \qquad \text{for all} \qquad i, j \in I, j \notin I'.$$

Remark 1.3.17. If $\sigma(i) \neq 1$ for some $i \in I$, the above algebra homomorphism is not a coalgebra homomorphism. ◊

Let V be a $U_q(\mathfrak{g})$-module. Given $\mu \in P_d$ and σ as above, the weight space of V of weight μ and type σ is the subspace

$$V_{\mu,\sigma} = \{v \in V : k_i v = \sigma(i)q_i^{\mu(h_i)}v, g_j v = q^{\mu(d_j)}v, \text{ for all } i, j \in I, j \notin I'\}.$$

As before, $x_i^\pm V_{\mu,\sigma} \subseteq V_{\mu\pm\alpha_i,\sigma}$ and, if $(\mu, \sigma) \neq (\mu', \sigma')$, then $V_{\mu,\sigma} \cap V_{\mu',\sigma'} = \{0\}$. Therefore, $\bigoplus_{\substack{\mu\in P_d\\ \sigma\in\{\pm1\}^I}} V_{\mu,\sigma}$ and $V_\sigma = \bigoplus_{\mu \in P_d} V_{\mu,\sigma}$ are submodules of V. V is said to be a weight-module if $V = \bigoplus_{\mu,\sigma} V_{\mu,\sigma}$. A weight-module V is said to be of *type* σ if $V = V_\sigma$. If $\sigma(i) = 1$ for all $i \in I$, and $V = V_\sigma$, V is said to be a module of *type* 1.

Given $\lambda \in P_d$ such that $\lambda(\hat{\mathfrak{h}}') = \{0\}$, notice that the $U_q(\mathfrak{g})$-module $V(\lambda, \sigma)$ given by the quotient of $U_q(\mathfrak{g})$ by the left ideal generated by $x_i^{\pm}, k_i - \sigma(i), g_j - q^{\lambda(d_j)}$ is a one-dimensional module of type σ. The following is easily established.

Proposition 1.3.18. Let V, W be $U_q(\mathfrak{g})$-modules and $\sigma, \sigma' : I \to \{1, -1\}$.

(i) If V is of type σ and W is of type σ', then $V \otimes W$ is of type $\sigma \circ \sigma'$. Moreover, if $\sigma \neq \sigma'$, then $\mathrm{Hom}_{U_q(\mathfrak{g})}(V, W) = \{0\}$.

(ii) If V is of type σ, then the pullback $V^{\sigma'}$ of V by σ' is of type $\sigma \circ \sigma'$. In particular, V^{σ} is of type 1. □

It follows from the above proposition that it suffices to study weight-module of type 1. Thus, henceforth, the expression weight-module will stand for weight-module of type 1 and the weight-spaces will be denoted by V_μ. As before, a nonzero vector in V_μ is called a weight-vector of weight μ. In fact, from this point on, the description of the constructions and of the statements of the results listed above in the classical case are transported to the quantum case in the obvious manner. We shall add a subscript q in the notation developed in the classical case to refer to the quantum analogues. Thus, the Verma module of highest-weight λ is denoted by $M_q(\lambda)$ while $V_q(\lambda)$ is the irreducible quotient, and the quantum analogues of category $\mathcal{O}$ and $\mathcal{O}^{\mathrm{int}}$ are denoted by $\mathcal{O}_q$ and $\mathcal{O}_q^{\mathrm{int}}$, respectively. The notation for characters will remain the same: $\mathrm{ch}(V)$.

Remark 1.3.19. Although all the results listed above in the classical case hold in the quantum case as well, some proofs are not developed in quite the same manner and, in some cases, the argumentation is very different. For instance, (1.3.6) does not follow immediately from the definition since the comultiplication is not cocommutative in the quantum setting and, in fact, is not true for any two given $U_q(\mathfrak{g})$-modules. In case these modules are in $\mathcal{O}_q^{\mathrm{int}}$, (1.3.6) follows from complete reducibility and (1.3.7). Alternatively, one can use the quasi-triangular structure of $U_q(\mathfrak{g})$ to deduce (1.3.6) whenever the action of the universal R-matrix on $V \otimes W$ is well-defined. A tensor category where (1.3.6) holds for any two objects is said to be a braided tensor category (the formal definition of a braiding is a little more technical than this). We also remark that, since S is an anti-automorphism, we have $(V \otimes W)^* \cong W^* \otimes V^*$ provided there exists such an isomorphism as vector spaces (which is the case if both V and W are finite-dimensional or, more generally, if $V \otimes W$ have finite-dimensional weight-spaces). ◊

1.4. Loop algebras and affine Kac-Moody algebras

We now review a second realization of affine Kac-Moody algebras and their quantum groups. It turns out that these alternate realizations are crucial for the development of the finite-dimensional representation theory of $\mathfrak{g}'$ and of $U_q(\mathfrak{g}')$.

Given a Lie algebra $\mathfrak{a}$ and an associative algebra A, the vector space $\mathfrak{a} \otimes A$ can be equipped with a Lie algebra structure by setting

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab \quad \text{for all} \quad x, y \in \mathfrak{a}, a, b \in A.$$

Notice that $\mathfrak{a} \otimes 1 = \{x \otimes 1 : x \in \mathfrak{a}\}$ is a subalgebra of $\mathfrak{a} \otimes A$ isomorphic to $\mathfrak{a}$ and, therefore, we regard $\mathfrak{a}$ as a subalgebra of $\mathfrak{a} \otimes A$. Under this identification, we keep denoting an element $x \in \mathfrak{a}$ by x instead of $x \otimes 1$.

Remark 1.4.1. When $A = \mathbb{F}[t, t^{-1}]$, the Lie algebra $\mathfrak{a} \otimes A$ is called the loop algebra of $\mathfrak{a}$. The terminology comes from the fact that $\mathbb{C}[t, t^{-1}]$ is the algebra of regular functions on the circle and, if $\mathfrak{a}$ is the Lie algebra of a Lie group G, then $\mathfrak{a} \otimes A$ is related to the Lie algebra of the group of functions from the circle to G (loops on G). More generally, if A is the algebra of regular functions on an algebraic variety X and $\mathfrak{a}$ is the Lie algebra of an algebraic group G, then $\mathfrak{a} \otimes A$ is related to the Lie algebra of the group of functions from V to G. In the case that $A = \mathbb{F}[t]$, $\mathfrak{a} \otimes A$ is called the current algebra over $\mathfrak{a}$. Multivariable analogues of current and loop algebras (also known as toroidal algebras) have also been studied frequently. We shall use the notation $\tilde{\mathfrak{a}}$ for the loop algebra over $\mathfrak{a}$ and $\mathfrak{a}[t]$ for the current algebra. Notice that we can regard $\tilde{\mathfrak{a}}$ as $\mathbb{Z}$-graded Lie algebra in the obvious way. ◊

For the remainder of the text, we fix C so that $\mathfrak{g}$ is finite-dimensional and let $\tilde{\mathfrak{g}}$ be the loop algebra over $\mathfrak{g}$. Consider the vector space $\hat{\mathfrak{g}}' = \tilde{\mathfrak{g}} \times \mathbb{F}$ and denote by c the element $(0, 1)$ so that we can write $\hat{\mathfrak{g}}' = \tilde{\mathfrak{g}} \oplus \mathbb{F}c$. There exists a unique Lie algebra structure on $\hat{\mathfrak{g}}'$ such that c is central ($[\tilde{\mathfrak{g}}, c] = 0$) and

$$[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s} + r\, \delta_{r,-s}(x, y)c \qquad \text{for all} \qquad x, y \in \mathfrak{g}, r, s \in \mathbb{Z}.$$

The $\mathbb{Z}$-gradation on $\tilde{\mathfrak{g}}$ extends to one on $\hat{\mathfrak{g}}'$ by setting the degree of c to be zero. Let $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}' \times \mathbb{F}$ and denote by d the element $(0, 1)$ so that $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}' \oplus \mathbb{F}d$. There exists a unique Lie algebra structure on $\hat{\mathfrak{g}}$ such that $\hat{\mathfrak{g}}' \times \{0\}$ is an ideal isomorphic to $\hat{\mathfrak{g}}'$ and d acts as the degree operator on $\hat{\mathfrak{g}}'$, i.e., $[d, x] = rx$ for all x in the r-th graded piece of $\hat{\mathfrak{g}}'$. Again, we extend the $\mathbb{Z}$-gradation to one on $\hat{\mathfrak{g}}$ by setting the degree of d to be zero. Set $\hat{\mathfrak{h}}' = \mathfrak{h} \oplus \mathbb{F}c$ and $\hat{\mathfrak{h}} = \hat{\mathfrak{h}}' \oplus \mathbb{F}d$.

Recall that θ denotes the maximal root of $\mathfrak{g}$. Choose $x_\theta^\pm \in \mathfrak{g}_{\pm\theta}$ such that $[x_\theta^+, x_\theta^-] = \theta^\vee$ and set $x_0{}^\pm = x_\theta^\mp \otimes t^{\pm 1}, h_0 = [x_0^+, x_0^-]$. Observe that

$$h_0 = [x_\theta{}^- \otimes t, x_\theta{}^+ \otimes t^{-1}] = [x_\theta{}^-, x_\theta{}^+] + (x_\theta{}^+, x_\theta{}^-)c = \frac{2c}{(\theta, \theta)} - \theta^\vee.$$

Remark 1.4.2. Because of the above, several authors find it convenient to renormalize the Killing form of $\mathfrak{g}$ so that $(\theta, \theta) = 2$ while in our normalization $(\theta, \theta) = 2r^\vee$. ◊

Definition 1.4.3. Let $\hat{I} = I \sqcup \{0\}$ and define the extend Cartan matrix matrix $\hat{C} = (\hat{c}_{ij})_{i,j \in \hat{I}}$ of C by setting $\hat{c}_{00} = 2$,

$$\hat{c}_{0j} = -\alpha_j(\theta^\vee) = -\frac{2(\alpha_j, \theta)}{(\theta, \theta)}, \quad \hat{c}_{i0} = -\theta(h_i) = -\frac{2(\alpha_i, \theta)}{(\alpha_i, \alpha_i)}, \quad \text{and} \quad \hat{c}_{ij} = c_{ij} \quad \text{for all} \quad i, j \in I. \qquad ◊$$

One easily checks that we have

$$[h_i, x_j{}^\pm] = \pm\hat{c}_{ij} x_j^\pm \qquad \text{and} \qquad \mathrm{ad}(x_i^\pm)^{1-\hat{c}_{ij}}(x_j^\pm) = 0 \qquad \text{for all} \qquad i, j \in \hat{I} \tag{1.4.1}$$

and, moreover:

Lemma 1.4.4. The matrix $\hat{C} = (\hat{c}_{ij})_{i,\, j \in \hat{I}}$ is a generalized Cartan matrix of affine type and s_0 is the lacing number $r^\vee$ of $\mathfrak{g}$. □

It follows that the subalgebra of $\hat{\mathfrak{g}}$ generated by $x_i^\pm, h_i, d, i \in I$, is a quotient of the the affine Kac-Moody algebra associated to $\hat{C}$. In fact, we have:

Theorem 1.4.5. The Lie algebra $\hat{\mathfrak{g}}$ is isomorphic to the affine Kac-Moody algebra associated to $\hat{C}$. □

Remark 1.4.6. All generalized Cartan matrices with Dynkin diagram in Table 1.1.2 are of the form $\hat{C}$ for some C with Dynkin diagram in Table 1.1.1. The affine Kac-Moody algebras of twisted type can be realized as subalgebras of the non-twisted ones. Namely, each non-trivial Dynkin diagram automorphism σ of $\mathfrak{g}$ gives rise to a Lie algebra automorphism of $\hat{\mathfrak{g}}$. The twisted affine algebra $\hat{\mathfrak{g}}^\sigma$ associated to σ is then the corresponding fixed point subalgebra of $\hat{\mathfrak{g}}$. We shall not consider the twisted affine algebras here. ◊

Regard the root lattice Q of $\mathfrak{g}$ as a subset of $\hat{\mathfrak{h}}^*$ by extending $\alpha_i, i \in I$, to an element of $\hat{\mathfrak{h}}^*$ by setting $\alpha_i(c) = \alpha_i(d) = 0$. Let δ be the unique element of $\hat{\mathfrak{h}}^*$ such that $\delta(\hat{\mathfrak{h}}') = 0$ and $\delta(d) = 1$ and set $\alpha_0 = \delta - \theta$. Given $\lambda \in \hat{\mathfrak{h}}^*$, let

$$\hat{\mathfrak{g}}_\lambda = \{x \in \hat{\mathfrak{g}} : [h, x] = \lambda(h)x, \text{ for all } h \in \hat{\mathfrak{h}}\},$$

and set $\hat{R} = \{\alpha \in \hat{\mathfrak{h}}^* : \hat{\mathfrak{g}}_\alpha \neq 0\} \setminus \{0\}$. One easily checks that $\hat{\mathfrak{g}}_0 = \hat{\mathfrak{h}}$,

$$\hat{\mathfrak{g}}_{k\delta} = \mathfrak{h} \otimes t^k, \quad \text{and} \quad \hat{\mathfrak{g}}_{\alpha+m\delta} = \mathfrak{g}_\alpha \otimes t^m \quad \text{for all} \quad k, m \in \mathbb{Z}, k \neq 0, \alpha \in R. \tag{1.4.2}$$

In particular,

$$\hat{R} = \{\alpha + k\delta \,:\, \alpha \in R,\ k \in \mathbb{Z}\} \cup \{k\delta \,:\, k \in \mathbb{Z} \setminus \{0\}\}$$

and $\hat{\mathfrak{g}} = \hat{\mathfrak{h}} \oplus \bigoplus_{\alpha \in \hat{R}} \hat{\mathfrak{g}}_\alpha$. The set $\{k\delta \,:\, k \in \mathbb{Z} \setminus \{0\}\}$ is the set of imaginary roots.

We now turn to the quantum setting and present a realization of $U_q(\hat{\mathfrak{g}})$ which resembles the above realization of the affine Kac-Moody algebra $\hat{\mathfrak{g}}$. However, in the quantum case, the description is not via a construction beginning from $U_q(\mathfrak{g})$, but rather another presentation in terms of generators and relations. This time the generators stand for "deformations" of the elements $x_{\alpha_i} \otimes t^r, h_i \otimes t^s \in \hat{\mathfrak{g}}, i \in I, r, s \in \mathbb{Z}, s \neq 0$. Notice that $U_q(\mathfrak{g})$ is isomorphic to the subalgebra of $U_q(\hat{\mathfrak{g}})$ generated by $x_i^\pm, k_i, i \in I$, and we shall identify $U_q(\mathfrak{g})$ with this subalgebra. The following theorem was proved in [2].

Theorem 1.4.7. The quantized universal enveloping algebra $U_q(\hat{\mathfrak{g}}')$ is isomorphic to the $\mathbb{F}$-associative algebra A_q generated by elements $x_{i,r}^\pm, h_{i,s}, k_i^\pm, c^{\pm 1/2}, i \in I, r, s \in \mathbb{Z}, s \neq 0$, subject to the following defining relations. The elements $c^{\pm 1/2}$ are central,

$$c^{+1/2}c^{-1/2} = 1 = k_ik_i^{-1}, \quad k_ik_j = k_jk_i, \quad k_ih_{j,s} = h_{j,s}k_i, \quad k_ix_{j,r}^\pm k_i^{-1} = q_i^{\pm c_{ij}}x_{j,r}^\pm, \quad i, j \in I, r, s \in \mathbb{Z}, s \neq 0,$$

$$\sum_{\sigma \in S_{1-c_{ij}}} \sum_{m=0}^{1-c_{ij}} (-1)^m \begin{bmatrix} 1-c_{ij} \\ m \end{bmatrix}_{q_i} x_{i,r_{\sigma(1)}}^\pm \dots x_{i,r_{\sigma(m)}}^\pm x_{j,s}^\pm x_{i,r_{\sigma(m+1)}}^\pm \dots x_{i,r_{\sigma(1-c_{ij})}}^\pm = 0, \quad i, j \in I, i \neq j, r_n, s \in \mathbb{Z},$$

$$[h_{i,r}, h_{j,s}] = \delta_{r,-s} \frac{[rc_{ij}]_{q_i}}{r} \frac{c^r - c^{-r}}{q_j - q_j^{-1}}, \quad i, j \in I, r, s \in \mathbb{Z} \setminus \{0\}, \quad \text{where} \quad c^{\pm r} := (c^{\pm 1/2})^{2r},$$

$$[h_{i,s}, x_{j,r}^\pm] = \pm \frac{1}{s} [sc_{ij}]_{q_i} x_{j,r+s}^\pm (c^{\mp 1/2})^{|r|}, \quad i, j \in I, r, s \in \mathbb{Z}, s \neq 0,$$

$$x_{i,r+1}^\pm x_{j,r}^\pm - q_i^{\pm c_{ij}} x_{j,s}^\pm x_{i,r+1}^\pm = q_i^{\pm c_{ij}} x_{i,r}^\pm x_{j,s+1}^\pm - x_{j,s+1}^\pm x_{i,s}^\pm, \quad i, j \in I, r, s \in \mathbb{Z},$$

$$[x_{i,r}^+, x_{j,s}^-] = \delta_{ij} \frac{c^{\frac{r-s}{2}} \psi_{i,r+s}^+ - \psi_{i,r+s}^- c^{\frac{s-r}{2}}}{q_i - q_i^{-1}}, \quad i, j \in I, r, s \in \mathbb{Z}, \quad \text{where} \quad c^{\frac{m}{2}} := (c^{+1/2})^m,$$

$\psi^{\pm}_{i,\mp m} = 0$ if $m > 0$, and $\psi^{\pm}_{i,\pm m}, m \geq 0$, are defined by the following equality of power series in u:

$$\sum_{m\geq 0} \psi^{\pm}_{i,\pm m} u^m = k_i^{\pm 1} \exp\left(\pm(q_i - q_i^{-1}) \sum_{s>0} h_{i,\pm s} u^s \right).$$

Moreover, for all $i \in I$, the isomorphism maps the generator $x_i^{\pm}$ of $U_q(\mathfrak{g})$ to the generator $x^{\pm}_{i,0}$ of A_q. □

Henceforth, we identify $U_q(\hat{\mathfrak{g}}')$ with A_q. We shall refer to the second line of relations for A_q as the loop analogue of the quantum Serre's relations.

Let $U_q(\tilde{\mathfrak{g}})$ be the quotient of $U_q(\hat{\mathfrak{g}}')$ by the ideal generated by $c^{1/2} - 1$. We will refer to $U_q(\tilde{\mathfrak{g}})$ as the quantum loop algebra of $\mathfrak{g}$ (it is most often called quantum affine algebra in the literature by abuse of terminology). We keep denoting the images of the generators of $U_q(\hat{\mathfrak{g}}')$ in $U_q(\tilde{\mathfrak{g}})$ by their original notation. Let $U_q(\tilde{\mathfrak{n}}^{\pm})$ be the subalgebra of $U_q(\tilde{\mathfrak{g}})$ generated by $x^{\pm}_{i,r}, i \in, r \in \mathbb{Z}$, respectively. Let also $U_q(\tilde{\mathfrak{h}})$ be the subalgebra of $U_q(\tilde{\mathfrak{g}})$ generated by the elements $k_i^{\pm 1}, h_{i,s}, i \in I, s \in \mathbb{Z} \setminus \{0\}$. Given $J \subseteq I$, let $U_q(\tilde{\mathfrak{g}}_J)$ be the subalgebra generated by $x^{\pm}_{i,r}, k_i, i \in J, r \in \mathbb{Z}$. For $J = \{i\}$ we use the simplified notation $U_q(\tilde{\mathfrak{g}}_i)$ (note this is isomorphic to $U_{q_i}(\tilde{\mathfrak{sl}}_2)$). Given $i \in I, r \in \mathbb{Z}$, let $U_q(\tilde{\mathfrak{g}}_{i,r})$ be the subalgebra of $U_q(\tilde{\mathfrak{g}})$ generated by k_i and $x^{\pm}_{i,\pm r}$ which is isomorphic to $U_{q_i}(\mathfrak{sl}_2)$. The next proposition also follows from the results of [2].

Proposition 1.4.8. The multiplication map induces an isomorphism of vector spaces $U_q(\tilde{\mathfrak{n}}^-) \otimes U_q(\tilde{\mathfrak{h}}) \otimes U_q(\tilde{\mathfrak{n}}^+) \to U_q(\tilde{\mathfrak{g}})$. □

The proof of the following lemma is straightforward.

Lemma 1.4.9. For every $s \in \mathbb{Z}$, there exists a unique algebra automorphism of $U_q(\tilde{\mathfrak{g}}')$ such that $k_i \mapsto k_i$ and $x^{\pm}_{i,r} \mapsto x^{\pm}_{i,r\pm s}$ for all $i \in I, r \in \mathbb{Z}$. Moreover, this automorphism is the identity when restricted to $U_q(\tilde{\mathfrak{h}})$. □

It will be convenient to consider the elements $\Lambda_{i,r}, i \in I, r \in \mathbb{Z}$, of $U_q(\tilde{\mathfrak{h}})$ defined by

$$\sum_{r=0}^{\infty} \Lambda_{i,\pm r} u^r = \exp\left(- \sum_{s=1}^{\infty} \frac{h_{i,\pm s}}{[s]_{q_i}} u^s \right).$$

In particular, $\Lambda_{i,0} = 1$ for all $i \in I$. Note that, setting $\Lambda_i^{\pm}(u) = \sum_{r=0}^{\infty} \Lambda_{i,\pm r} u^r$ and $\Psi_i^{\pm}(u) = \sum_{m\geq 0} \psi^{\pm}_{i,\pm m} u^m$, we have

$$\Psi_i^{\pm}(u) = k_i^{\pm 1} \frac{\Lambda_i^{\pm}(u q_i^{\mp 1})}{\Lambda_i^{\pm}(u q_i^{\pm 1})} \tag{1.4.3}$$

where the division is that of formal power series in u with coefficients in $U_q(\tilde{\mathfrak{h}})$. One easily checks that (1.4.3) is equivalent to

$$\Lambda_{i,\pm r} = \frac{\mp k_i^{\mp 1}}{q_i^r - q_i^{-r}} \sum_{t=1}^{r} q_i^{\pm(r-t)} \psi^{\pm}_{i,\pm t} \Lambda_{i,\pm(r-t)} \tag{1.4.4}$$

and to

$$\psi^{\pm}_{i,\pm r} = \mp k_i^{\pm 1}(q_i^r - q_i^{-r})\Lambda_{i,\pm r} - \sum_{t=1}^{r-1} q_i^{\pm(r-t)}\psi_{i,\pm t}\Lambda_{i,\pm(r-t)}. \tag{1.4.5}$$

Note that $U_q(\tilde{\mathfrak{h}})$ is generated by $k_i^{\pm 1}$ and $\Lambda_{i,r}$, $i \in I, r \in \mathbb{Z}$. In fact, one can check that the elements $\Lambda_{i,r}, i \in I, r \in \mathbb{Z}$ are algebraically independent and, therefore, the subalgebras $\mathbf{\Lambda}^{\pm}$ of $U_q(\tilde{\mathfrak{h}})$ generated by $\Lambda_{i,\pm r}, i \in I, r > 0$, are polynomial algebras on these elements. In particular, the multiplication map induces an isomorphism of commutative algebras:

$$\mathbf{\Lambda}^- \otimes U_q(\mathfrak{h}) \otimes \mathbf{\Lambda}^+ \to U_q(\tilde{\mathfrak{h}}). \tag{1.4.6}$$

The Hopf algebra structure on $U_q(\hat{\mathfrak{g}}')$ induces one on $U_q(\tilde{\mathfrak{g}})$. However, a precise formula for the comultiplication of the generators $x^{\pm}_{i,r}, h_{i,s}$, and $\Lambda_{i,r}$ is not known. It is also not true that the subalgebras $U_q(\tilde{\mathfrak{n}}^{\pm}), U_q(\tilde{\mathfrak{h}})$, and $U_q(\tilde{\mathfrak{g}}_J)$ are Hopf subalgebras of $U_q(\tilde{\mathfrak{g}})$. For notation convenience, we set $U_q(\tilde{\mathfrak{n}}^{\pm})^0 = U_q(\tilde{\mathfrak{n}}^{\pm}) \cap U_q(\tilde{\mathfrak{g}})^0$ and so on. The next proposition gives partial information on the comultiplication in terms of the loop like generators. The proof can be found in [2, 3, 26, 27].

Proposition 1.4.10. For $i \in I$, let $U_q(\tilde{\mathfrak{n}}^{\pm}(i))$ be the subalgebra of $U_q(\tilde{\mathfrak{n}}^{\pm})$ generated by $x^{\pm}_{j,r}$ with $j \neq i$ and $r \in \mathbb{Z}$.

(i) Modulo $(U_q(\tilde{\mathfrak{h}})U_q(\tilde{\mathfrak{n}}^-)^0) \otimes (U_q(\tilde{\mathfrak{h}})U_q(\tilde{\mathfrak{n}}^+)^0)$ we have $\Delta(h_i, s) = h_{i,s} \otimes 1 + 1 \otimes h_{i,s}$,

$$\Delta(\Lambda_{i,\pm r}) = \sum_{s=0}^{r} \Lambda_{i,\pm s} \otimes \Lambda_{i,\pm(r-s)} \qquad \text{and} \qquad \Delta(\psi^{\pm}_{i,\pm r}) = \sum_{s=0}^{r} \psi^{\pm}_{i,\pm s} \otimes \psi^{\pm}_{i,\pm(r-s)}.$$

(ii) Modulo $(U_q(\tilde{\mathfrak{h}})U_q(\tilde{\mathfrak{n}}^-)^0) \otimes (U_q(\tilde{\mathfrak{h}})(U_q(\tilde{\mathfrak{n}}^+)^0)^2) + (U_q(\tilde{\mathfrak{h}})U_q(\tilde{\mathfrak{n}}^-)^0) \otimes (U_q(\tilde{\mathfrak{h}})U_q(\tilde{\mathfrak{n}}^+(i))^0)$ we have

$$\Delta(x^+_{i,r}) = x^+_{i,r} \otimes 1 + k_i \otimes x^+_{i,r} + \sum_{s=1}^{r} \psi^+_{i,s} \otimes x^+_{i,r-s} \quad \text{if} \quad r \geq 0,$$

$$\Delta(x^+_{i,-r}) = k_i^{-1} \otimes x^+_{i,-r} + 1 \otimes x^+_{i,-r} + \sum_{s=1}^{r-1} \psi^-_{i,-s} \otimes x^+_{i,-r+s} \quad \text{if} \quad r > 0.$$

(iii) Modulo $(U_q(\tilde{\mathfrak{h}})(U_q(\tilde{\mathfrak{n}}^-)^0)^2) \otimes (U_q(\tilde{\mathfrak{h}})U_q(\tilde{\mathfrak{n}}^+)^0) + (U_q(\tilde{\mathfrak{h}})U_q(\tilde{\mathfrak{n}}^-)^0) \otimes (U_q(\tilde{\mathfrak{h}})U_q(\tilde{\mathfrak{n}}^+(i))^0)$ we have

$$\Delta(x^-_{i,r}) = x^-_{i,r} \otimes k_i + 1 \otimes x^-_{i,r} + \sum_{s=1}^{r-1} x^-_{i,r-s} \otimes \psi^+_{i,s} \quad \text{if} \quad r > 0,$$

$$\Delta(x^-_{i,-r}) = x^-_{i,-r} \otimes k_i^{-1} + 1 \otimes x^-_{i,-r} + \sum_{s=1}^{r} x^-_{i,-r+s} \otimes \psi^-_{i,-s} \quad \text{if} \quad r \geq 0. \qquad \square$$

The next lemma is of crucial importance in the study of finite-dimensional representations of $U_q(\tilde{\mathfrak{g}})$. For convenience, we introduce the notation of quantum divided powers: $(x^{\pm}_{i,r})^{(m)} := \frac{(x^{\pm}_{i,r})^m}{[m]_i!}$. Also, given $i \in I$ and $s, m \in \mathbb{Z}, m \geq 0$, define

$$X^-_{i,s;\pm}(u) = \sum_{r \geq 1} x^-_{i,\pm(r+s)}u^r \qquad \text{and} \qquad (X^-_{i,s;\pm}(u))^{(m)} = \frac{1}{[m]_i!}(X^-_{i,s;\pm}(u))^m.$$

Lemma 1.4.11. For every $i \in I$ and $s \in \mathbb{Z}$, we have

$$(x_{i,\mp s}^{+})^{(l)}(x_{i,\pm(s+1)}^{-})^{(m)} = (-k_i^{\pm 1})^l \left((X_{i,s;\pm}^{-}(u))^{(m-l)}\Lambda_i^{\pm}(uq_i^{\pm 1})\right)_m$$

modulo elements in $U_q(\tilde{\mathfrak{g}})U_q(\tilde{\mathfrak{n}}^+)^0$. Here $(X_{i,s}^-(u))^{(m-l)}$ is understood to be zero if $m < l$ and the subindex m on the right-hand side means the coefficient of u^m in the given power series.

Proof. The case $(x_{i,0}^+)^{(l)}(x_{i,1}^-)^{(m)}$ was proved in [27, Section 5] and the case $(x_{i,0}^+)^{(l)}(x_{i,-1}^-)^{(m)}$ is proved similarly. The general case follows from these by applying the algebra automorphism of Lemma 1.4.9. □

Remark 1.4.12. One can define elements $\Lambda_{i,r}$ in the classical context as well by replacing quantum numbers by usual ones in the definition above, where $h_{i,s}$ is replaced by $h_i \otimes t^s$. Then (1.4.6) holds again and, in fact, $U(\tilde{\mathfrak{h}})$ is the associative commutative algebra freely generated by the elements $h_i, \Lambda_{i,r}, i \in I, r \in \mathbb{Z}, r \neq 0$. It is not difficult to see that $\Delta(\Lambda_{i,\pm r}) = \sum_{s=0}^{r} \Lambda_{i,\pm s} \otimes \Lambda_{i,\pm r \mp s}$ in this case. The classical version of Lemma 1.4.11 (whose statement is recovered from the quantum one in the obvious manner) was proved in [40]. ◊

2. Basic Finite-Dimensional Representation Theory of Affine Algebras

Recall that in Subsection 1.4 we fixed a Cartan matrix C so that $\mathfrak{g}$ is a finite-dimensional simple Lie algebra. Recall also that $\hat{\mathfrak{g}}$ denotes the affine Kac-Moody algebra associated to the extended matrix $\hat{C}$ and that $\tilde{\mathfrak{g}}$ is the underlying loop algebra over $\mathfrak{g}$. This section is dedicated to the study of the basic facts of the finite-dimensional representation theory of $\hat{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}$ as well of their quantum groups. The main goal is to classify the irreducible representations. Along the way, it will become natural to introduce the notion of ℓ-weight spaces and the associated character theory as well as the concepts of Weyl modules.

2.1. Simple modules in the classical setting

To shorten notation, we shall write $x_{i,r}^{\pm}$ and $h_{i,r}$ for the elements $x_i^{\pm} \otimes t^r$ and $h_i \otimes t^r, i \in I, r \in \mathbb{Z}$, of $\tilde{\mathfrak{g}}$, respectively.

Let V be a nonzero finite-dimensional $\hat{\mathfrak{g}}'$-module. Then, as an $\mathfrak{h}$-module, we have $V = \bigoplus_{\mu \in P} V_\mu$ and the relation $[h_j, x_{i,r}^{\pm}] = \pm\alpha_i(h_j)x_{j,r}^{\pm}, i, j \in I$, implies

(2.1.1) $$x_{i,r}^{\pm} V_\mu \subseteq V_{\mu\pm\alpha_i} \qquad \text{for all} \qquad i \in I, r \in \mathbb{Z}.$$

In particular, since $x_0^{\pm} \in \mathfrak{g}_{\mp\theta} \otimes t^{\pm 1}$, we have

(2.1.2) $$x_0^{\pm} V_\mu \subseteq V_{\mu\pm\theta}.$$

It now easily follows that V is integrable. If λ is a maximal weight of V, then $\lambda \in P^+$ and $x_{i,r}^{+} V_\lambda = \{0\}$ for all $i \in I, r \in \mathbb{Z}$. In particular, if W is the $\hat{\mathfrak{g}}'$-submodule of V generated by V_λ, then

(2.1.3) $$W_\mu \neq \{0\} \qquad \text{if and only if} \qquad \mu \in \mathrm{wt}(\lambda).$$

and

(2.1.4) $$(x_{i,r}^{-})^{\lambda(h_i)+1} V_\lambda = \{0\} \qquad \text{for all} \qquad i \in I, r \in \mathbb{Z}.$$

Proposition 2.1.1. The central element c acts trivially in every finite-dimensional $\hat{\mathfrak{g}}'$-module.

Proof. Let V be a nonzero finite-dimensional $\hat{\mathfrak{g}}'$-module. By Proposition 1.3.3, h_0 also act semisimply on V and, hence, so does c. Therefore, we may assume that V is irreducible. Let $\lambda \in P^+$ be a maximal weight of V and $v \in V_\lambda \setminus \{0\}$. Since V is irreducible and c is central, it follows from Schur's Lemma that the c by multiplication by a fixed scalar on V. Hence, it suffices to show that $cv = 0$.

Let a be the eigenvalue of c on V. Given $i \in I, r \in \mathbb{Z}$, consider the subalgebra $\hat{\mathfrak{g}}_{i,r}$ generated by $x_{i,\pm r}^{\pm}$ which is isomorphic to $\mathfrak{sl}_2$. Let $x^{\pm}, h$, denote the usual basis of $\mathfrak{sl}_2$ and consider the isomorphism $\mathfrak{sl}_2 \to \hat{\mathfrak{g}}_{i,r}$ determined by $x^{\pm} \mapsto x_{i,r}^{\pm}$. In particular, $h \mapsto h_i + \frac{r}{s_i}c$. Regard V as an $\mathfrak{sl}_2$-module by means of this isomorphism and notice that

$$hv = \left(\lambda(h_i) + \frac{ra}{s_i}\right)v.$$

On the other hand, since $x^+v = 0$, the $\mathfrak{sl}_2$-submodule of V generated by V must a simple finite-dimensional module of highest-weight $\lambda(h_i) + \frac{ra}{s_i}$. Therefore,

$$\lambda(h_i) + \frac{ra}{s_i} \in \mathbb{Z}_{\geq 0} \qquad \text{for all} \qquad r \in \mathbb{Z}.$$

This implies $a = 0$ as desired. □

Proposition 2.1.2. Let V be a simple finite-dimensional $\hat{\mathfrak{g}}$-module. Then, V is one-dimensional and $\hat{\mathfrak{g}}'V = \{0\}$.

Proof. Let v be an eigenvector of the action of d with eigenvalue a. Then, since $[d, h_{i,r}] = r, i \in I, r \in \mathbb{Z}$, $h_{i,r}v$ is also an eigenvector of the action of d with eigenvalue $a + r$. Since V is finite-dimensional, it follows that $h_{i,r}$ acts nilpotently on V for $r \neq 0$. By the previous proposition, c acts trivially on V and, hence, V is $\mathbb{Z}$-graded module for $\tilde{\mathfrak{g}} \oplus \mathbb{F}d$. Since $\tilde{\mathfrak{h}}$ is abelian, we have

$$h_{i,r}V_\mu \subseteq V_\mu \tag{2.1.5}$$

and there must be a nonzero $v \in V_\mu$ which is a common eigenvector for the actions of $h_{i,r}, i \in I, r \in \mathbb{Z}$. In particular, $h_{i,r}v = \Lambda_{i,r} = 0$ if $r \neq 0$. Let $\lambda \in P^+$ be a maximal weight of V, v be an eigenvector for $\tilde{\mathfrak{h}}$. By (2.1.1), given $i \in I$, there exists $m > 0$ such that $(x_{i,r}^-)^m v = 0$. Then, by lemma 1.4.11,

$$0 = (x_{i,-s}^+)^{(m-1)}(x_{i,s+1}^-)^{(m)}v = \left(x_{i,s+1}^-\Lambda_{i,m-1} + x_{i,s+2}^-\Lambda_{i,m-2} + \cdots + x_{i,s+m}^-\right)v = x_{i,s+m}^-v.$$

It follows that $x_{i,r}^-v = 0$ for all $i \in I, r \in \mathbb{Z}$. By considering the subalgebra $\tilde{\mathfrak{g}}_{i,r}$, it follows that $\lambda = 0$. Hence, $\tilde{\mathfrak{g}}V = 0$, and V is generated by the action of d on V. Since V is simple, it must be one-dimensional. □

Henceforth, we are left to study finite-dimensional representations of $\tilde{\mathfrak{g}}$. We start with looking at evaluation representations. Namely, given $a \in \mathbb{F}^\times$, let $\mathrm{ev}_a : \tilde{\mathfrak{g}} \to \mathfrak{g}$ be the evaluation map $x \otimes f(t) \mapsto f(a)x$ which is easily seen to be a Lie algebra homomorphism. Then, if V is a $\mathfrak{g}$-module, we can consider the $\tilde{\mathfrak{g}}$-module $V(a)$ obtained by pulling-back the action of $\mathfrak{g}$ to $\tilde{\mathfrak{g}}$ via ev_a. Modules of the form $V(a)$ are called evaluation modules. Notice that $V(a)$ is simple if and only if V is simple. We will denote by $V(\lambda, a)$ the evaluation module constructed from $V(\lambda), \lambda \in P^+$.

Theorem 2.1.3. Let $\lambda_1, \dots \lambda_m \in P^+ \setminus \{0\}, a_1, \dots, a_m \in \mathbb{F}^\times$. Then, $V(\lambda_1, a_1) \otimes \cdots \otimes V(\lambda_m, a_m)$ is irreducible if and only if $a_i \neq a_j$ for all $i \neq j$.

Proof. We write the proof for $m = 2$ and let as exercise for the reader to write the details in general. Thus, to simplify notation, write $a = a_1, b = a_2, \lambda = \lambda_1, \mu = \lambda_2$. Let also u, v be highest-weight vectors for $V(\lambda)$ and $V(\mu)$, respectively. Fix a basis $\left\{u_j : j \in J\right\}$ of $V(\lambda)$ and $\{v_k : k \in K\}$ of $V(\mu)$ formed by weight vectors.

Suppose $a \neq b$ and let W be a nontrivial irreducible submodule of $V(\lambda, a) \otimes V(\mu, b)$. Let ν is a maximal weight of W and w be a nonzero vector in W_ν which is an eigenvector for the action of $\tilde{\mathfrak{h}}$ (cf. (2.1.5)). Then, $\tilde{\mathfrak{n}}^+w = 0$ and $W = U(\tilde{\mathfrak{n}}^-)w$. We want to show that w is a scalar multiple of $u \otimes v$. Since $\left\{u_j \otimes v_k : j \in J, k \in K\right\}$ is a basis of $V(\lambda) \otimes V(\mu)$ we can write

$$w = \sum_{j,k} a_{j,k} u_r \otimes v_k, \qquad \text{for some } a_{j,k} \in \mathbb{F}.$$

Let $x \in \mathfrak{n}^+$ and notice that, since $xw = (x \otimes t)w = 0$, we have

$$\sum_{j,k} a_{j,k}(xu_j) \otimes v_k = -\sum_{j,k} a_{j,k}u_j \otimes (xv_k) \quad \text{and} \quad a\sum_{j,k}(xu_j) \otimes v_k = -b\sum_{j,k} u_j \otimes (xv_k).$$

The hypothesis $a \neq b$ then implies that

$$\sum_{j,k} a_{j,k}(xu_j) \otimes v_k = \sum_{j,k} a_{j,k}u_j \otimes (xv_k) = 0. \tag{2.1.6}$$

The linear independence of $\{v_k : k \in K\}$ implies that, for each $k \in K$, we have

$$0 = \sum_{j \in J} a_{j,k}(xu_j) = x\left(\sum_{j \in J} a_{j,k}u_j\right).$$

Since $V(\lambda)$ is irreducible, it follows that $\sum_{j\in J} a_{j,k}u_j$ is a scalar multiple of u. Similarly, for each $j \in J$, we get that $\sum_{k\in K} a_{j,k}v_k$ is a scalar multiple of v. This proves that w is a scalar multiple of $u \otimes v$. Before proving the converse, let us comment on the case $m > 2$. For instance, if $m = 3$, we would need to use that $(x \otimes t^m)w = 0$ for $m = 0, 1, 2$ and then, the analogue of (2.1.6) would follow from the fact that the Vandermonde matrix $\begin{bmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{bmatrix}$ is nonsingular if $a_i \neq a_j$ for $i \neq j$.

Now assume $a = b$. By Proposition 1.3.12, we can choose $\nu < \lambda+\mu$ such that $[V(\lambda)\otimes V(\mu) : \nu] \neq 0$. In other words, there exists $w \in V(\lambda) \otimes V(\mu)$ which is a highest-weight vector of weight ν. Write

$$w = \sum_{j,k} a_{j,k}u_j \otimes v_k \qquad \text{for some} \qquad a_{j,k} \in \mathbb{F},$$

and observe that

$$(x \otimes t^m)w = a^m xw = 0 \qquad \text{and} \qquad (h \otimes t^m)w = a^m \nu(h)w \qquad \text{for all} \qquad x \in \mathfrak{n}^+, h \in \mathfrak{h}, m \in \mathbb{Z}.$$

It follows that the $\tilde{\mathfrak{g}}$-submodule W of $V(\lambda, a) \otimes V(\mu, a)$ generated by w satisfies

$$W_\omega \neq 0 \qquad \text{only if} \qquad \omega < \nu.$$

In particular, W is a proper submodule. □

Remark 2.1.4. In the classic theory finite-dimensional representations of $\mathfrak{g}$ or $U_q(\mathfrak{g})$, the tensor product of two non trivial simple modules is never simple (Proposition 1.3.12). On the other hand, the above Theorem gives plenty of examples of simple tensor products within the finite-dimensional representation theory of $\tilde{\mathfrak{g}}$. One can expand the argument of the above proof to obtain the following. If $V = V(\lambda) \otimes V(\mu)$ for some $\lambda, \mu \in P^+$ and $a \in \mathbb{F}^\times$, then

$$V(\lambda, a) \otimes V(\mu, a) \cong \bigoplus_{\nu \in P^+} V(\nu, a)^{\oplus[V:\nu]}.$$

◊

The remainder of this subsection is dedicated to proving that every simple finite-dimensional $\tilde{\mathfrak{g}}$-module is isomorphic to a unique tensor product of evaluation modules (up to re-ordering), thus completing the classification of the simple $\tilde{\mathfrak{g}}$-modules.

Since ev_a is a Lie algebra map, it can be uniquely extended to a Hopf algebra map $U(\tilde{\mathfrak{g}}) \to U(\mathfrak{g})$ which will also be denoted by ev_a. One can easily checks that

(2.1.7) $$\mathrm{ev}_a(\Lambda_{i,r}) = (-a)^r \binom{h_i}{|r|} \qquad \text{where} \qquad \binom{h_i}{s} = \frac{h_i(h_i-1)\dots(h_i-(s-1))}{s!}.$$

In particular, if V is a $\mathfrak{g}$-module and $v \in V_\mu$ for some $\mu \in P$, we have the following identity of formal power series in the variable u with coefficients in $V(a)$:

(2.1.8) $$\Lambda_i^\pm(u)\, v = \left(\sum_{r\geq 0} (-a)^{\pm r} \binom{\mu(h_i)}{r} u^r\right) v.$$

The above can be expressed in a more convenient way as follows. For $\mu \in P$ and $a \in \mathbb{F}^\times$, let $\boldsymbol{\omega}_{\mu,a} \in \mathbb{F}(u)^I$ be the I-tuple of rational functions whose i-th component $(\boldsymbol{\omega}_{\mu,a})_i(u)$ is $(1-au)^{\mu(h_i)}$. We identify the rational function $(1-au)^{-1}$ with the geometric power series $\sum_{r\geq 0} a^r u^r \in \mathbb{F}[[u]]$. This way, every rational rational function $f(u) \in \mathbb{F}(u)$ such that $f(0) = 1$ can be identified with a unique element of $\mathbb{F}[[u]]$. One can now easily check that (2.1.8) implies that

$$\Lambda_i^+(u)\, v = (\boldsymbol{\omega}_{\mu,a})_i(u)\, v.$$

The action of $\Lambda_i^-(u)$ on $V(a)_\mu$ can also be described in this way as follows. Given a polynomial $f(u) = 1 + c_1 u + \cdots + c_n u^n \in \mathbb{F}[u]$ of degree n, let $f^-(u) = c_n^{-1} u^n f(u^{-1})$. Thus, writing $f^-(u) = 1 + c_{-1}u + \cdots + c_{-n}u^n$, we have

(2.1.9) $$c_n c_{-r} = c_{n-r} \qquad \text{for all} \qquad r = 0, 1, \dots, n.$$

Alternatively, if $f(u) = \prod_{r=1}^n (1 - a_r u)$, then $f^-(u) = \prod_{r=1}^n (1 - a_r^{-1} u)$. The assignment $f \mapsto f^-$ can be extended from polynomials to rational functions in the obvious way. Then, we can consider the I-tuple of rational functions $\boldsymbol{\omega}_{\mu,a}^-$ whose i-th entry is $(\boldsymbol{\omega}_{\mu,a})_i^-$ and (2.1.8) is equivalent to

(2.1.10) $$\Lambda_i^\pm(u)\, v = (\boldsymbol{\omega}_{\mu,a})_i^\pm(u)\, v$$

where, for notational convenience, we set $f^+(u) = f(u)$.

The set $\mathbb{F}[[u]]^I$ of I-tuples of power series is a ring under coordinate-wise addition and multiplication. Let $\mathscr{P}$ be the multiplicative subgroup generated by $\boldsymbol{\omega}_{\mu,a}, \mu \in P, a \in \mathbb{F}^\times$. Since $\mathbb{F}$ is algebraically closed, this coincides with the subgroup generated by $\boldsymbol{\omega}_{\omega_i,a}, i \in I, a \in \mathbb{F}^\times$. We simplify notation and set $\boldsymbol{\omega}_{i,a} = \boldsymbol{\omega}_{\omega_i,a}$.

Definition 2.1.5. The elements $\boldsymbol{\omega}_{i,a}$ are called fundamental ℓ-weights and the abelian group $\mathscr{P}$ is called the ℓ-weight lattice of $\mathfrak{g}$. The submonoid generated by the fundamental ℓ-weights will be denoted by $\mathscr{P}^+$. The elements of $\mathscr{P}^+$ are called dominant ℓ-weights or Drinfeld polynomials. ◊

Notice that there exists a unique group homomorphism

(2.1.11) $$\mathrm{wt} : \mathscr{P} \to P \qquad \text{such that} \qquad \mathrm{wt}(\boldsymbol{\omega}_{\mu,a}) = \mu \qquad \text{for all} \qquad \mu \in P, a \in \mathbb{F}^\times.$$

The prefix ℓ is chosen here to suggest that these concepts should be thought of as "loop analogues" of their classic counterparts.

We can identify $\mathscr{P}$ with a subset of $U(\tilde{\mathfrak{h}})^*$ as follows. Let $\boldsymbol{\mu} \in \mathscr{P}$, identify the i-th rational function of $\boldsymbol{\mu}^{\pm}$ with a formal power series as explained above and write $\boldsymbol{\mu}_i^{\pm}(u) = \sum_{r\geq 0} \boldsymbol{\mu}_{i,r}^{\pm} u^r$. Since $U(\tilde{\mathfrak{h}})$ is the commutative associative algebra freely generated by the elements $h_i, \Lambda_{i,r}, i \in I, r \in \mathbb{Z}, r \neq 0$, (Remark 1.4.12), there exists a unique algebra map $U(\tilde{\mathfrak{h}}) \to \mathbb{F}$ such that

$$h_i \mapsto \mathrm{wt}(\boldsymbol{\mu})(h_i) \qquad \text{and} \qquad \Lambda_{i,\pm r} \mapsto \boldsymbol{\mu}_{i,r}^{\pm} \qquad \text{for all} \qquad i \in I, r \in \mathbb{Z}. \tag{2.1.12}$$

Then, given a $\tilde{\mathfrak{g}}$-module V and $\boldsymbol{\mu} \in \mathscr{P}$, set

$$V_{\boldsymbol{\mu}} = \{v \in V : (x - \boldsymbol{\mu}(x))^n\, v = 0 \text{ for all } x \in U(\tilde{\mathfrak{h}}) \text{ and } n \gg 0\}. \tag{2.1.13}$$

Definition 2.1.6. Let V be a $\tilde{\mathfrak{g}}$-module. A nonzero vector of $V_{\boldsymbol{\mu}}$ is called an ℓ-weight vector and $V_{\boldsymbol{\mu}}$ is referred to as an ℓ-weight space of V. V is said to be an ℓ-weight module if

$$V = \bigoplus_{\boldsymbol{\mu} \in \mathscr{P}} V_{\boldsymbol{\mu}}.$$

If $V_{\boldsymbol{\mu}} \neq 0$, $\boldsymbol{\mu}$ is said to be an ℓ-weight of V. The set of all ℓ-weights of V will be denoted by $\mathrm{wt}_\ell(V)$. $\Diamond$

Let V and W be $\tilde{\mathfrak{g}}$-modules and $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathscr{P}$. Using the formula for the comultiplication of $\Lambda_{i,r}$ (Remark 1.4.12) it is not difficult, to check that we have the following (cf. Proposition 2.3.3):

$$V_{\boldsymbol{\mu}} \otimes W_{\boldsymbol{\nu}} \subseteq (V \otimes W)_{\boldsymbol{\mu\nu}}. \tag{2.1.14}$$

Let us return to the study of evaluation representations and their tensor products. Given a $\mathfrak{g}$-module V and $a \in \mathbb{F}^\times$, it follows from (2.1.10) that

$$V(a)_\mu = V_{\boldsymbol{\omega}_{\mu,a}}. \tag{2.1.15}$$

It then follows from (2.1.14) that the tensor product of ℓ-weight modules is again an ℓ-weight module.

Remark 2.1.7. While $\tilde{\mathfrak{h}}$ acts semisimply on an evaluation module constructed from a weight module, and hence also on tensor products of evaluation modules, we shall see later on (Example 2.2.10) that the same is not true on a general ℓ-weight module even if it is finite-dimensional. $\Diamond$

Let $\lambda, \mu \in P^+$, and fix highest-weight vectors of v and w for $V(\lambda)$ and $V(\mu)$, respectively. Consider the evaluation modules $V(\lambda, a)$ and $V(\mu, b)$ for some $a, b \in \mathbb{F}^\times$ and notice that $v \otimes w$ is an eigenvector for the action of $\tilde{\mathfrak{h}}$ with eigenvalues given by the ℓ-weight $\boldsymbol{\omega}_{\lambda,a}\boldsymbol{\omega}_{\mu,a}$. Moreover, $\tilde{\mathfrak{n}}^+(v \otimes w) = 0$. This motivates the following definition

Definition 2.1.8. An ℓ-weight vector v is said to be a highest ℓ-weight vector if v is an eigenvector for the action of $\tilde{\mathfrak{h}}$ and $\tilde{\mathfrak{n}}^+ v = 0$. A $\tilde{\mathfrak{g}}$-module V is said to be a highest ℓ-weight module if it is generated by a highest ℓ-weight vector. $\Diamond$

The proof of the following proposition is straightforward.

Proposition 2.1.9.

(i) Every highest ℓ-weight module is a weight module. Moreover, if $\boldsymbol{\mu}$ is the highest ℓ-weight of a highest ℓ-weight module V, then $V_\nu \neq 0$ only if $\nu \leq \mathrm{wt}(\boldsymbol{\mu})$.

(ii) Every highest ℓ-weight module has a unique proper submodule and, hence, a unique irreducible quotient. In particular, every highest-ℓ-weight module is indecomposable.

(iii) Two highest ℓ-weight modules are isomorphic only if they have the same highest ℓ-weight. □

One can consider Verma type highest ℓ-weight-modules. Namely, given $\boldsymbol{\mu} \in U(\tilde{\mathfrak{h}})^*$, let $M(\boldsymbol{\mu})$ be the universal highest ℓ-weight module of highest ℓ-weight $\boldsymbol{\mu}$. In other words, $M(\boldsymbol{\mu})$ is the quotient of $U(\tilde{\mathfrak{g}})$ by the left ideal generated by $\tilde{\mathfrak{n}}^+$ and $x - \boldsymbol{\mu}(x)$ for all $x \in U(\tilde{\mathfrak{h}})$. Then, as a $U(\tilde{\mathfrak{n}}^-)$-module, $M(\boldsymbol{\mu})$ is isomorphic to its free rank one module and, hence, is nonzero. We will denote by $V(\boldsymbol{\mu})$ the unique irreducible quotient of $M(\boldsymbol{\mu})$. The following now follows immediately from Theorem 2.1.3 and the above discussion.

Corollary 2.1.10. Let $\lambda_1, \dots \lambda_m \in P^+ \setminus \{0\}, a_1, \dots, a_m \in \mathbb{F}^\times$ distinct, and $\boldsymbol{\lambda} = \prod_{j=1}^m \boldsymbol{\omega}_{\lambda_j, a_j}$. Then, $V(\boldsymbol{\lambda}) \cong V(\lambda_1, a_1) \otimes \cdots \otimes V(\lambda_m, a_m)$. In particular, $V(\boldsymbol{\lambda})$ is finite-dimensional for all $\boldsymbol{\lambda} \in \mathscr{P}^+$. □

Remark 2.1.11. Notice that, since $x_0^+ \in \tilde{\mathfrak{g}}_{-\theta} \otimes t \subseteq \tilde{\mathfrak{n}}^-$, the highest-$\ell$-weight vector of $M(\boldsymbol{\mu})$ is not a highest-weight vector in the classic sense. In fact, if $\boldsymbol{\lambda} \in \mathscr{P}^+$ is not the I-tuple $\mathbf{1}$ of constant polynomials, the irreducible module $V(\boldsymbol{\lambda})$ cannot have a highest-weight vector in the classic sense since, otherwise, the action of $\tilde{\mathfrak{g}}$ could be lifted to an action of $\hat{\mathfrak{g}}$ contradicting Proposition 2.1.2. Evidently, $V(\mathbf{1})$ is the trivial representation of $\tilde{\mathfrak{g}}$. The reader should have noticed that we did not state in Proposition 2.1.9 that a highest ℓ-weight module is an ℓ-weight module. In the classic context of highest-weight modules, this was a trivial consequence of (1.3.2) which is in turn a consequence of the fact that the elements $x_i^\pm$ are eigenvectors for the adjoint action of $\mathfrak{h}$. Notice however that the elements $x_{i,r}^\pm$ are not eigenvectors for the adjoint action of $\tilde{\mathfrak{h}}$. In particular, there is no loop analogue of (1.3.2) for a general ℓ-weight module, even for finite-dimensional ones. In fact, one easily checks that, if v is a highest-ℓ-weight vector for $V(\lambda), \lambda \in \mathscr{P}^+$, then $x_{i,r}^- v$ may not be an ℓ-weight vector. ♢

Lemma 2.1.12. Suppose $\mathfrak{g} = \mathfrak{sl}_2$ and let $I = \{i\}$. For every $m, r \in \mathbb{Z}_{\geq 0}$ such that $0 < r \leq m$, there exists a polynomial $f_{m,r} \in \mathbb{Z}[t_1, \dots, t_{m+1}]$ satisfying the following property. For every finite-dimensional $\tilde{\mathfrak{g}}$-module V such that $\tilde{\mathfrak{n}}^+ V_{m\omega_i} = 0$ and $\Lambda_{i,m}$ acts bijectively on $V_{m\omega_i}$, we have

$$\Lambda_{i,-s} v = f_{m,s}(\Lambda_{i,1}, \dots, \Lambda_{i,m}, \Lambda_{i,m}^{-1}) v \quad \text{for all} \quad v \in V_\lambda, i \in I, 0 < s \leq \lambda(h_i),$$

where $\Lambda_{i,m}^{-1}$ is any left-inverse for the action of $\Lambda_{i,m}$ on $V_{m\omega_i}$.

Proof. Suppose V satisfies the above condition and $v \in V_\lambda$. By Lemma 1.4.11 and (2.1.4), for every $s > 1$, we have

$$0 = (x_{i,0}^+)^{(m)} (x_{i,1}^-)^{(m+s)} v = (-1)^m \left((x_{i,1}^-)^{(s)} \Lambda_{i,m} + \sum_{j=1}^m Y_{s,j} \Lambda_{i,m-j} \right) v \tag{2.1.16}$$

where $Y_{s,j}$ is a $\mathbb{Z}$-linear combination of elements of the form $(x_{i,1}^-)^{(p_1)} \cdots (x_{i,m+1}^-)^{(p_{m+1})}$ (with $\sum_k p_k = s$ and $\sum_k k p_k = s + j$) which does not depend neither on V nor on v. Since $-s < s + j - 2s < m$, it is not difficult to see that $(x_{i,-2}^+)^{(s)} Y_{s,j} \in U(\tilde{\mathfrak{g}}) U(\tilde{\mathfrak{n}}^+)^0 + H_{s,j}$, where $H_{s,j}$ is linear combination of monomials of the form $\Lambda_{i,r_1} \cdots \Lambda_{i,r_m}$ such that $-s < r_j < m$. Moreover, $(x_{i,-2}^+)^{(s)}(x_{i,1}^-)^{(s)} \in (-1)^s\Lambda_{i,-s} + U(\tilde{\mathfrak{g}})U(\tilde{\mathfrak{n}}^+)^0$ by Lemma 1.4.11. Plugging this into (2.1.16) and using that $\Lambda_{i,r}v \in V_{m\omega_i}$ for all $r \in \mathbb{Z}$, we get

$$0 = (x_{i,-2}^+)^{(s)}\left((x_{i,1}^-)^{(s)}\Lambda_{i,m}v + \sum_{j=1}^m Y_{s,j}\Lambda_{i,m-j}v\right) = (-1)^s\Lambda_{i,-s}\Lambda_{i,m}v + \sum_{j=1}^m \Lambda_{i,m-j}H_{s,j}v,$$

which implies

$$\Lambda_{i,-s}\Lambda_{i,m}v = (-1)^s \sum_{j=1}^m \Lambda_{i,m-j}H_{s,j}v, \tag{2.1.17}$$

Since $H_{s,j}$ involves the elements $\Lambda_{i,r}$ with $r > -s$ only, an easy induction on s using (2.1.17) completes the proof. □

We are ready to complete the classification of the finite-dimensional simple $\tilde{\mathfrak{g}}$-modules.

Theorem 2.1.13. Let V be a finite-dimensional $\tilde{\mathfrak{g}}$-module and v be an eigenvector for action of $\tilde{\mathfrak{h}}$ such that $\tilde{\mathfrak{n}}^+v = 0$. Then, v is a highest-ℓ-weight vector and its ℓ-weight is in $\mathscr{P}^+$. In particular, if V is irreducible, $V \cong V(\boldsymbol{\lambda})$ for some $\boldsymbol{\lambda} \in \mathscr{P}^+$.

Proof. The last statement is clear from the previous since, if $\lambda \in P^+$ is a maximal weight of V and v is an eigenvector for action of $\tilde{\mathfrak{h}}$ in V_λ, then v generates a submodule of V which is a highest-ℓ-weight module with highest ℓ-weight in $\mathscr{P}^+$.

Let λ be the weight of V and let $\omega_{i,r} \in \mathbb{F}, i \in I, r \in \mathbb{Z}$, be the eigenvalue of the action of $\Lambda_{i,r}$ on v. Since $\mathfrak{n}^+v = \{0\}$ and V is finite-dimensional, we must have $\lambda \in P^+$. Let $f_i(u) = \sum_{r\geq 0}\omega_{i,r}u^r$ and $g_i(u) = \sum_{r\geq 0}\omega_{i,-r}u^r$. We need to show that:

(1) $\omega_{i,r} = 0$ if $|r| > \lambda(h_i)$.

(2) $\omega_{i,\pm\lambda(h_i)} \neq 0$.

(3) $\omega_{i,\lambda(h_i)}\omega_{i,-s} = \omega_{i,\lambda(h_i)-s}$ for all $s = 1, \ldots, \lambda(h_i)$.

Let W be the submodule generated by v. To prove (1) and (2), given $s \in \mathbb{Z}$, consider the algebra $\tilde{\mathfrak{g}}_{i,s} \cong \mathfrak{sl}_2$ and let W_s be the the $\tilde{\mathfrak{g}}_{i,s}$-submodule of V generated by v. Then, $[x_{i,s}^+, x_{i,-s}^-] = h_i$ and W_s is a highest-weight-module for $\mathfrak{sl}_2$ with highest weight $\lambda(h_i)$. This implies $(x_{i,-s}^-)^m v = 0$ if $m > \lambda(h_i)$. Now, applying Lemma 1.4.11 with $l = m$, we get

$$(x_{i,\mp s}^+)^{(m)}(x_{i,\pm(s+1)}^-)^{(m)}v = (-1)^m\Lambda_{i,\pm m}v.$$

In particular, if $m > \lambda(h_i)$, (1) follows since $(x_{i,\mp s}^+)^{(m)}(x_{i,\pm(s+1)}^-)^{(m)}v \in W_{\lambda+\alpha_i} = 0$. For proving (2), by considering $m = \lambda(h_i)$ above, we are left to show that $(x_{i,\mp s}^+)^{(m)}(x_{i,\pm(s+1)}^-)^{(m)}v \neq 0$. In Remark 1.3.5 we observed that $(x_{i,\pm(s+1)}^-)^{(m)}v$ is a nonzero vector of $W_{\mp(s+1)}$. Moreover, $x_{i,\pm s}^-(x_{i,\pm(s+1)}^-)^{(m)}v \in W_{\lambda-(m+1)\alpha_i} = \{0\}$ since $\lambda - (m+1)\alpha_i \notin \mathrm{wt}(\lambda)$. In other words, $(x_{i,\pm(s+1)}^-)^{(m)}v$ is a lowest-weight vector for $\tilde{\mathfrak{g}}_{i,\mp s}$ of weight $-\lambda(h_i)$. By the analogous remark for lowest-weight modules, $(x_{i,\mp s}^+)^{(m)}(x_{i,\pm(s+1)}^-)^{(m)}v \neq 0$.

Now (3) follows from the previous lemma applied to the algebra $\tilde{\mathfrak{g}}_i$. Indeed, let $\boldsymbol{\lambda} \in \mathscr{P}^+$ be the ℓ-weight whose i-th component is $\sum_{r\geq 0}\omega_{i,r}u^r$. Notice that both W and $V(\boldsymbol{\lambda})$ satisfy the condition of the lemma on the weight space λ. Thus, the action of $\Lambda_i^-(u)$ on both W_λ and V_λ is obtained from that of $\Lambda_i^+(u)$ by means of the same polynomial on $\omega_{i,1}, \cdots, \omega_{i,\lambda(h_i)}, \omega_{i,\lambda(h_i)}^{-1}$. □

Corollary 2.1.14. Every finite-dimensional $\tilde{\mathfrak{g}}$-module is an ℓ-weight module.

Proof. Since every finite-dimensional representation has a composition series, it suffices to prove the corollary for the irreducible ones. But these are tensor products of evaluation modules. □

2.2. The quantum setting and Weyl modules

As in the classical case, one shows that the only simple finite-dimensional $U_q(\hat{\mathfrak{g}})$-modules are one-dimensional and that the central element c acts as the identity operator on any finite-dimensional $U_q(\hat{\mathfrak{g}}')$-module. In particular, $c^{1/2}$ acts as multiplication by ± 1. As in the discussion about type σ and type 1-modules, one sees that the modules on which $c^{1/2}$ acts as multiplication by -1 are obtained from those on which it acts as multiplication by 1 after tensoring with a one-dimensional module (cf. Proposition 1.3.18). Therefore, the study of finite-dimensional $U_q(\hat{\mathfrak{g}}')$-modules reduces to that of $U_q(\tilde{\mathfrak{g}})$-modules (which are of type 1 as $U_q(\mathfrak{g})$-modules).

The main goal of this subsection is to obtain the classification of the simple finite-dimensional $U_q(\tilde{\mathfrak{g}})$-modules. In the classical case, we have seen that we have two points of view for describing this classification: in terms of tensor products of evaluation modules or in terms of Drinfeld polynomials. In the quantum case, the former is no longer an option because, unless $\mathfrak{g} = \mathfrak{sl}_n$, there is no quantum analogue of evaluation maps. As a consequence, the study of the structure of the finite-dimensional representations of $U_q(\tilde{\mathfrak{g}})$ is much harder in the quantum case. Still, the classification of the simple modules in terms of Drinfeld polynomials can be carried out in essentially the same way. However, we shall consider a broader context and introduce other important class of $U_q(\tilde{\mathfrak{g}})$-modules: the Weyl modules in the sense of Chari and Pressley. We let $\mathscr{I}$ denote the category of $U_q(\tilde{\mathfrak{g}})$-modules which are integrable weight-modules (of type 1) as $U_q(\mathfrak{g})$-modules.

We begin observing that if $V \in \mathscr{I}$, since $U_q(\tilde{\mathfrak{h}})$ is commutative, we have

$$\Lambda_{i,r} V_\mu \subseteq V_\mu \qquad \text{for all} \qquad i \in I, r \in \mathbb{Z}, \mu \in P. \tag{2.2.1}$$

Also, (2.1.1) remains valid.

Lemma 2.2.1. Let $V \in \mathscr{I}$, $\lambda \in \mathrm{wt}(V)$ be such that $x^+_{i,r} V_\lambda = \{0\}$ for all $i \in I, r \in \mathbb{Z}$, and $W = U_q(\tilde{\mathfrak{g}}) V_\lambda$. Then:

(i) $\lambda \in P^+$ and $\mathrm{wt}(W) = \mathrm{wt}(\lambda)$. In particular, $(x^-_{i,r_1}) \cdots (x^-_{i,r_m}) V_\lambda = \{0\}$ for all $i \in I, r \in \mathbb{Z}$ if $m > \lambda(h_i)$.

(ii) $\Lambda_{i,\pm r} V_\lambda = 0$ for all $i \in I, r > \lambda(h_i)$.

(iii) $\Lambda_{i,\pm\lambda(h_i)}$ act as a linear monomorphism on V_λ.

(iv) $(\Lambda_{i,\lambda(h_i)} \Lambda_{i,-s} - \Lambda_{i,\lambda(h_i)-s}) V_\lambda = \{0\}$ for all $i \in I, 0 \le s \le \lambda(h_i)$.

Proof. It follows from (2.1.1) and (2.2.1) that λ is the unique maximal element of $\mathrm{wt}(W)$. Since V is integrable, any $v \in V_\lambda$ generates a finite-dimensional $U_q(\mathfrak{g})$-submodule of V. Hence, $\lambda \in P^+$ and the second statement of part (i) then follows from Corollary 1.3.11. Parts (ii) and (iii) are proved similarly to items (1) and (2) in the proof of Theorem 2.1.13. We cannot prove part (iv) in the same way we proved item (3) in the proof of Theorem 2.1.13 since we have not constructed any representation satisfying the hypothesis of the lemma yet. We use an alternate approach which could have been used in the classical context as well by using the classical analogues of (1.4.4) and (1.4.5).

Let $v \in V_\lambda$ and write $n = \lambda(h_i)$. Using Lemma 1.4.11 with $l = n$ and $m = l + 1$ we get

$$\sum_{r=0}^{n} (q_i^{n-r} x_{i,r}^- \Lambda_{i,n-r}) v = 0. \tag{2.2.2}$$

Apply $x_{i,-s}^+$ with $s \geq 0$ to (2.2.2) and use the relation $(q_i - q_i^{-1})[x_{i,s}^+, x_{i,r}^-] = \psi_{i,r-s}^+ - \psi_{i,r-s}^-$ to obtain

$$\sum_{r=0}^{s} q_i^{n-r} \psi_{i,r-s}^- \Lambda_{i,n-r} v = \sum_{r=s}^{n} q_i^{n-r} \psi_{i,r-s}^+ \Lambda_{i,n-r} v. \tag{2.2.3}$$

By (1.4.4) with $r = n - s$, the right-hand side of (2.2.3) is

$$q_i^{n-s} k_i \Lambda_{i,n-s} v + \sum_{t=1}^{n-s} q_i^{n-s-t} \psi_{i,t}^+ \Lambda_{i,n-s-t} v = (q_i^{2n-s} - q_i^n (q_i^{n-s} - q_i^{-(n-s)})) \Lambda_{i,n-s} v = q_i^s \Lambda_{i,n-s} v.$$

Plugging this in (2.2.3) we get

$$\sum_{r=0}^{s-1} q_i^{n-r} \psi_{i,r-s}^- \Lambda_{i,n-r} v = (q_i^s - q_i^{-s}) \Lambda_{i,n-s} v. \tag{2.2.4}$$

We now proceed recursively on $s = 1, \ldots, n$. For $s = 1$, the left-hand side of (2.2.4) is $q_i^n \psi_{i,-1}^- \Lambda_{i,n} v = q_i^n (k_i^{-1}(q_i - q_i^{-1})\Lambda_{i,-1}) \Lambda_{i,n} v = (q_i - q_i^{-1}) \Lambda_{i,-1} \Lambda_{i,n} v$ where we used (1.4.5) in the first equality. It follows from (2.2.4) that $(\Lambda_{i,-1}\Lambda_{i,n} - \Lambda_{i,n-1}) v = 0$ as claimed. Now, fix $s > 1$ and assume $\Lambda_{i,n}\Lambda_{i,-r} v = \Lambda_{i,n-r} v$ for all $0 \leq r < s$. The left hand side of (2.2.4) is

$$\begin{aligned}
\sum_{r=0}^{s-1} q_i^{n-r} \psi_{i,r-s}^- \Lambda_{i,n-r} v &= q_i^n \psi_{i,-s}^- \Lambda_{i,n} v + \sum_{r=1}^{s-1} q_i^{n-r} \psi_{i,r-s}^- \Lambda_{i,n-r} v = \\
&= q_i^n \psi_{i,-s}^- \Lambda_{i,n} v + \sum_{r=1}^{s-1} q_i^{n-r} \psi_{i,r-s}^- \Lambda_{i,n} \Lambda_{i,-r} v = \\
&= q_i^n \Lambda_{i,n} \left(\psi_{i,-s}^- + \sum_{r=1}^{s-1} q_i^{-r} \psi_{i,r-s}^- \Lambda_{i,-r} \right) v = \\
&= q_i^n \Lambda_{i,n} \left(k_i^{-1} (q_i^s - q_i^{-s}) \Lambda_{i,-s} \right) v = (q_i^s - q_i^{-s}) \Lambda_{i,n} \Lambda_{i,-s} v
\end{aligned}$$

where we used (1.4.5) in the last line. Plugging this in (2.2.4) the proof is complete. □

Definition 2.2.2. For $\lambda \in P^+$, the global Weyl module $W_q(\lambda)$ of highest-weight λ is the $U_q(\tilde{\mathfrak{g}})$-module generated by a vector v satisfying the defining relations of having weight λ and $x_{i,r}^+ v = (x_i^-)^{\lambda(h_i)+1} v = 0$ for all $i \in I, r \in \mathbb{Z}$. ◊

Proposition 2.2.3. For every $\lambda \in P^+$, $W_q(\lambda) \in \mathscr{I}$. Moreover, every $V \in \mathscr{I}$ which is generated by a highest-weight vector of weight λ is a quotient of $W_q(\lambda)$.

Proof. The second statement is clear from Lemma 2.2.1(i) and the definition of $W_q(\lambda)$. Clearly $xv \in W_q(\lambda)_\lambda$ for all $x \in U_q(\tilde{\mathfrak{h}})$. Therefore, $W_q(\lambda) = U_q(\tilde{\mathfrak{n}}^-)V_\lambda$ is a weight-module and λ is the unique maximal element of $\mathrm{wt}(W_q(\lambda))$. This and (2.1.1) immediately imply that $x_{i,r}^+$ act locally nilpotently on $W_q(\lambda)$ for all $i \in I, r \in \mathbb{Z}$. It remains to show that x_i^- act locally nilpotently for all $i \in I$. Thus, we have to show that, for all $v \in W_q(\lambda)_\lambda, m \geq 0, i_1, \dots i_m \in I, r_1, \dots, r_m \in \mathbb{Z}$, and $i \in I$, there exists $n > 0$ such that $(x_i^-)^n x_{i_m,r_m}^- \cdots x_{i_1,r_1}^- v = 0$. This can be proved by induction on $m \geq 0$ similarly to the proof of Lemma 1.3.2 using the loop analogue of the quantum Serre's relations (cf. [59, Lemma 5.7]). □

Remark 2.2.4. One can define global Weyl modules $W(\lambda)$ in the classical context as well. Notice that if $\boldsymbol{\lambda} \in \mathscr{P}^+$ is such that $\mathrm{wt}(\boldsymbol{\lambda}) = \lambda$, then $V(\boldsymbol{\lambda})$ is a quotient of $W(\lambda)$ showing that $W(\lambda) \neq \{0\}$. In order to show that $W_q(\lambda) \neq \{0\}$, observe that it suffices to show this for $\mathfrak{g} = \mathfrak{sl}_2$. In that case, we shall see in Subsection 2.3 that there exists an algebra map $U_q(\tilde{\mathfrak{g}}) \to U_q(\mathfrak{g})$ which is the identity on $U_q(\mathfrak{g})$. The pull-back of $V_q(\lambda)$ by this map is then a nonzero quotient of $W_q(\lambda)$ showing that $W_q(\lambda) \neq 0$. One can also think of $W(\lambda)$ as a quotient of the $\tilde{\mathfrak{g}}$-module induced from the $\mathfrak{g}$-module $V(\lambda)$. It follows from Lemma 2.2.1 that $\mathrm{wt}(W(\lambda)) = \mathrm{wt}(\lambda)$ and, hence, the elements $x_0^\pm$ also act locally nilpotently on $W(\lambda)$, i.e., $W(\lambda)$ is an integrable $\hat{\mathfrak{g}}'$-module. Moreover, since $W(\lambda)$ is a quotient of $U(\tilde{\mathfrak{g}})$ by a $\mathbb{Z}$-graded ideal, we can regard $W(\lambda)$ as a $\hat{\mathfrak{g}}$-module. Evidently, all these comments apply to the quantum setting as well. ◊

We can regard $W_q(\lambda)$ as a right module for $U_q(\tilde{\mathfrak{h}})$ as follows:

$$(xv)\, y = (xy)\, v, \qquad \text{for all} \qquad x \in U_q(\tilde{\mathfrak{g}}), y \in U_q(\tilde{\mathfrak{h}}), v \in W_q(\lambda)_\lambda.$$

One easily checks that this is a well defined action.

Theorem 2.2.5. For all $\lambda \in P^+$, $W_q(\lambda)$ is finitely generated as a right $U_q(\tilde{\mathfrak{h}})$-module.

Proof. Set $V = W_q(\lambda)$ and let $v \in W_q(\lambda)_\lambda$ be a generator of $W_q(\lambda)$. Since $\mathrm{wt}(V) \subseteq \mathrm{wt}(\lambda)$, $W_q(\lambda)$ has finitely many weight spaces and it remains to show that all weight spaces are finitely generated as $U_q(\tilde{\mathfrak{h}})$-modules. It suffices to show that, given $\mu < \lambda$, V_μ is generated by elements of the form

$$(x_{i_m,r_m}^-) \cdots (x_{i_1,r_1}^-) yv \quad \text{with} \quad y \in U_q(\tilde{\mathfrak{h}}), \quad \sum_{j=1}^m \alpha_{i_j} = \lambda - \mu, \quad 0 \leq r_j < \lambda(h_{i_j}) + j - 1. \tag{2.2.5}$$

This will be proved by induction on $m \geq 1$.

Let $m = 1, i = i_1, r = r_1$. Observe that Lemmas 1.4.11 and 2.2.1 imply $\sum_{t=0}^{\lambda(h_i)} q_i^t x_{i,\lambda(h_i)+s-t}^- \Lambda_{i,t} v = 0$ for all $s \in \mathbb{Z}$. We rewrite this in two equivalent ways:

$$x_{i,\lambda(h_i)+s}^- v = \sum_{t=1}^{\lambda(h_i)} q_i^t x_{i,\lambda(h_i)+s-t}^- \Lambda_{i,t} v \qquad \text{and} \qquad x_{i,s}^- v = \left(\sum_{t=0}^{\lambda(h_i)-1} q_i^t x_{i,\lambda(h_i)+s-t}^- \Lambda_{i,t} \right) \Lambda_{i,-\lambda(h_i)} v. \tag{2.2.6}$$

To obtain the second version of (2.2.6), we used that $(\Lambda_{i,\lambda(h_i)} v)\Lambda_{i,-\lambda(h_i)} = v$. If $r \geq \lambda(h_i)$, we let $s = r - \lambda(h_i)$ in the first version of (2.2.6) and proceed by induction on r to get the claim. If $r < 0$, we let $s = r$ in the second version of (2.2.6) and obtain the claim by induction on $|r|$.

Now, consider a vector of the form $x_{i,r}^- x_{i_m,r_m}^- \cdots x_{i_1,r_1}^- v$ with $m \geq 1$ and assume, by induction hypothesis on m, that $0 \leq r_j \leq \lambda(h_{i_j}) + j - 1$, for all $j = 1, \dots, m$. Recall the defining relation:

$$x_{i,r}^- x_{j,s}^- - q_i^{-c_{ij}} x_{j,s}^- x_{i,r}^- = q_i^{-c_{ij}} x_{i,r-1}^- x_{j,s+1}^- - x_{j,s+1}^- x_{i,r-1}^-. \tag{2.2.7}$$

Suppose first that $r > \lambda(h_i) + m$ and proceed by induction on r using (2.2.7). Namely, letting $j = i_m$, and $s = r_m$ in 2.2.7, we get

$$x^-_{i,r}x^-_{i_m,r_m} = q_i^{-c_{ii_m}}(x^-_{i,r-1}x^-_{i_m,r_m+1} + x^-_{i_m,r_m}x^-_{i,r}) - x^-_{i_m,r_m+1}x^-_{i,r-1}. \tag{2.2.8}$$

Therefore,

$$\begin{aligned}(x^-_{i,r}x^-_{i_m,r_m})x^-_{i_{m-1},r_{m-1}}\cdots x^-_{i_1,r_1}yv &= \underbrace{-x^-_{i_m,r_m+1}x^-_{i,r-1}x^-_{i_{m-1},r_{m-1}}\cdots x^-_{i_i,r_1}yv}_{a} + \\ & q_i^{-c_{i,i_m}}(\underbrace{x^-_{i,r-1}x^-_{i_m,r_m+1}x^-_{i_{m-1},r_{m-1}}\cdots x^-_{i_1,r_1}yv}_{b} + \underbrace{x^-_{i_m,r_m}x^-_{i,r}x^-_{i_{m-1},r_{m-1}}\cdots x^-_{i_1,r_1}yv}_{c}).\end{aligned}$$

We want to show that a, b, and c are in the span of vectors of the form (2.2.5). For a, by induction hypothesis on m, $x^-_{i,r-1}x^-_{i_{m-1},r_{m-1}}\cdots x^-_{i_i,r_1}yv$ is in the span of vectors of this form. Since $0 \le r_m < \lambda(h_{i_m}) + m - 1$, we have $0 \le r_m + 1 < \lambda(h_{i_m}) + m$ as desired. The conclusion for c is reached similarly. For b, by induction hypothesis on m, $x^-_{i_m,r_m+1}x^-_{i_{m-1},r_{m-1}}\ldots x^-_{i_1,r_1}v$ is also in the span of vectors of the form (2.2.5). An obvious induction on r completes the argument.

Finally, if $r < 0$, we use (2.2.7) with $r + 1$ in place of r, $j = i_m$, and $s = r_m - 1$ to get

$$x^-_{i,r}x^-_{i_m,r_m} = q_i^{-c_{ii_m}}(x^-_{i,r+1}x^-_{i_m,r_m-1} + x^-_{i_m,r_m}x^-_{i,r}) - x^-_{i_m,r_m-1}x^-_{i,r+1}. \tag{2.2.9}$$

Now one proceeds as in the previous case using a further induction on $|r|$. □

Remark 2.2.6. Notice that, in the classical case, the proof of the above theorem implies that $W(\lambda) = U(\mathfrak{g}[t])v$. ◊

The next corollary is immediate.

Corollary 2.2.7. Let $\lambda \in P^+$ and V be a quotient of $W_q(\lambda)$ such that V_λ is finite-dimensional. Then, V is finite-dimensional. □

Now, let V be a simple finite-dimensional $U_q(\tilde{\mathfrak{g}})$-module. As in the classical case, if λ is a maximal weight of V, then there exists $v \in V_\lambda$ which is an eigenvector for the action of $U_q(\tilde{\mathfrak{h}})$. Since, V is certainly in $\mathscr{I}$, it must be a quotient of $W_q(\lambda)$. In particular, V_λ is a one-dimensional quotient of $W_q(\lambda)_\lambda$. But $W_q(\lambda)_\lambda$ is generated by the action of the subalgebra generated $\Lambda_{i,r}, i \in I, r = 1, \ldots \lambda(h_i)$. Let ann_λ be the kernel of the representation of this algebra on $W_q(\lambda)_\lambda$ and A_λ be its quotient by ann_λ. Then, when regarded as a module for A_λ, V_λ is isomorphic to $A_\lambda/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of A_λ. Since A_λ is a finitely generated commutative algebra, $\mathfrak{m}$ must be generated by elements of the form $\Lambda_{i,r} - a_{i,r}$ for some $a_{i,r} \in \mathbb{F}$. Moreover, $a_{i,\lambda(h_i)} \neq 0$. By letting $\boldsymbol{\lambda} \in \mathscr{P}^+$ be the element whose i-th polynomial is $1 + \sum_{r=1}^{\lambda(h_i)} a_{i,r}u^r$, we have an injective map from the set $\mathrm{specm}(A_\lambda)$ of maximal ideals of A_λ to $\mathscr{P}^+$.

Proposition 2.2.8. The map $\mathrm{specm}(A_\lambda) \to \mathscr{P}^+$ constructed above is bijective.

Remark 2.2.9. We postpone the proof of this proposition to Subsection 2.3. It easily follows from Corollary 2.3.4. Notice that for $\mathfrak{g} = \mathfrak{sl}_2$ the proposition is equivalent to saying that the homomorphism from the polynomial algebra $\mathbb{F}[t_1, \cdots, t_{m-1}][t_m^{\pm 1}]$ to A_λ sending t_r to $\Lambda_{i,r}$ is an isomorphism. ◊

Definition 2.2.10. Let $\boldsymbol{\lambda} \in \mathscr{P}^+, \lambda = \mathrm{wt}(\boldsymbol{\lambda})$, and $\mathfrak{m}$ be the ideal of A_λ associated to $\boldsymbol{\lambda}$. The local Weyl module $W_q(\boldsymbol{\lambda})$ is the quotient of $W_q(\lambda)$ by the submodule generated by $\mathfrak{m}W_q(\lambda)_\lambda$. If V is a quotient of $W_q(\boldsymbol{\lambda})$, then V is said to be a highest-ℓ-weight module of highest ℓ-weight $\boldsymbol{\lambda}$. $\diamondsuit$

Given $\boldsymbol{\lambda} \in \mathscr{P}^+$, let $\boldsymbol{\lambda}_i(u) = \sum_{r\geq 0} \lambda_{i,r}u^r$ and $\lambda = \mathrm{wt}(\boldsymbol{\lambda})$. Notice that $W_q(\boldsymbol{\lambda})$ is isomorphic to the quotient of $U_q(\tilde{\mathfrak{g}})$ by the ideal generated by $x_{i,r}^+, (x_i^-)^{\lambda(h_i)+1}, k_i - q_i^{\lambda(h_i)}, \Lambda_{i,s} - \lambda_{i,s}, \Lambda_{i,\lambda(h_i)}\Lambda_{i,-s} - \Lambda_{i,\lambda(h_i)-s}$, for all $i \in I, r, s \in \mathbb{Z}, 0 < s \leq \lambda(h_i)$. Since $W_q(\boldsymbol{\lambda})_\lambda$ is one-dimensional by construction, $W_q(\boldsymbol{\lambda})$ is finite-dimensional by Corollary 2.2.7.

One easily proves the quantum analogue of Proposition 2.1.9. We denote by $V_q(\boldsymbol{\lambda})$ the irreducible quotient of $W_q(\boldsymbol{\lambda}), \boldsymbol{\lambda} \in \mathscr{P}^+$. The next proposition is now immediate.

Proposition 2.2.11. The assignment $\boldsymbol{\lambda} \mapsto V_q(\boldsymbol{\lambda})$ induces a bijection from $\mathscr{P}^+$ to the set of isomorphism classes of simple finite-dimensional $U_q(\tilde{\mathfrak{g}})$-modules. $\square$

Since every finite-dimensional module has a composition series, given a finite-dimensional $U_q(\tilde{\mathfrak{g}})$-module V and $\boldsymbol{\lambda} \in \mathscr{P}^+$, we can consider the multiplicity $[V : \boldsymbol{\lambda}]$ of $V_q(\boldsymbol{\lambda})$ as a simple factor of V (and similarly in the classical setting). Evidently, $[V : \boldsymbol{\lambda}] = 1$ if V is a highest-ℓ-weight module of highest ℓ-weight $\boldsymbol{\lambda}$.

Example 2.2.12. Let $\mathfrak{g} = \mathfrak{sl}_2, I = \{i\}$, and $\boldsymbol{\lambda} = \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{i,b}$ for some $a, b \in \mathbb{F}^\times$. Then, $\mathrm{wt}(\boldsymbol{\lambda}) = 2\omega_i$ and $\mathrm{wt}(W(\boldsymbol{\lambda})) = \{2\omega_i, 0, -2\omega_i\}$. Also, $\dim(W(\boldsymbol{\lambda})_{\pm 2\omega_i}) = 1$ and, from the proof of Theorem 2.2.5, we see that $W(\boldsymbol{\lambda})_0$ is panned by $x_i^- v, x_{i,1}^- v$ where v is a nonzero element of $W(\boldsymbol{\lambda})_{2\omega_i}$. One can check that these two vectors are linearly independent and, hence, $\dim(W(\boldsymbol{\lambda})) = 4$ (similarly in the quantum case).

If $a \neq b$, we have $V(\boldsymbol{\lambda}) \cong V(\omega_i, a) \otimes V(\omega_i, b)$ which is 4-dimensional and, therefore, the Weyl module $W(\boldsymbol{\lambda})$ is irreducible. It follows that the ℓ-weights of $W(\boldsymbol{\lambda})$ are $\boldsymbol{\lambda}, \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{i,b}^{-1}, \boldsymbol{\omega}_{i,a}^{-1}\boldsymbol{\omega}_{i,b}$, and $\boldsymbol{\lambda}^{-1}$. All the ℓ-weight spaces are one-dimensional. It is interesting to notice that neither $x_i^- v$ nor $x_{i,1}^- v$ are ℓ-weight vectors (for instance, they are not eigenvectors for the action of $\Lambda_{i,1} = -h_{i,1}$). In fact, $x_{i,r}^- v$ is not an ℓ-weight vector for every $r \in \mathbb{Z}$.

If $a = b$, then the irreducible quotient of $W(\boldsymbol{\lambda})$ is the evaluation module $V(2\omega_i, a)$ which is 3-dimensional. Thus, we have a non-split short exact sequence of representations

$$0 \to \mathbb{F} \to W(\boldsymbol{\lambda}) \to V(\boldsymbol{\lambda}) \to 0 \tag{2.2.10}$$

where $\mathbb{F}$ stands for the trivial representation of $\tilde{\mathfrak{g}}$, i.e., the simple module associated to the constant Drinfeld polynomial $\mathbf{1}$. In particular, this shows that the category of finite-dimensional $\tilde{\mathfrak{g}}$-modules is not semisimple (we will see a similar non-split short exact sequence in the quantum case in the next subsection). It is not difficult to check that $W(\boldsymbol{\lambda})_0 = W(\boldsymbol{\lambda})_{\mathbf{1}}$. Therefore, the vectors $x_{i,r}^- v, r \in \mathbb{Z}$, are ℓ-weight vectors. However, one can check that they are still not eigenvectors for the action of $\Lambda_{i,1}$ - only generalized eigenvectors. Therefore, $\tilde{\mathfrak{h}}$ does not act semisimply on $W(\boldsymbol{\lambda})$. $\diamondsuit$

For later use, we record the following trivially established lemma (which also holds in the classical setting with the same proof).

Lemma 2.2.13. Let $\boldsymbol{\lambda} \in \mathscr{P}^+, v$ a highest-ℓ-weight vector of $V_q(\boldsymbol{\lambda})$, and $J \subseteq I$. Then, the $U_q(\tilde{\mathfrak{g}}_J)$-submodule of V generated by v is irreducible. $\square$

2.3. Basic character theory

Similarly to the classical case, we identify the ℓ-weight lattice $\mathscr{P}$ with a subset of $U_q(\tilde{\mathfrak{h}})^*$. Namely, given $\boldsymbol{\mu} \in \mathscr{P}$, we define $\boldsymbol{\mu}(\Lambda_{i,r})$ as in (2.1.12), while we set $\boldsymbol{\mu}(k_i) = q_i^{\mathrm{wt}(\boldsymbol{\mu})(h_i)}$ and require $\boldsymbol{\mu}$ to be an algebra homomorphism $U_q(\tilde{\mathfrak{h}}) \to \mathbb{F}$ as before. Once this identification is done, one can define the notions of ℓ-weight vectors and modules as in Definition 2.1.6 and the ones of highest ℓ-weight vectors and modules as in Definition 2.1.8 (one should replace the requirement $\tilde{\mathfrak{n}}v = 0$ by $U_q(\tilde{\mathfrak{n}}^+)^0 v = 0$). Notice that this agree with the definition of highest-ℓ-weight modules in the sense of of Definition 2.2.10 in case $\boldsymbol{\mu} \in \mathscr{P}^+$. The notation $\mathrm{wt}_\ell(V)$ is also defined as before. Notice that if V is an ℓ-weight module, then

$$V_\mu = \bigoplus_{\substack{\boldsymbol{\mu} \in \mathscr{P} \\ \mathrm{wt}(\boldsymbol{\mu}) = \mu}} V_{\boldsymbol{\mu}} \qquad \text{for all} \qquad \mu \in P. \tag{2.3.1}$$

One of the main goals of this section is to prove the quantum analogue of Corollary 2.1.14. Namely, we want to prove that every finite-dimensional (type 1) $U_q(\tilde{\mathfrak{g}})$-module is an ℓ-weight module. Before doing that, let us define the ℓ-analogue of the notion of characters.

Definition 2.3.1. Let V be an ℓ-weight module with finite-dimensional ℓ-weight spaces. The qcharacter of V is the function $\mathrm{qch}(V) : \mathscr{P} \to \mathbb{Z}, \boldsymbol{\mu} \mapsto \dim(V_{\boldsymbol{\mu}})$. Given $\chi \in \mathbb{Z}^{\mathscr{P}}$, let $\mathrm{wt}_\ell(\chi) = \{\boldsymbol{\mu} \in \mathscr{P} : \chi(\boldsymbol{\mu}) \neq 0\}$ (in particular, $\mathrm{wt}_\ell(V) = \mathrm{wt}_\ell(\mathrm{qch}(V))$). A dominant ℓ-weight $\boldsymbol{\lambda}$ is said to be q-minuscule if $\mathrm{wt}_\ell(V_q(\boldsymbol{\lambda})) \cap \mathscr{P}^+ = \{\boldsymbol{\lambda}\}$. In that case $V_q(\boldsymbol{\lambda})$ is said to be an ℓ-minuscule module. ◊

Remark 2.3.2. The reason for the choice of terminology qcharacter instead of ℓ-character is historical and will be explained in Subsection 2.4. Evidently, the notion makes sense in the classical context as well (and some other contexts) where the terminology q sounds strange. However, notice that the qcharacter of a simple finite-dimensional $\tilde{\mathfrak{g}}$-module is easily deduced from its character due to (2.1.15), (2.1.14), and Corollary 2.1.10. Thus, the notion is really interesting only in the quantum case. However, as in the case of characters, one easily checks that the qcharacter determines the multiplicities of the simple factors of a finite-dimensional module V and vice-versa (something the character alone does not do even in the classical context). We shall denote by $[V : \boldsymbol{\lambda}]$ the multiplicity of $V_q(\boldsymbol{\lambda})$ as a simple factor of V.

The notion of q-minuscule weight generalizes that of minuscule weights. However, we shall see that we do not have a characterization of q-minuscule weights in terms of the action of a group on $\mathscr{P}$ as we have for minuscule in terms of the Weyl group action on P. It turns out that, if q and q' are not roots of unit, then $\boldsymbol{\lambda}$ is q-minuscule if and only if it is q'-minuscule. One can also think of defining 1-minuscule elements of $\mathscr{P}^+$ by requiring that the set of ℓ-weights of the corresponding simple $\tilde{\mathfrak{g}}$-module has a unique dominant ℓ-weight. However, using Corollary 2.1.10, it is not difficult to see that $\boldsymbol{\lambda}$ is 1-minuscule if and only if it is of the form $\boldsymbol{\lambda} = \prod_{j=1}^m \boldsymbol{\omega}_{i_j,a_j}$ with $m \geq 0, a_j \in \mathbb{F}^\times$ all distinct, and $i_j \in I$ such that ω_{i_j} is minuscule. In other words, $V(\boldsymbol{\lambda})$ is a tensor product of evaluation modules (at distinct evaluation parameters) of minuscule representations of $\mathfrak{g}$. Thus, the classical notion of a minuscule ℓ-weight is also not too interesting. We shall see below that there are many more q-minuscule ℓ-weights. Because of this, we shall drop the dependence of q in the terminology and simply say $\boldsymbol{\lambda}$ is a minuscule ℓ-weight. ◊

As in the case of characters, it is convenient to regard $\mathrm{qch}(V)$ as a formal sum

$$\mathrm{qch}(V) = \sum_{\mu \in \mathscr{P}} \dim(V_{\mu})\boldsymbol{\mu},$$

where we identify $\boldsymbol{\mu}$ with the characteristic function of the subset $\{\boldsymbol{\mu}\} \subseteq \mathscr{P}$. If V is finite-dimensional, this allows us to regard $\mathrm{qch}(V)$ as an element of the group ring $\mathbb{Z}[\mathscr{P}]$. Notice that, by (2.3.1), the character of an ℓ-weight module with finite-dimensional weight spaces is given by

$$\mathrm{ch}(V) = \sum_{\mu \in \mathscr{P}} \dim(V_{\mu}) e^{\mathrm{wt}(\mu)}. \tag{2.3.2}$$

Since the formula for the comultiplication of the elements $\Lambda_{i,r}$ is more complicated than in the classical setting (indeed not known precisely), (2.1.14) is no longer true in the quantum setting. Still, the following proposition (which is easily deduced from (2.1.14) in the classical case), remains true.

Proposition 2.3.3. Let V and W be finite-dimensional ℓ-weight modules. Then, $V \otimes W$ is an ℓ-weight module and $\mathrm{qch}(V \otimes W) = \mathrm{qch}(V)\mathrm{qch}(W)$.

Proof. Choose a basis α of V and a basis β of W, both consisting of ℓ-weight vectors. Say, $v_j \in V_{\boldsymbol{\mu}_j}$ and $w_k \in W_{\boldsymbol{\mu}_k}$, for some $\boldsymbol{\mu}_j, \boldsymbol{\nu}_k \in \mathscr{P}$. We will write the proof under the assumption that all ℓ-weight vectors are in fact eigenvectors for the action of $U_q(\tilde{\mathfrak{h}})$ and leave it to the reader to provide the details for the case of generalized eigenvectors. Order α in such a way that $\mathrm{wt}(\boldsymbol{\mu}_j) < \mathrm{wt}(\boldsymbol{\mu}_{j'})$ implies $j > j'$. Similarly, order β in such way that $\mathrm{wt}(\boldsymbol{\nu}_k) < \mathrm{wt}(\boldsymbol{\nu}_{k'}) \Rightarrow k > k'$. Also, order $\alpha \otimes \beta$ so that $(j,k) > (j',k')$ if $k > k'$.

Given $i \in I, r > 0$, Proposition 1.4.10(i) implies

$$\Lambda_{i,r}(v_j \otimes w_k) = \sum\nolimits_{s=0}^{r} (\Lambda_{i,s} v_j) \otimes (\Lambda_{i,r-s} w_k) + m_{j,k},$$

where $m_{j,k}$ is a linear combination of elements $v_{j'} \otimes w_{k'}$ with $k' < k$. Let $(\boldsymbol{\mu}_j)_i(u) = \sum_{s \geq 0} \boldsymbol{\mu}_{j,i,s} u^s$ for some $\boldsymbol{\mu}_{j,i,s} \in \mathbb{F}$, and, similarly, let $(\boldsymbol{\nu}_k)_i(u) = \sum_{s \geq 0} \boldsymbol{\nu}_{k,i,s} u^s$. Thus, $\Lambda_{i,s} v_j = \boldsymbol{\mu}_{j,i,s} v_j$ and $\Lambda_{i,s} w_k = \boldsymbol{\nu}_{k,i,s} w_k$. Therefore,

$$\Lambda_{i,r}(v_j \otimes w_k) = \left(\sum_{s=0}^{r} \boldsymbol{\mu}_{j,i,s} \boldsymbol{\nu}_{k,i,r-s} \right) v_j \otimes w_k + m_{j,k}.$$

It follows that the matrix of the action of $\Lambda_{i,r}$ on $V \otimes W$ with respect to the basis $\alpha \otimes \beta$ is upper triangular with $\sum_{s=0}^{r} \boldsymbol{\pi}_{j,i,s} \boldsymbol{\varpi}_{k,i,r-s}$ in the diagonal entry corresponding to (j,k). On the other hand, $(\boldsymbol{\mu}_j)_i(u)(\boldsymbol{\nu}_k)_i(u) = \sum_{r \geq 0} \sum_{s=0}^{r} \boldsymbol{\mu}_{j,i,s} \boldsymbol{\nu}_{k,i,r-s} u^r$. □

The next corollary follows immediately.

Corollary 2.3.4. Let $\boldsymbol{\lambda} = \boldsymbol{\mu}\boldsymbol{\nu}$ for some $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathscr{P}^+$. Then, $V_q(\boldsymbol{\lambda})$ is a quotient of the submodule of $V = V_q(\boldsymbol{\mu}) \otimes V_q(\boldsymbol{\nu})$ generated by $V_{\mathrm{wt}(\lambda)}$. □

Corollary 2.3.5. Let $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathscr{P}^+$. Then, $V_q(\boldsymbol{\mu}) \otimes V_q(\boldsymbol{\nu})$ is irreducible if and only if $V_q(\boldsymbol{\nu}) \otimes V_q(\boldsymbol{\mu})$ is irreducible.

Proof. Let $V = V_q(\boldsymbol{\mu}) \otimes V_q(\boldsymbol{\nu})$, $W = V_q(\boldsymbol{\nu}) \otimes V_q(\boldsymbol{\mu})$, $\boldsymbol{\lambda} = \boldsymbol{\mu\nu}$, and $\lambda = \mathrm{wt}(\boldsymbol{\lambda})$. Then, the submodules of V and W generated by V_λ and W_λ, respectively, are quotients of $W_q(\boldsymbol{\lambda})$ by Corollary 2.3.4. Since V and W have the same dimension, either both are irreducible or none. □

Thus, in order to prove that $V_q(\boldsymbol{\lambda})$ is an ℓ-weight module, it suffices to show that the fundamental representations $V_q(\boldsymbol{\omega}_{i,a})$ is an ℓ-weight module. Moreover, it is clear that it suffices to do this in the case that $\mathfrak{g} = \mathfrak{sl}_2$. In that case, as observed earlier, we have quantum analogues of evaluation maps. In particular, $V_q(\boldsymbol{\omega}_{i,a})$ is a simple evaluation module. We will be able to describe precisely the qcharatcer of all simple evaluation modules. For doing that, it will be convenient to introduce ℓ-analogues of simple roots. We take the chance and introduce them for any $\mathfrak{g}$ before restricting our attention the $\mathfrak{sl}_2$ case.

Definition 2.3.6. Given $i \in I, a \in \mathbb{F}^\times, r \in \mathbb{Z}_{\geq 0}$, let

$$\boldsymbol{\omega}_{i,a,r} = \prod_{s=1}^{r} \boldsymbol{\omega}_{i,aq_i^{r+1-2s}} \qquad \text{and} \qquad \boldsymbol{\alpha}_{i,a} = \boldsymbol{\omega}_{i,aq_i,2}\left(\prod_{j\neq i} \boldsymbol{\omega}_{j,aq_i,-c_{ji}}\right)^{-1}.$$

The elements $\boldsymbol{\alpha}_{i,a}$ are called quantum simple ℓ-roots and the subgroup $\mathcal{Q}_q$ of $\mathcal{P}$ generated by them is called the quantum ℓ-root lattice of $\mathfrak{g}$. Denote by $\mathcal{Q}_q^+$ the submonoid generated by the simple ℓ-roots and by $\mathcal{Q}_q^-$ the submonoid by their inverses. Define a partial order on $\mathcal{P}$ by $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$ if $\boldsymbol{\lambda\mu}^{-1} \in \mathcal{Q}_q^+$. ◊

Remark 2.3.7. Observe that the classic simple roots are given in terms of the fundamental weights by the formula: $\alpha_i = 2\omega_i - \sum_{j\neq i}(-c_{ji})\omega_j$. This implies $\mathrm{wt}(\boldsymbol{\alpha}_{i,a}) = \alpha_i$ which gives a partial motivation for the above definition of simple ℓ-roots. A more complete motivation, explaining the choices of the other parameters appearing in the definition is given by the next proposition (cf. (1.1.3)). One can also consider the ℓ-root lattice in the classical context which will be denoted by $\mathcal{Q}$. In that case, $\boldsymbol{\alpha}_{i,a} = \boldsymbol{\omega}_{\alpha_i,a}$. ◊

The proof of the next lemma is straightforward.

Lemma 2.3.8. The ℓ-root lattice is the free abelian group generated by the simple ℓ-roots. □

Proposition 2.3.9. There exists a unique action of the braid group $\mathcal{B}$ of $\mathfrak{g}$ on $\mathcal{P}$ such that

$$(T_i(\boldsymbol{\mu}))_i(u) = \left(\boldsymbol{\mu}_i\left(q_i^2 u\right)\right)^{-1} \qquad \text{and} \qquad (T_i(\boldsymbol{\mu}))_j(u) = \boldsymbol{\mu}_j(u)\prod_{r=0}^{-c_{ij}-1} \boldsymbol{\mu}_i\left(q^{s_i-c_{ij}-1-2r}u\right)$$

for all $\boldsymbol{\mu} \in \mathcal{P}, i, j \in I, i \neq j$. In particular, T_w acts by a group homomorphism for all $w \in \mathcal{W}$,

$$\mathrm{wt}(T_w(\boldsymbol{\mu})) = w(\mathrm{wt}(\boldsymbol{\mu})) \qquad \text{and} \qquad \boldsymbol{\alpha}_{i,a} = \boldsymbol{\omega}_{i,a}(T_i(\boldsymbol{\omega}_{i,a}))^{-1}$$

for all $w \in \mathcal{W}, \boldsymbol{\mu} \in \mathcal{P}, i \in I, a \in \mathbb{F}^\times$.

Proof. The checking of the first statement is straightforward using the defining relations of the braid group. It is clear from the above expression for the action of T_i that $T_i(\boldsymbol{\mu\nu}) = T_i(\boldsymbol{\mu})T_i(\boldsymbol{\nu})$ for all $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{P}$ showing that $\mathcal{B}$ acts by group homomorphisms on $\mathcal{P}$. Thus, it suffices to show that $\mathrm{wt}(T_w(\boldsymbol{\mu})) = w(\mathrm{wt}(\boldsymbol{\mu}))$ when $w = r_i$ for some $i \in I$ and $\boldsymbol{\mu}$ is a fundamental ℓ-weight which is immediately verified. The above expression for $\boldsymbol{\alpha}_{i,a}$ is easily verified. □

Corollary 2.3.10. Let $j \in I$ and $\boldsymbol{\mu} = \prod_{i\in I, a\in\mathbb{F}^\times} \boldsymbol{\omega}_{i,a}^{p_{i,a}} \in \mathscr{P}$ where $p_{i,a} \in \mathbb{Z}$. Then, $T_j(\boldsymbol{\mu}) = \boldsymbol{\mu} \prod_{a\in\mathbb{F}^\times} \boldsymbol{\alpha}_{j,a}^{-p_{j,a}}$. In particular, if $\boldsymbol{\mu}_j(u)$ is a polynomial, $T_j(\boldsymbol{\mu}) \le \boldsymbol{\mu}$.

Proof. Immediate from the fact that the action is by group homomorphism and $T_j(\boldsymbol{\omega}_{i,a}) = \boldsymbol{\omega}_{i,a}\boldsymbol{\alpha}_{i,a}^{-\delta_{ij}}$. □

Let us formally state the quantum analogue of Corollary 2.1.14.

Theorem 2.3.11. Every finite-dimensional (type 1) $U_q(\tilde{\mathfrak{g}})$-module is an ℓ-weight module.

As in the classical case, it suffices to prove Theorem 2.3.11 for the simple modules and, as we have already observed, it suffices to prove it for the fundamental modules for $\mathfrak{g} = \mathfrak{sl}_2$. Before restricting our attention to the $\mathfrak{sl}_2$ case, we record the following corollary.

Corollary 2.3.12. Let $\mathscr{G}_q$ be the Grothendieck ring of the category of finite-dimensional type 1 $U_q(\tilde{\mathfrak{g}})$-modules. The assignment $V \mapsto \mathrm{qch}(V)$ induces a ring homomorphism $\mathrm{qch} : \mathscr{G}_q \to \mathbb{Z}[\mathscr{P}]$. Moreover, $\mathscr{G}_q$ is commutative and is generated by the classes of the fundamental representations $V_q(\boldsymbol{\omega}_{i,a}), i \in I, a \in \mathbb{F}^\times$.

Proof. Let V be a finite-dimensional (type 1) $U_q(\tilde{\mathfrak{g}})$-module. Since V is an ℓ-weight module, the multiplicities $[V : \boldsymbol{\lambda}]$ are completely determined by $\mathrm{qch}(V)$ (Remark 2.3.2). In particular, there exists a unique homomorphism of additive abelian groups $\mathscr{G}_q \to \mathbb{F}[\mathscr{P}]$ such that the class of is mapped to $\mathrm{qch}(V)$. Proposition 2.3.3 then implies that this is a ring homomorphism and that $\mathscr{G}_q$ is commutative. The last statement is immediate from Corollary 2.3.4. □

For the remainder of the subsection, we set $\mathfrak{g} = \mathfrak{sl}_2$ and let i be the unique element of I.

Proposition 2.3.13. Given $a \in \mathbb{F}^\times$, there exists a unique algebra homomorphism $\mathrm{ev}_a : U_q(\tilde{\mathfrak{sl}}_2) \to U_q(\mathfrak{sl}_2)$ such that

$$\mathrm{ev}_a(x_{i,r}^+) = (ak_i)^r x_i^+, \qquad \mathrm{ev}_a(x_{i,r}^-) = a^r x_i^- k_i^r, \qquad \mathrm{ev}_a(k_i) = k_i, \qquad \text{for all} \qquad r \in \mathbb{Z}.$$

Proof. The uniqueness is clear since the elements $x_{i,r}^\pm, k_i, r \in \mathbb{Z}$, generate $U_q(\tilde{\mathfrak{sl}}_2)$. For proving the existence one just needs to check that the defining relations are preserved by the above assignments. It is easier to work with Chevalley-Kac generators for doing that and then deduce the above formulas using the isomorphism of Theorem 1.4.7. It remains to define ev_a on the Chevalley-Kac generators $x_0^\pm$. Thus, set $\mathrm{ev}_a(x_0^\pm) = (qa^{-1})^{\mp 1} x_i^\mp$. The checking of the relations is then straightforward. One then uses that the isomorphism of Theorem 1.4.7 (composed with the projection onto $U_q(\tilde{\mathfrak{sl}}_2)$) maps the elements $x_i^\pm, k_i$ to themselves while $x_0^+ \mapsto x_{i,1}^- k_i^{-1}$ and $x_0^- \mapsto k_i x_{i,-1}^-$. It immediately follows that $\mathrm{ev}_a(x_{i,-1}^+) = (ak_i)^{-1} x_i^+$ and $\mathrm{ev}_a(x_{i,1}^-) = a x_i^- k_i$. Since $[x_i^+, x_{i,1}^-] = h_{i,1}$, we get

$$\mathrm{ev}_a(h_{i,1}) = a(x_i^+ x_i^- k_i - x_i^- k_i x_i^+) = a[x_i^+, x_i^-]_{q^2} k_i \tag{2.3.3}$$

where $[x, y]_p := xy - pyx$ is the p-deformed commutator. Since $x_{i,r+1}^\pm = [h_{i,1}, x_i^\pm]$, an easy induction on $r \ge 0$ using (2.3.3) proves the stated formulas for $\mathrm{ev}_a(x_{i,r}^\pm)$ with $r \ge 0$. For $r \le 0$, one proceeds similarly using that $h_{i,-1} = [x_{i,-1}^+, x_i^-]$. □

To shorten notation, given $m \in \mathbb{Z}_{\geq 0}$ we denote by $V_q(m)$ the simple $U_q(\mathfrak{g})$-module with highest-weight $m\omega_i$. Similarly, we denote by V_m the corresponding weight space of a $U_q(\mathfrak{g})$-module V. Let also $V_q(m,a)$ be the pullback of $V_q(m)$ by ev_a, $a \in \mathbb{F}^\times$.

Theorem 2.3.14. For all $m \in \mathbb{Z}_{\geq 0}, a \in \mathbb{F}^\times$, we have $V_q(m,a) \cong V_q(\boldsymbol{\omega}_{i,a,m})$. Moreover, $V_q(m,a)$ is an ℓ-weight module and

$$\mathrm{qch}(V_q(m,a)) = \boldsymbol{\omega}_{i,a,m} \sum_{r=0}^{m} \left(\prod_{s=0}^{r-1} \boldsymbol{\alpha}_{i,aq^{m-1-2s}} \right)^{-1}.$$

Proof. Let v_m be a highest-weight vector of $V_q(m)$ and $v_m^j = (x_i^-)^{(j)} v_m, 0 \leq j \leq m$. Then, $\{v_m^j : j = 0, \ldots, m\}$ is a basis of $V_q(m)$ and one can easily check that

$$x_i^+ v_m^j = [m+1-j]_q v_m^{j-1} \qquad \text{and} \qquad x_i^- v_m^j = [j+1]_q v_m^{j+1}. \tag{2.3.4}$$

Set $\boldsymbol{\mu}_{m,j} = \boldsymbol{\omega}_{i,a,m} \left(\prod_{s=0}^{j-1} \boldsymbol{\alpha}_{i,aq^{m-1-2s}} \right)^{-1}$. Notice that $\boldsymbol{\mu}_{m,j}(u)$ is the rational function

$$\boldsymbol{\mu}_{m,j}(u) = \frac{(1-aq^{-(m-1)}u)(1-aq^{-(m-1)+2}u)\cdots(1-aq^{m-1-2j}u)}{(1-aq^{m+1-2(j-1)}u)\cdots(1-aq^{m-1}u)(1-aq^{m+1}u)}. \tag{2.3.5}$$

All the statements follow if we show that $v_m^j \in V_{\boldsymbol{\mu}_{m,j}}$ for all $j = 0, \ldots, m$. By (1.4.3), this is equivalent to showing that

$$\Psi_i^+(u)\, v_m^j = q^{m-2j} \frac{\boldsymbol{\mu}_{m,j}(q^{-1}u)}{\boldsymbol{\mu}_{m,j}(qu)} \qquad \text{and} \qquad \Psi_i^-(u)\, v_m^j = q^{-m+2j} \frac{\boldsymbol{\mu}_{m,j}(qu)}{\boldsymbol{\mu}_{m,j}(q^{-1}u)}. \tag{2.3.6}$$

The computations for proving each of these identities are analogous, so we focus on the former. By Proposition 2.3.13 and (2.3.4) we have

$$\begin{aligned} x_{i,r}^+ v_m^j &= (ak_i)^r x_i^+ v_m^j = (ak_i)^r [m+1-j]_q v_m^{j-1} = \left(aq^{m-2(j-1)}\right)^r [m+1-j]_q v_m^{j-1}, \\ x_{i,r}^- v_m^j &= a^r x_i^- k_i^r v_m^j = \left(aq^{m-2j}\right)^r [j+1]_q v_m^{j+1}. \end{aligned} \tag{2.3.7}$$

For $r > 0$, this implies

$$(q-q^{-1})^{-1} \psi_{i,r}^+ v_m^0 = [x_{i,r}^+, x_{i,0}^-] v_m^0 = x_{i,r}^+ x_{i,0}^- v_m^0 = x_{i,r}^+ v_m^1 = (aq^m)^r [m]_q v_m^0.$$

Hence,

$$\psi_{i,r}^+ v_m^0 = (q-q^{-1})(aq^m)^r [m]_q v_m^0. \tag{2.3.8}$$

Notice that under our identification of formal power series with rational functions, the series $\sum_{r\geq 1}(aq^m u)^r$ corresponds to $\frac{aq^m u}{1-aq^m u}$. Therefore,

$$\Psi_i^+(u)\, v_m^0 = \left(k_i + [m]_q (q-q^{-1}) \sum_{r\geq 1} (aq^m)^r \right) v_m^0 = q^m \left(\frac{1-aq^{-m}u}{1-aq^m u} \right) v_m^0,$$

and (2.3.6) with $j = 0$ as well as the first statement of the theorem follow. The general case of (2.3.6) can be proved similarly. Namely, one uses (2.3.7) to obtain the general version of (2.3.8) and proceed as in the case $j = 0$ to get (2.3.6) from the expression (2.3.5) for $\boldsymbol{\mu}_{m,j}(u)$. We leave the details for the reader. □

Remark 2.3.15. It follows from the above formulas that all quantum evaluation modules are ℓ-minuscule showing that there are many more minuscule ℓ-weights than minuscule weights. Notice however that the qcharacter of the evaluation modules are not invariant under the braid group action on $\mathscr{P}$ in general (in fact, that is the case only for the trivial representation). $\Diamond$

Corollary 2.3.16. Let V be a tensor product of simple finite-dimensional $U_q(\tilde{\mathfrak{sl}}_2)$-modules, say $V = V_q(\boldsymbol{\lambda}_1) \otimes \cdots \otimes V_q(\boldsymbol{\lambda}_m)$ for some $m \geq 1, \boldsymbol{\lambda}_j \in \mathscr{P}^+$, and $\boldsymbol{\lambda} = \prod_{j=1}^m \boldsymbol{\lambda}_j$. Then, $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V)$ only if $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$.

Proof. By Proposition 2.3.3, it suffices to consider the case $m = 1$. Write $\boldsymbol{\lambda} = \prod_{j=1}^n \boldsymbol{\omega}_{i,a_j}$. Then, by Corollary 2.3.4, $V_q(\boldsymbol{\lambda})$ is a quotient of $V = V_q(1, a_1) \otimes \cdots \otimes V_q(1, a_n)$. Again, Proposition 2.3.3 implies that it suffices to consider the case $n = 1$. But this follows immediately from the previous theorem. $\square$

Remark 2.3.17. It is natural to expect that the definition of the simple ℓ-roots was made in such a way that the following holds for arbitrary $\mathfrak{g}$: if V is a highest-ℓ-weight module of highest-ℓ-weight $\boldsymbol{\lambda} \in \mathscr{P}^+$ and $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V)$, then $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$. This is true, but the proof (which will be given in Section 4) is amazingly harder than one may expect at this point. Even in the classical case this is not so immediate, although much easier to prove than in the quantum case. The classical version of the previous corollary (for general $\mathfrak{g}$) easily follows from (2.1.15) and Corollary 2.1.10. $\Diamond$

It is natural to expect that, once we have quantum analogues of evaluation modules in the $\mathfrak{sl}_2$ case, that we should be able to describe the simple modules as tensor products of evaluation as in the classical case. It is not difficult to see that, given $\boldsymbol{\lambda} \in \mathscr{P}^+$, there exist unique $m \geq 0, a_j \in \mathbb{F}^\times$, and $r_j > 0, j = 1, \ldots, m$, such that

$$\boldsymbol{\lambda} = \prod_{j=1}^{m} \boldsymbol{\omega}_{i,a_j,r_j} \qquad \text{and} \qquad \frac{a_j}{a_k} \neq q^{\pm(r_j + r_k - 2p)} \qquad \text{for all} \qquad j, k, 0 \leq p < \min\{r_j, r_k\}. \tag{2.3.9}$$

Definition 2.3.18. The factorization (2.3.9) is called the q-factorization of $\boldsymbol{\lambda}$ and the factors $\boldsymbol{\omega}_{i,a_j,r_j}$ are called the q-factors of $\boldsymbol{\lambda}$. An ordered pair $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ of elements of $\mathscr{P}^+$ is said to be in q-resonant order if

$$\frac{a_j}{b_k} \neq q^{-(r_j + s_k - 2p)} \qquad \text{for all} \qquad 0 \leq p < r_j, 1 \leq j \leq m, 1 \leq k \leq n$$

where $\{\boldsymbol{\omega}_{i,a_j,r_j} \mid 1 \leq j \leq m, a_j \in \mathbb{F}^\times, r_j > 0\}$ and $\{\boldsymbol{\omega}_{i,b_j,s_j} \mid 1 \leq j \leq n, b_j \in \mathbb{F}^\times, s_j > 0\}$ are the sets of q-factors of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, respectively. The pair $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is said to be in weak q-resonant order if

$$\frac{a_j}{b_k} \neq q^{-(r_j + s_k - 2p)} \qquad \text{for all} \qquad 0 \leq p < \min\{r_j, s_k\}, 1 \leq j \leq m, 1 \leq k \leq n.$$

The polynomials $\boldsymbol{\lambda}, \boldsymbol{\mu}$ are said to be in general position if both $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ and $(\boldsymbol{\mu}, \boldsymbol{\lambda})$ are in weak q-resonant order. An m-tuple $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_m)$ of elements of $\mathscr{P}^+$ is said to be in (weak) q-resonant order if $(\boldsymbol{\lambda}_r, \boldsymbol{\lambda}_s)$ is in (weak) q-resonant order for all $r < s$. The family $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_m$, is in general position if they are pairwise in general position. $\Diamond$

Remark 2.3.19. The definition of the above concepts for general $\mathfrak{g}$ will be given in Subsection 3.2 using the above definition for the $\mathfrak{sl}_2$-case and the braid group action on $\mathscr{P}$. In the $\mathfrak{sl}_2$-case, it is not difficult to check that $\boldsymbol{\lambda}, \boldsymbol{\mu}$ are in general position if and only if the q-factorization of $\boldsymbol{\lambda}\boldsymbol{\mu}$ is the multiplication of their q-factorizations (and similarly for general families). Notice, if $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is in q-resonant order, it is also in weak q-resonant order. $\Diamond$

Theorem 2.3.20. Let $\lambda \in \mathscr{P}^+$ and $\lambda = \prod_{j=1}^m \omega_{i,a_j,r_j}$ be its q-factorization. Then, $V_q(\lambda) \cong V_q(a_1, r_1) \otimes \cdots \otimes V_q(a_m, r_m)$.

We postpone the proof of this theorem to Subsection 3.2.

Corollary 2.3.21. Let $\lambda, \mu \in \mathscr{P}^+$. Then $V_q(\lambda) \otimes V_q(\mu)$ is irreducible if and only if λ and μ are in general position.

Proof. Let $\lambda = \prod_{j=1}^m \omega_{i,a_j,r_j}$ and $\mu = \prod_{j=1}^n \omega_{i,b_j,s_j}$ be the corresponding q-factorizations. Then, λ, μ are in general position if and only if the family $\omega_{i,a_1,r_1}, \dots, \omega_{i,a_m,r_m}, \omega_{i,b_1,s_1}, \dots, \omega_{i,b_n,s_n}$ is in general position. In that case, it follows from the theorem that

$$V_q(\lambda\mu) \cong \left(V_q(\omega_{i,a_1,r_1}) \otimes \cdots V_q(\omega_{i,a_m,r_m})\right) \otimes \left(V_q(\omega_{i,b_1,s_1}) \otimes \cdots V_q(\omega_{i,b_n,s_n})\right) \cong V_q(\lambda) \otimes V_q(\mu).$$

Conversely, if $V_q(\lambda) \otimes V_q(\mu)$ is irreducible, i.e., $V_q(\lambda) \otimes V_q(\mu) \cong V_q(\lambda\mu)$, it follows from the theorem and Corollary 2.3.5 that $V_q(\lambda)$ is isomorphic to the tensor product of the modules $V_q(\omega_{i,a_j,r_j})$ and $V_q(\omega_{i,b_k,s_k})$ in any order. If λ and μ were not in general position, there would be j, k such that ω_{i,a_j,r_j} and ω_{i,b_k,s_k} are not in general position.

Therefore, it suffices to consider the case $m = n = 1$ and prove that if $\omega_{i,a,r}$ and $\omega_{i,b,s}$ is not in general position, then $V_q(a, r) \otimes V_q(b, s)$ is reducible. One easily checks that the q-factorization of $\lambda = \omega_{i,a,r}\omega_{i,b,s}$ is of the form $\omega_{i,a',r'}\omega_{i,b',s'}$ for some $a', b' \in \mathbb{F}^\times$ and where $\min\{r', s'\} < \min\{r, s\}$. Then, it follows from the theorem that $V_q(\lambda)$ is isomorphic to $V_q(r') \otimes V_q(s')$ when regarded as a $U_q(\mathfrak{g})$-module. On the other hand, $V_q(r, a) \otimes V_q(b, s)$ is isomorphic to $V_q(r) \otimes V_q(s)$ when regarded as a $U_q(\mathfrak{g})$-module. One easily checks that $V_q(r') \otimes V_q(s')$ and $V_q(r) \otimes V_q(s)$ do not have the same character and, hence, cannot be isomorphic. □

Remark 2.3.22. One easily checks that the theorem can be deduced from the corollary, i.e., the statements are equivalent. ◊

Example 2.3.23. Let $a \in \mathbb{F}^\times$ and consider $\lambda = \omega_{i,a}^2$ whose unique q-factor is $\omega_{i,a}$ with multiplicity 2. It follows from Theorem 2.3.20 that $V_q(\lambda) \cong V(1, a) \otimes V(1, a)$ whose qcharacter is

$$\mathrm{qch}(V_q(\omega_{i,a}^2)) = \omega_{i,a}^2 + 2\omega_{i,a}\omega_{i,aq^2}^{-1} + \omega_{i,aq^2}^{-2}.$$

Therefore, contrary to the classic theory of minuscule weights, there are ℓ-minuscule modules having ℓ-weight spaces of dimension higher than one. ◊

Example 2.3.24. We give an example of simple $U_q(\tilde{\mathfrak{sl}}_2)$-module which is not ℓ-minuscule. Let $\lambda = \omega_{i,a}^2\omega_{i,aq^2} = \omega_{i,aq,2}\omega_{i,a}$ for some $a \in \mathbb{F}^\times$. By Theorem 2.3.20, $V_q(\lambda) \cong V(2, aq) \otimes V(1, a)$ and, hence,

$$\mathrm{qch}(V_q(\lambda)) = \omega_{i,a}^2\omega_{i,aq^2} + \omega_{i,a} + \omega_{i,a}^2\omega_{i,aq^4}^{-1} + 2\omega_{i,a}\omega_{i,aq^2}^{-1}\omega_{i,aq^4}^{-1} + (\omega_{i,aq^2}^2\omega_{i,aq^4})^{-1}.$$

◊

Let us now take a look at the quantum counterpart of Example 2.2.12.

Example 2.3.25. Let $a, b \in \mathbb{F}^\times$ and $\lambda \in \mathscr{P}^+$ be the polynomial $\lambda_i(u) = (1 - au)(1 - bu) = \omega_{i,a}\omega_{i,b}$. As in the classical case, we see that $W_q(\lambda)$ is 4-dimensional. Moreover, by Theorem 2.3.20, if $b \neq aq^{\pm 2}$, then $V_q(\lambda) \cong V_q(1, a) \otimes V_q(1, b)$ which is 4-dimensional. It then follows from Corollary 2.3.4 that $W_q(\lambda) \cong V_q(\lambda)$.

We will show in the next section that, if $b \neq aq^2$, then $W_q(\lambda) \cong V_q(1, a) \otimes V_q(1, b)$. In fact, we will show in Subsection 3.2 that the right-hand side is a highest-ℓ-weight module and, hence, a quotient of $W_q(\lambda)$. Since both modules have the dimension 4, the isomorphism follows. Notice that this implies that, for every $a, b \in \mathbb{F}^\times$, either $W_q(\lambda) \cong V_q(1, a) \otimes V_q(1, b)$ or $W_q(\lambda) \cong V_q(1, b) \otimes V_q(1, a)$. In particular,

$$\mathrm{qch}(W_q(\lambda)) = \omega_{i,a}\omega_{i,b} + \omega_{i,aq^2}^{-1}\omega_{i,b} + \omega_{i,a}\omega_{i,bq^2}^{-1} + (\omega_{i,aq^2}\omega_{i,bq^2})^{-1}.$$

Hence, the ℓ-weight spaces of $W_q(\lambda)$ are all one-dimensional unless $a = b$. Moreover, $\mathrm{wt}_\ell(W_q(\lambda))$ has a unique dominant ℓ-weight unless $b = aq^{\pm 2}$. Suppose $b = aq^{-2}$ so that $V_q(\lambda) \cong V_q(2, aq^{-1})$ which is 3-dimensional. Therefore, the kernel of the canonical projection $W_q(\lambda) \to V_q(\lambda)$ is the trivial representation and we have obtained the quantum version of (2.2.10). Noticing that $\alpha_{i,b} = \omega_{i,b}\omega_{i,a} = \lambda$, this sequence can be written in the form $0 \to V_q(\mathbf{1}) \to W_q(\alpha_{i,b}) \to V_q(\alpha_{i,b}) \to 0$. ◊

Remark 2.3.26. One can define the notion of lowest-ℓ-weight modules in a similar manner by exchanging the roles of $U_q(\tilde{\mathfrak{n}}^+)$ and $U_q(\tilde{\mathfrak{n}}^-)$. Evidently, similar results for lowest-ℓ-weight modules can be proved. In particular, every finite-dimensional highest-ℓ-module is also a lowest-ℓ-weight module. Indeed, since the lowest weight is $w_0(\lambda)$ where λ is the highest weight, the lowest-weight space is 1-dimensional and, hence, must be spanned by an eigenvector for the action of $U_q(\tilde{\mathfrak{h}})$. The $U_q(\mathfrak{g})$-submodule generated by the lowest-weight space is irreducible and contains the highest-weight space. Therefore, it generates the whole module. Notice that, in the $\mathfrak{sl}_2$ case, Theorem 2.3.14 implies that the lowest-ℓ-weight of $V_q(\omega_{i,a,m})$ is $\omega_{i,aq^2,m}^{-1} = T_i(\omega_{i,a,m})$. Theorem 2.3.20 together with Proposition 2.3.3 then implies that the lowest-ℓ-weight of $V_q(\lambda)$ is $T_i(\lambda)$. Since $r_i = w_0$ in this case, it follows that $T_i(\lambda)$ is the lowest ℓ-weight of any highest-ℓ-module with highest ℓ-weight λ. We shall compute the the lowest-ℓ-weight of any highest-ℓ-weight module for general $\mathfrak{g}$ later on (Proposition 3.1.2). ◊

2.4. Bibliographical notes

1. Classification of simple modules

The classification of the simple finite-dimensional $\tilde{\mathfrak{g}}$-modules in terms of tensor products of evaluation modules follows from the work of V. Chari and A. Pressley [9, 23]. The classification in terms of Drinfeld polynomials arose only when the quantum case started to be studied and is also due to Chari and Pressley. The $\mathfrak{sl}_2$ case was studied in [24] where Theorem 2.3.20 was also proved. The proof of Proposition 2.3.13 presented here is also from [24] although a more general version for $\mathfrak{g} = \mathfrak{sl}_n$ was previously proved by Jimbo [60]. The classification in terms of Drinfeld polynomials in general follows from the $\mathfrak{sl}_2$ case and was treated in the book [25]. The setting of quantum groups at roots of unity was considered in [27]. The term Drinfeld polynomials comes from a similar result on the classification of finite-dimensional representations of Yangians obtained by Drinfeld. The terminology highest-ℓ-weight module came only later on (for instance, the terminology "pseudo-highest-weight module" is used in the book [25]). The assumption that the ground field $\mathbb{F}$ is algebraically closed is used here to ensure the existence of eigenvalues for the action of the elements $\Lambda_{i,r}$. One can obtain the classification of simple module over a non-algebraically closed field by using Galois groups (cf. [57]).

The classification of simple finite-dimensional $\tilde{\mathfrak{g}}$-modules has been recently extended to the broader context of algebras of the form $\mathfrak{g} \otimes A$ in [14] and to the even broader context of equivariant map algebras (which includes the twisted affine algebras) in [81]. Since the choice of an evaluation parameter is equivalent to the choice of a maximal ideal of $\mathbb{F}[t, t^{-1}]$, it is not surprising that the classification is given in terms of a generalization of evaluation modules obtained by choosing maximal ideals of A. The case of quantum groups at roots of unity were treated in [27] (Lusztig's version) and [4] (Kac-DeConcini's version). It is interesting to note that one can define an algebra $A_q(\mathfrak{g})$ given by generators and relations as in Theorem 1.4.7 starting from any symmetrizable Kac-Moody algebra $\mathfrak{g}$. These algebras are called Drinfeld affinizations and, in the special case that $\mathfrak{g}$ is an affine Kac-Moody algebra, they are also known as quantum toroidal algebras (since their $q \to 1$ limit is a toroidal algebra over a simple finite-dimensional Lie algebra). One can then study the category of representations of $A_q(\mathfrak{g})$ which lie in the category $\mathscr{O}_q^{\rm int}$ associated to the Kac-Moody algebra $\mathfrak{g}$. The classification of the simple modules is also given in terms of Drinfeld polynomials and was obtained in [46].

2. Weyl modules

The above notions of Weyl modules were introduced by Chari and Pressley in [28]. The global Weyl modules were originally called maximal integrable modules and the local Weyl modules were simply called Weyl modules. The present terminology of local and global Weyl modules were introduced by B. Feigin and S. Loktev in [33]. The choice of the terminology Weyl modules comes from the finite-dimensional representation theory of algebraic groups in positive characteristic where the Weyl modules are the universal highest-weight modules while, in our context, the local Weyl modules are the universal finite-dimensional highest-ℓ-weight modules. The original motivation for the term comes from another perspective since, at the time, the language of ℓ-weight modules was not used. Namely, in the context of algebraic groups in characteristic p, Weyl's approach to construct what are nowadays named Weyl modules, was by a process of reduction modulo p of the modules $V(\lambda)$. Chari and Pressley conjectured something similar for the local Weyl modules. Namely, assume $\mathbb{F}$ is the algebraic closure of $\mathbb{K}(q)$ for some field $\mathbb{K}$ such that q is transcendent over $\mathbb{K}$, i.e., $\mathbb{K}(q)$ is the field of rational functions in one variable q. Then, the conjecture was that, if $\boldsymbol{\lambda} \in \mathscr{P}^+$ is such that that $\boldsymbol{\lambda}_i(u) \in \mathbb{K}[u]$ for all $i \in I$, then the Weyl module $W(\boldsymbol{\lambda})$ could be obtained as the $q \to 1$ limit of the irreducible module $V_q(\boldsymbol{\lambda})$ (which is isomorphic to $W_q(\boldsymbol{\lambda})$ in this case). This conjecture was proved still in [28] in the $\mathfrak{sl}_2$ case. For $\mathfrak{g} = \mathfrak{sl}_n$ it was proved by Chari and Loktev in [17] by finding a PBW-like basis of $W(\boldsymbol{\lambda})$ while for any simply laced $\mathfrak{g}$ it was proved by G. Fourier and P. Littelmann in [35] by establishing a relation with the theory of Demazure modules. Both of these proofs establish relations with the notion of fusion products defined by Feigin and Loktev in [32]. This relation with Demazure modules was very recently generalized to an arbitrary simple Lie algebra $\mathfrak{g}$ by W. Naoi in [76] with the help of the theory of crystals, thus finalizing the proof of the conjecture. It is interesting to remark that it is very easy to see that the $q \to 1$ limit of $V_q(\boldsymbol{\lambda})$ is a quotient of $W(\boldsymbol{\lambda})$. Thus, in order to prove the conjecture one "just" needs to prove that $\dim(W(\boldsymbol{\lambda})) \le \dim(V_q(\boldsymbol{\lambda}))$. The proof of the conjecture actually implies that the character of the Weyl modules depends only on $\mathrm{wt}(\boldsymbol{\lambda})$ (and not on the choice of q or $\mathbb{F}$). More precisely, we have the following corollary which will be used below.

Corollary 2.4.1. Let $\boldsymbol{\lambda} \in \mathscr{P}^+$ and $\lambda = \mathrm{wt}(\boldsymbol{\lambda})$. Write $\lambda = \sum_{i \in I} m_i \omega_i$ and let $c_i = \mathrm{ch}(W_q(\boldsymbol{\omega}_{i,1}))$. Then, $\mathrm{ch}(W_q(\boldsymbol{\lambda})) = \mathrm{ch}(W(\boldsymbol{\lambda})) = \prod_{i \in I} c_i^{m_i}$. □

There is also a proof based on the theory of global crystal basis of M. Kashiwara which has been suggested by H. Nakajima. Namely, the global Weyl modules are isomorphic to the level zero extremal-weight modules in the sense of Kashiwara [62, 63] which have global crystal bases. The argument essentially uses the fact that these bases are "well behaved" with respect to the defining relations of the global Weyl modules and, therefore, the $q \to 1$ limit of a quantum global Weyl module is a classical global Weyl module. Moreover, the results of J. Beck and H. Nakajima [5] imply that the global Weyl module $W_q(\lambda)$ is free of finite-rank as a right-module for the algebra A_λ mentioned in Proposition 2.2.8. This was previously proved in the $\mathfrak{sl}_2$ case still in [28] and for any simply laced $\mathfrak{g}$ by Nakajima in [72] using the geometry of quiver varieties. A similar result in the context of the algebras $\mathfrak{g} \otimes A$ was proved in [14]. This structure is compatible with the crystal theory so that the $q \to 1$ limit of the global Weyl $W_q(\lambda)$ remains free of the same rank as a right-module for A_λ (whose classical limit is itself!). It is clear from the way that the local Weyl modules are obtained from the global ones that the dimension of a local Weyl module is exactly the rank of the global Weyl module as a right A_λ-module, thus proving the conjecture. We plan to survey this argument in a more complete manner elsewhere. The advantage of this method over the aforementioned ones is that one can consider general limits $q \to \xi$ with $\xi \in \mathbb{F}^\times$, which allows one to go to the root of unity setting (in Lusztig's sense). Moreover, it also allows to perform change of base field and, therefore, one can have similar results in positive characteristic. In particular, this proves a similar conjecture we made in [55] in the context of hyper loop algebras (the study of these algebras essentially corresponds to that of algebraic loop groups). The theory of Weyl modules has just been expanded to the setting of equivariant map algebras [34].

3. Character theory

The study of ℓ-weight modules essentially begun in the paper [38] where E. Frenkel and N. Reshetikhin introduced the notion of qcharacters, although the terminology ℓ-weight spaces and so on was not used. The original definition of qcharacters was rather more complicated than the one we gave above. It was motivated by the study of deformed $\mathcal{W}$-algebras and used the concept of transfer matrices which involves the R-matrix of $U_q(\hat{\mathfrak{g}})$. Because of the quantum nature of the original definition, the name q-character was chosen. It was proved in [36, Proposition 2.4] that the definition given above coincides with the definition of [38]. Although the above definition does not sound quantum in nature nor does it acknowledge what the value of q is, for historical reasons the terminology qcharacter has been kept (and the prefix q may not be the name of the quantization parameter). There is also two different notations for the elements $\boldsymbol{\omega}_{i,a}$ and $\boldsymbol{\alpha}_{i,a}$ in use in the literature. Namely, the elements $\boldsymbol{\alpha}_{i,a}$ are denoted by A_{i,aq_i} in [38] and several other papers which follow the notation developed there. Also, the elements $\boldsymbol{\omega}_{i,a}$ are denoted there by $Y_{i,a}$ and the elements of $\mathcal{P}$ are referred to as monomials (of the qcharacters) instead of ℓ-weights. In fact, the systematic use of the terminology ℓ-weight lattice and ℓ-root lattice was initiated in [19], where the notations $\boldsymbol{\alpha}_{i,a}$ and $\boldsymbol{\omega}_{i,a}$ were introduced and the definition of $\boldsymbol{\alpha}_{i,a}$ using the braid group action on $\mathcal{P}$ as in Proposition 2.3.9 was given (the first statement of that proposition appeared previously in [7, 12] in a different form). We find the notation $\boldsymbol{\omega}_{i,a}$ and $\boldsymbol{\alpha}_{i,a}$ more suggestive, making the parallel with the classic theory of weights and roots much more evident.

The quest for understanding the qcharacters of the simple modules remains ongoing. The first attempt to obtain a general procedure for calculating them was given in [36] and is now known as the Frenkel-Mukhin algorithm which we shall present in Subsection 4.1. Therefore, we leave further historical comments related to qcharacters to Subsection 4.4. We remark that the terminology minuscule for ℓ-weights started to be used systematically more recently and that the terminology "special module" was in practice for sometime in place of ℓ-minuscule module (frequently one simply says "minuscule module" meaning ℓ-minuscule). We shall make some remarks about the terminologies "general position" and q-resonant order in Subsection 3.4. The terminology q-factorization is also recent and appeared for the first time in [15]. As mentioned in Remark 2.3.2, the theory of qcharacters is trivial in the classical setting. However, if the ground field is not algebraically closed, the story is a little more interesting and is nicely described using Galois groups. This was done in a joint work with D. Jakelić [56] in the broader context of hyper loop algebras. The same ideas of [56] can be used in the quantum setting as well. In the root of unity setting, the theory of qcharacters was studied in [37, 45, 58]. The classic characters of finite-dimensional representations of the simple Lie algebra $\mathfrak{g}$ (or of $U_q(\mathfrak{g})$) also give rise to a ring homomorphism from the associated Grothendieck ring to the integral group ring $\mathbb{Z}[P]$. The image of the classic character homomorphism is the subring of $\mathbb{Z}[P]$ of elements left invariant by the Weyl group action. As we have seen, the qcharacters of the simple modules are not invariant under the action of the braid group on $\mathscr{P}$. The image of the qcharacter ring homomorphism was described in [36] as the intersection of the kernels of some operators introduced in [38] called screening operators.

3. Tensor Products of Simple Modules

The main goal of this subsection is to study a few results concerning the tensor structure of the category of finite-dimensional modules for the quantum affine algebra $U_q(\tilde{\mathfrak{g}})$. The main problem we address here is the one of finding a sufficient condition for a tensor product of simple modules to be highest-ℓ-weight. It will be necessary to have some knowledge about the dual module of a simple module and, thus, we begin by studying duality. As an application of the results on tensor products, we finish the section presenting the block decomposition of the category.

3.1. Duality

Recall that if V is a $\tilde{\mathfrak{g}}$-module, then V^* is turned into a $\tilde{\mathfrak{g}}$-module by means of (1.3.9) and similarly in the quantum setting. In this subsection we study some results about dual representations which will be useful tools in the study of other topics such as tensor products and extensions. The quantum setting is rather more complicated since it is also not known a precise formula for the antipode on the loop-like generators. In fact, we shall see in Remark 3.1.8 below that $(V^*)^*$ is not usually isomorphic to V for a finite-dimensional $U_q(\tilde{\mathfrak{g}})$-module V. This is a very different scenario than that of the finite-dimensional representation theory of $U_q(\mathfrak{g})$. It is also different than the classical context because, since $S^2 = 1$, the double dual of any finite-dimensional $\tilde{\mathfrak{g}}$-module is isomorphic to the original module.

The loop analogue of Proposition 1.3.13 for finite-dimensional representations is easily established both in the classical and quantum settings. One of our goals is to establish the analogues of Corollary 1.3.15. We start with the classical case which is simpler. In particular, since $S(h_{i,r}) = -h_{i,r}$ for all $i \in I, r \in \mathbb{Z}$, we have

(3.1.1) $$S(\Lambda_i^{\pm}(u)) = (\Lambda_i^{\pm}(u))^{-1} \qquad \text{for all} \qquad i \in I,$$

where the inverse is that of formal power series with coefficients in $U(\tilde{\mathfrak{h}})$. This immediately implies the classical loop analogue of (1.3.10):

(3.1.2) $$(V_{\boldsymbol{\mu}})^* = (V^*)_{\boldsymbol{\mu}^{-1}} \qquad \text{for all} \qquad \boldsymbol{\mu} \in \mathscr{P}.$$

Let $\lambda \in P^+, a \in \mathbb{F}^\times$, and $V = V(\lambda, a) = V(\boldsymbol{\omega}_{\lambda,a})$. By (2.1.15), for all $w \in \mathscr{W}$, $V_{w(\lambda)} = V_{\boldsymbol{\omega}_{w(\lambda),a}}$ is spanned by the vector v_w for some choice of highest-weight vector v of $V(\lambda)$. In particular, $V(\lambda, a)^*$ is a lowest-ℓ-weight module of lowest ℓ-weight $\boldsymbol{\omega}_{w_0(\lambda),a}$. One easily checks that there exists a unique action of the Weyl group on $\mathscr{P}$ by group homomorphisms such that

(3.1.3) $$w(\boldsymbol{\omega}_{\mu,a}) = \boldsymbol{\omega}_{w(\mu),a} \qquad \text{for all} \qquad w \in \mathscr{W}, \mu \in P, a \in \mathbb{F}^\times.$$

Given $\boldsymbol{\lambda} = \prod_{j=1}^m \boldsymbol{\omega}_{\lambda_j,a_j} \in \mathscr{P}^+$, it follows from Corollary 2.1.10 and Proposition 2.3.3 that $V(\boldsymbol{\lambda})$ is a lowest-ℓ-weight module of lowest ℓ-weight $w_0(\boldsymbol{\lambda})$. More generally, if V is a highest-ℓ-weight module of highest ℓ-weight $\boldsymbol{\lambda}$, then

(3.1.4) $$V_{w(\mathrm{wt}(\boldsymbol{\lambda}))} = V_{w(\boldsymbol{\lambda})} \qquad \text{for all} \qquad w \in \mathscr{W}.$$

We are ready to establish the loop analogue of Corollary 1.3.15 in the classical context.

Proposition 3.1.1. Let $\boldsymbol{\lambda} \in \mathscr{P}^+$. Then, $V(\boldsymbol{\lambda})^* \cong V(w_0(\boldsymbol{\lambda})^{-1})$.

Proof. Since $V(\boldsymbol{\lambda})^*$ is irreducible, we must have $V(\boldsymbol{\lambda})^* \cong V(\boldsymbol{\mu})$ for some $\boldsymbol{\mu} \in \mathscr{P}^+$. To shorten notation, write $V = V(\boldsymbol{\lambda}), \lambda = \mathrm{wt}(\boldsymbol{\lambda})$, and $\mu = \mathrm{wt}(\boldsymbol{\mu})$. Let $v \in V_{w_0(\lambda)}$ and $f \in (V^*)_{-w_0(\lambda)}$ be nonzero. Since $(V^*)_{-\nu} = (V_\nu)^*$ for all $\nu \in P$ by (1.3.10) and $\dim(V_{w_0(\lambda)}) = \dim((V^*)_{-w_0(\lambda)}) = 1$, we must have $f(v) \neq 0$. Also,$V_{w_0(\lambda)} = V_{w_0(\boldsymbol{\lambda})}$ by (3.1.4). Moreover, since $-w_0(\lambda)$ is the unique maximal weight of V^*, we must have $(V^*)_{-w_0(\lambda)} = (V^*)_{\boldsymbol{\mu}}$. Therefore, both v and f are ℓ-weight vectors with ℓ-weights $w_0(\boldsymbol{\lambda})$ and $\boldsymbol{\mu}$, respectively. Hence, $f \otimes v$ is an ℓ-weight vector of ℓ-weight $\boldsymbol{\mu}(w_0(\boldsymbol{\lambda}))$ by (2.1.14). Therefore, we are left to show that $\boldsymbol{\mu}(w_0(\boldsymbol{\lambda})) = \mathbf{1}$, i.e., that $f \otimes v$ is an ℓ-weight vector of ℓ-weight $\mathbf{1}$.

For doing that, notice that the canonical linear map $V^* \otimes V \to \mathbb{F}, f \otimes v \mapsto f(v)$, is a homomorphism of representations (this is true for any Hopf algebra, but in general we cannot state the same for the map $V \otimes V^* \to \mathbb{F}$). But $\mathbb{F}$ is the trivial representation, i.e., $\mathbb{F} \cong V_q(\mathbf{1})$. Therefore, since $f(v) \neq 0$, $f(v)$ is an ℓ-weight vector of ℓ-weight $\mathbf{1}$ and, hence, the same must hold for $f \otimes v$. □

We now turn to the quantum case. Since the precise formula for $S(\Lambda_{i,r})$ is not known in this case, we do not have a quantum analogue of (3.1.2). However, it is natural to expect that the quantum analogue of (3.1.4) should obtained by replacing the Weyl group action on $\mathscr{P}$ defined in (3.1.3) by the braid group action on $\mathscr{P}$ as defined in Proposition 2.3.9. This is what we prove next.

Proposition 3.1.2. Let V be a highest-ℓ-weight module of highest ℓ-weight $\boldsymbol{\lambda} \in \mathscr{P}^+$. Then, $V_{T_w(\boldsymbol{\lambda})} = V_{w(\mathrm{wt}(\boldsymbol{\lambda}))}$ for all $w \in \mathscr{W}$. In particular, the lowest ℓ-weight of V is $T_{w_0}(\boldsymbol{\lambda})$.

Proof. The second statement is immediate from the first (cf. Remark 2.3.26).

Let $\lambda = \mathrm{wt}(\boldsymbol{\lambda})$, $v \in V_\lambda \setminus \{0\}$, and $v_w \in V_{w\lambda} \setminus \{0\}$ be defined as in (1.3.4). Then, since $V_{w(\lambda)} = \mathbb{F}v_w$, v_w is necessarily an ℓ-weight vector (say $v_w \in V_{\boldsymbol{\mu}}$) which is an eigenvector for the action of $U_q(\tilde{\mathfrak{h}})$. We need to show that $\boldsymbol{\mu} = T_w(\boldsymbol{\lambda})$, i.e., that $\Lambda_i^+(u)v_w = (T_w(\boldsymbol{\lambda}))_i(u)v_w$ for all $i \in I$. This will be proved by induction on $\ell(w) \geq 0$ which clearly starts if $\ell(w) = 0$.

Let $w \in \mathscr{W}$ and $i \in I$ be such that $l(r_iw) = l(w) + 1$. By induction hypothesis, assume that $v_w \in V_{T_w(\boldsymbol{\lambda})}$. We need to show that $v_{r_iw} \in V_{T_iT_w(\boldsymbol{\lambda})}$. Notice that Proposition 1.1.21(iii) implies $x_{i,r}^+ v_w = 0$ for all $r \in \mathbb{Z}$, i.e., v_w is a highest-ℓ-weight vector for the subalgebra $U_q(\tilde{\mathfrak{g}}_i) \cong U_q(\tilde{\mathfrak{sl}}_2)$. In particular, since by induction hypothesis $\boldsymbol{\mu} = T_w(\boldsymbol{\lambda})$, it follows that

$$(T_w(\boldsymbol{\lambda}))_i(u) \in \mathbb{F}[u] \tag{3.1.5}$$

and, moreover, $v_{r_iw} = (x_i^-)^m v_w$ where $m = w(\lambda)(h_i)$. Thus, we are left to show that

$$\Lambda_j^+(u)(x_i^-)^m v_w = (T_iT_w(\boldsymbol{\lambda}))_j(u)(x_i^-)^m v_w \qquad \text{for all} \qquad j \in I. \tag{3.1.6}$$

Notice that, for $j = i$, this follows from Remark 2.3.26.

Recall that $[h_{j,r}, x_{j'}^-] = -\frac{[rc_{jj'}]_{q_j}}{r} x_{j',r}^-$ for all $j, j' \in I, r \in \mathbb{Z}$. In particular,

$$[h_{j,r}, x_i^-] = \frac{[rc_{ji}]_{q_j}}{[2r]_{q_i}}[h_{i,r}, x_i^-] \qquad \text{for all} \qquad j \in I, r \in \mathbb{Z}. \tag{3.1.7}$$

Let $\mu_{j,r}$ be the eigenvalue of the action of $h_{i,r}$ on v_w. The, by definition of $\Lambda_{i,r}$ we have

$$\boldsymbol{\mu}_j(u) = \exp\left(-\sum_{s>0}\frac{\mu_{j,s}}{[s]_{q_j}}u^s\right).$$

Using (3.1.7) we get:

$$\begin{aligned} h_{j,s}v_{r_iw} &= h_{j,s}(x_i^-)^m v_w = (x_i^-)^m(\mu_{j,s}v_w) + \frac{[sc_{ji}]_{q_j}}{[2s]_{q_i}}[h_{i,s},(x_i^-)^m]v_w \\ &= \mu_{i,s}v_{r_iw} + \frac{[sc_{ji}]_{q_j}}{[2s]_{q_i}}(h_{i,s}(x_i^-)^m v_w - (x_i^-)^m(\mu_{i,s}v_w)) \\ &= \left(\mu_{j,s} - \frac{[sc_{ji}]_{q_j}}{[2s]_{q_i}}\mu_{i,s} + \frac{[sc_{ji}]_{q_{i_j}}}{[2s]_{q_i}}h_{i,s}\right)v_{r_iw}. \end{aligned}$$

By (3.1.6) with $j = i$, we know that $h_{i,s}v_{r_iw} = -\mu_{i,s}q_i^{2s}v_{r_iw}$. Plugging this above, we get

$$h_{j,s}v_{r_iw} = \left(\mu_{j,s} - [sc_{ji}]_{q_i}\mu_{i,s}(1+q_i^{-2s})\right)v_{r_iw}.$$

Setting $\tilde{\mu}_{j,s} := \mu_{j,s} - [sc_{ji}]_{q_i}\mu_{i,s}(1+q_i^{-2s})$, it follows that

$$\Lambda_j^+(u)v_{r_iw} = \exp\left(-\sum_{s>0}\frac{\tilde{\mu}_{j,s}}{[s]_{q_j}}u^s\right)v_{r_iw}. \tag{3.1.8}$$

A simple comparison of (3.1.8) with the definition of $(T_i(\boldsymbol{\mu}))_j(u)$ completes the proof of (3.1.6). □

Corollary 3.1.3. Let $\boldsymbol{\lambda} \in \mathscr{P}^+$ and $w \in \mathscr{W}$. Then $T_w(\boldsymbol{\lambda}) \le \boldsymbol{\lambda}$.

Proof. Immediate from Corollary 2.3.10 and (3.1.5). □

The following analogue of Corollary 1.3.8 is now immediate.

Corollary 3.1.4. Suppose $\boldsymbol{\lambda} \in \mathscr{P}^+$ is such that $\lambda = \mathrm{wt}(\boldsymbol{\lambda})$ is minuscule. Then $\mathrm{qch}(V_q(\boldsymbol{\lambda})) = \sum_{w\in\mathscr{W}_\lambda} T_w(\boldsymbol{\lambda})$. In particular, if $V_q(\boldsymbol{\lambda})_{\boldsymbol{\mu}} \neq \{0\}$, $\boldsymbol{\mu} \le \boldsymbol{\lambda}$. □

Remark 3.1.5. The classical analogue of the above corollary also follows easily by replacing the braid group action by the Weyl group action on $\mathscr{P}$. We have already remarked that the qcharacter of an ℓ-minuscule module is not invariant under the braid group action on $\mathscr{P}$ in general. We shall see later that the Frenkel-Mukhin algorithm can be used for computing the qcharacter of ℓ-minuscule modules. ◊

Example 3.1.6. Let $\mathfrak{g} = \mathfrak{sl}_n$, $i \in I$, and $a \in \mathbb{F}^\times$. Since every fundamental weight of $\mathfrak{sl}_n$ is minuscule, we can use Corollary 3.1.4 to express the qcharacter of $V_q(\boldsymbol{\omega}_{i,a})$. In particular, if $V_q(\boldsymbol{\omega}_{i,a})_{\boldsymbol{\mu}} \neq \{0\}$, $\boldsymbol{\mu} \le \boldsymbol{\lambda}$. For $n = 3$, we have $\mathscr{W}_{\omega_i} = \{r_i, r_jr_i\}$ where $i \neq j$. Then,

$$\mathrm{qch}(V_q(\boldsymbol{\omega}_{i,a})) = \boldsymbol{\omega}_{i,a} + \boldsymbol{\omega}_{i,a}\boldsymbol{\alpha}_{i,a}^{-1} + \boldsymbol{\omega}_{i,a}\boldsymbol{\alpha}_{i,a}^{-1}\boldsymbol{\alpha}_{j,aq}^{-1} = \boldsymbol{\omega}_{i,a} + \boldsymbol{\omega}_{i,aq^2}^{-1}\boldsymbol{\omega}_{j,aq} + \boldsymbol{\omega}_{j,aq^3}^{-1}.$$

◊

For notational convenience, given $\lambda \in \mathscr{P}^+$, set

$$\lambda^* = T_{w_0}(\lambda)^{-1}. \tag{3.1.9}$$

The proof of the next Corollary is similar to that of Proposition 3.1.1. However, one needs to take the same extra care taken in the proof of Proposition 2.3.3 since the tensor product of two ℓ-weight vectors may not be an ℓ-weight vector in the quantum case.

Corollary 3.1.7. Let $\lambda \in \mathscr{P}^+$. Then, $V_q(\lambda)^* \cong V_q(\lambda^*)$. □

Remark 3.1.8. Recall in Remark 1.1.22 that w_0 defines an involution on I. A straightforward but tedious computation working with the known reduced expressions for w_0 can be used to show that

$$(\lambda^*)_i(u) = \lambda_{w_0(i)}\left(q^{r^\vee h^\vee} u\right).$$

It follows that $(V_q(\lambda)^*)^* \cong V_q(\mathrm{sh}_{q^{2r^\vee h^\vee}}(\lambda))$, where sh_a, $a \in \mathbb{F}^\times$, is the group automorphism of $\mathscr{P}$ induced from the ring automorphism of $\mathbb{F}[u]$ given by the shift $u \mapsto au$. In the classical case, we could have set the notation $\lambda^* = (w_0(\lambda))^{-1}$. This is in fact consistent with the quantum notation since, by setting $q = 1$ in the definition of the braid group action on $\mathscr{P}$, we recover (3.1.3).

One can also use the formulas in Theorem 2.3.14 together with Corollary 3.1.7 to see that (3.1.2) is indeed false in the quantum setting. Theorem 2.3.14 can also be used to see that the qcharacter of the finite-dimensional $U_q(\tilde{\mathfrak{g}})$-modules are not invariant under the braid group action in general. In other words, we have no quantum loop analogue of Proposition 1.3.7. In the classical context, such a loop analogue under the Weyl group action defined by (3.1.3) is easily deduced from (2.1.14), (2.1.15), Corollary 2.1.10, and (3.1.2). ◊

Example 3.1.9. Let us return to Example 2.3.25. Recall that if $\lambda = \omega_{i,a}\omega_{i,b}$ and $b \neq aq^2$, then $W_q(\lambda) \cong V_q(1,a) \otimes V_q(1,b)$. Moreover, if $b = aq^{-2}$, we have a short exact sequence $0 \to V_q(\mathbf{1}) \to W_q(\lambda) \to V_q(\lambda) \to 0$. Therefore, we also have a short exact sequence $0 \to V_q(\lambda^*) \to (W_q(\lambda))^* \to V_q(\mathbf{1}) \to 0$. Using Remark 1.3.19 and that $\boldsymbol{\mu}^* = \mathrm{sh}_{q^2}(\boldsymbol{\mu})$ for all $\boldsymbol{\mu} \in \mathscr{P}^+$, the latter exact sequence can be rewritten as

$$0 \to V_q(\mathrm{sh}_{q^2}(\lambda)) \to V_q(1,a) \otimes V_q(1,q^2a) \to V_q(\mathbf{1}) \to 0.$$

In particular, $V_q(1,a) \otimes V_q(1,q^2a)$ is not a highest-ℓ-weight module while $V_q(1,q^2a) \otimes V_q(1,a)$ is isomorphic to $W_q(\mathrm{sh}_{q^2}(\lambda))$. This shows that, in our category, it may happen that $V \otimes W$ is not isomorphic to $W \otimes V$. In other words, the category is not braided. ◊

We end this subsection recording the following lemma to be used in the next subsection.

Lemma 3.1.10. Suppose V and V^* are finite-dimensional highest-ℓ-weight modules. Then, V is irreducible.

Proof. Let W be the irreducible quotient of V and consider the associated exact sequences $0 \to U \to V \to W \to 0$ and $0 \to W^* \to V^* \to U^*$. Then, W^* is irreducible and contains the lowest-weight space of V^*. Since the characters are $\mathscr{W}$-invariant, it also contains the highest-weight space. Since V^* is generated by its highest-weight space by hypothesis, it follows that $W^* = V^*$ showing that V^* (and therefore also V) is irreducible. □

3.2. Highest-ℓ-weight tensor products of simple modules

The main goal of this subsection is to establish a sufficient condition for a tensor product of simple $U_q(\tilde{\mathfrak{g}})$-modules to be a highest-ℓ-weight module. Along the way we prove Theorem 2.3.20.

Remark 3.2.1. Notice that, in the classical setting, such condition is easily obtained from Theorem 2.1.3 and Remark 2.1.4. In fact, it follows that in the classical setting a tensor product of simple modules is completely reducible. Theorem 3.2.5 below shows that the picture is very different in the quantum setting where we have plenty examples of tensor products of simple modules which are a indecomposable but reducible. $\diamondsuit$

We begin with the following lemma (whose proof is easily transported to the classical setting).

Lemma 3.2.2. Let $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathscr{P}^+$, $\lambda = \mathrm{wt}(\boldsymbol{\lambda})$, and $\lambda' = \mathrm{wt}(\boldsymbol{\lambda}')$. Suppose V and V' are highest-ℓ-weight modules with highest ℓ-weights $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$, respectively. Let also $v \in V_\lambda \setminus \{0\}$ and $v' \in V'_{\lambda'} \setminus \{0\}$. Then, $V \otimes V'$ is generated by $(v_{w_0} \otimes v')$. In particular, if $(v_{w_0} \otimes v')$ is in the submodule of $V \otimes V'$ generated by $v \otimes v'$, then $V \otimes V'$ is a highest-ℓ-weight module.

Proof. Let W be the submodule of $V \otimes V'$ generated by $v_{w_0} \otimes v'$. Since $w_0(\lambda)$ is the lowest weight of V, $x^-_{i,r} v_{w_0} = 0$ for all $i \in I, r \in \mathbb{Z}$. Then, by Proposition 1.4.10, we have

$$x^-_{i,r}(v_{w_0} \otimes v') = v_{w_0} \otimes (x^-_{i,r} v').$$

Since $V' = U_q(\tilde{\mathfrak{n}}^-)v'$, it follows that $v_{w_0} \otimes V' \subseteq W$. Next, we will show that $v \otimes V' \subseteq W$. Recall that if $w_0 = r_{i_l} \dots r_{i_1}$ is a reduced expression for w_0 and m_j, $j = 1, \dots, l$, are given by (1.3.3), then v is a nonzero multiple of $(x^+_{i_1})^{m_1} \dots (x^+_{i_l})^{m_l} v_{w_0}$. Therefore, it suffices to show that

$$(x^+_{i_m} \cdots x^+_{i_1} v_{w_0}) \otimes V' \subseteq W \qquad \text{for all} \qquad m \geq 0, i_j \in I, j = 1, \dots, m.$$

We prove this by induction on $m \geq 0$ which clearly starts at $m = 0$. Assume $m \geq 0$, let $u = x^+_{i_m} \dots x^+_{i_1} v_{w_0}$ and assume, by induction hypothesis, that $u \otimes V' \subseteq W$. Then, by Proposition 1.2.4,

$$x^+_i(u \otimes u') = (x^+_i u) \otimes u' + (k_i u) \otimes (x^+_i u') \qquad \text{for all} \qquad i \in I, u' \in V'.$$

Notice $k_i u = q^a u$ for some $a \in \mathbb{Z}$ and, hence, by induction hypothesis, $(k_i u) \otimes (x^+_i u') \in W$. Since W is a submodule of V and $u \otimes u' \in W$ by induction hypothesis, we have $x^+_i(u \otimes u') \in W$ and, therefore, $(x^+_i u) \otimes u' \in W$ for all $i \in I, u' \in V'$. This proves the inductive step.

Finally, it suffices to show that

$$(x^{\epsilon_m}_{i_m} \dots x^{\epsilon_1}_{i_1} v) \otimes V \subseteq W \qquad \text{for all} \qquad m \geq 0, i_j \in \hat{I}, \epsilon_j \in \{+,-\}, j = 1, \dots, m.$$

This is done exactly as in the previous step. $\square$

We now give the general definition of q-resonant ordering.

Definition 3.2.3. Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^+$. The ordered pair $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is said to be in (weak) q-resonant order if there exists a reduced expression $w_0 = r_{i_l} \cdots r_{i_1}$ for w_0 such that $\left(\left(T_{i_{j-1}} \cdots T_{i_1}(\boldsymbol{\lambda})\right)_{i_j}(u), \boldsymbol{\mu}_{i_j}(u)\right)$ is in (weak) q_{i_j}-resonant order in the sense of Definition 2.3.18 for all $j = 1, \dots, l$. An m-tuple $(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_m)$ of Drinfeld polynomials is said to be in (weak) q-resonant order if $(\boldsymbol{\lambda}_r, \boldsymbol{\lambda}_s)$ is in (weak) q-resonant order for all $1 \leq r < s \leq m$. $\diamondsuit$

Remark 3.2.4. Since q is not a root of unity, it is not difficult to show that, if $(\boldsymbol{\lambda},\boldsymbol{\mu})$ is in q-resonant order, then $(\boldsymbol{\lambda}_i(u),\boldsymbol{\mu}_i(u))$ is in (weak) q_i-resonant order in the sense of Definition 2.3.18 for all $i\in I$. Notice that (3.1.5) implies that $\left(T_{i_{j-1}}\cdots T_{i_1}(\boldsymbol{\lambda})\right)_{i_j}(u)$ is indeed a polynomial and, hence, the above definition makes sense. ◊

The main goal of this subsection is to prove the following theorem.

Theorem 3.2.5. Let $m>0$ and $\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_m\in\mathscr{P}^+$. If $(\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_m)$ is in q-resonant order, then $V_q(\boldsymbol{\lambda}_1)\otimes\cdots\otimes V_q(\boldsymbol{\lambda}_m)$ is a highest-ℓ-weight module.

Before venturing into the proof, let us obtain a consequence of Theorem 3.2.5 and Corollary 2.4.1 which describes the structure of the local Weyl modules in terms of the structure of the fundamental modules. We need the following lemma which is not difficult to establish.

Lemma 3.2.6. Let $m>0, a_j\in\mathbb{F}^\times, j=1,\dots,m$. If $\frac{a_j}{a_s}\notin q^{\mathbb{Z}_{>0}}$ for $j>s$, then $(\boldsymbol{\omega}_{i_1,a_1,r_1},\cdots,\boldsymbol{\omega}_{i_m,a_m,r_m})$ is in q-resonant order for any choice of $i_j\in I, r_j\in\mathbb{Z}_{\geq 0}$. In particular, for any choice of $a_j\in\mathbb{F}^\times, j=1,\dots,m$, there exists $\sigma\in S_m$ such that $(\boldsymbol{\omega}_{i_1,a_{\sigma(1)},r_1},\cdots,\boldsymbol{\omega}_{i_m,a_{\sigma(m)},r_m})$ is in q-resonant order for any choice of $i_j\in I, r_j\in\mathbb{Z}_{\geq 0}$. □

Corollary 3.2.7. Let $\boldsymbol{\lambda}\in\mathscr{P}^+$ and write $\boldsymbol{\lambda}=\left(\prod_{j=1}^m\boldsymbol{\omega}_{i_j,a_j}\right)$ for some $m\geq 0$, $i_j\in I$ and $a_j\in\mathbb{F}^\times, j=1,\dots,m$. Then, there exists $\sigma\in S_m$ such that:

$$W_q(\boldsymbol{\lambda})\cong V_q(\boldsymbol{\omega}_{i_{\sigma(1)},a_{\sigma(1)}})\otimes\cdots\otimes V_q(\boldsymbol{\omega}_{i_{\sigma(m)},a_{\sigma(m)}}).$$

Proof. Let σ be as in the previous lemma. Then, by Theorem 3.2.5, $V_q(\boldsymbol{\omega}_{i_{\sigma(1)},a_{\sigma(1)}})\otimes\cdots\otimes V_q(\boldsymbol{\omega}_{i_{\sigma(m)},a_{\sigma(m)}})$ is a quotient of $W_q(\boldsymbol{\lambda})$. As explained in the paragraph after Corollary 2.4.1, the dimension of $W_q(\boldsymbol{\lambda})$ is equal to the rank of the global Weyl module $W_q(\mathrm{wt}(\boldsymbol{\lambda}))$ as a right $A_{\mathrm{wt}(\boldsymbol{\lambda})}$-module. But Corollary 2.4.1 implies that this rank is exactly the dimension of $V_q(\boldsymbol{\omega}_{i_{\sigma(1)},a_{\sigma(1)}})\otimes\cdots\otimes V_q(\boldsymbol{\omega}_{i_{\sigma(m)},a_{\sigma(m)}})$. □

Remark 3.2.8. Notice that Corollary 3.2.7 together with Proposition 2.3.3 implies that the qcharacter of a quantum local Weyl module can be given as a product of fundamental qcharacters. Notice that, in the classical setting, local Weyl module may not be a tensor product of fundamental modules. ◊

Example 3.2.9. Let $\mathfrak{g}=\mathfrak{sl}_3$ and denote by i and j the two distinct elements of I. Let us study the Weyl module $W_q(\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,b})$ with $a,b\in\mathbb{F}^\times$. As remarked above, we have

$$\mathrm{qch}(W_q(\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,b}))=\mathrm{qch}(V_q(\boldsymbol{\omega}_{i,a}))\,\mathrm{qch}(V_q(\boldsymbol{\omega}_{j,b})).$$

Recall from Example 3.1.6 that

$$\mathrm{qch}(V_q(\boldsymbol{\omega}_{i,a}))=\boldsymbol{\omega}_{i,a}+\boldsymbol{\omega}_{i,aq^2}^{-1}\boldsymbol{\omega}_{j,aq}+\boldsymbol{\omega}_{j,aq^3}^{-1}\quad\text{and}\quad\mathrm{qch}(V_q(\boldsymbol{\omega}_{j,b}))=\boldsymbol{\omega}_{j,b}+\boldsymbol{\omega}_{j,bq^2}^{-1}\boldsymbol{\omega}_{i,bq}+\boldsymbol{\omega}_{i,bq^3}^{-1}. \tag{3.2.1}$$

Therefore,

$$\begin{aligned}\mathrm{qch}(W_q(\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,b}))=&\ \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,b}+\boldsymbol{\omega}_{i,aq^2}^{-1}\boldsymbol{\omega}_{j,aq}\boldsymbol{\omega}_{j,b}+\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{i,bq}\boldsymbol{\omega}_{j,bq^2}^{-1}\\&+\boldsymbol{\omega}_{j,aq^3}^{-1}\boldsymbol{\omega}_{j,b}+\boldsymbol{\omega}_{i,bq^3}^{-1}\boldsymbol{\omega}_{i,a}+\boldsymbol{\omega}_{i,bq}\boldsymbol{\omega}_{i,aq^2}^{-1}\boldsymbol{\omega}_{j,aq}\boldsymbol{\omega}_{j,bq^2}^{-1}\\&+\boldsymbol{\omega}_{i,aq^2}^{-1}\boldsymbol{\omega}_{i,bq^3}^{-1}\boldsymbol{\omega}_{j,aq}+\boldsymbol{\omega}_{i,bq}\boldsymbol{\omega}_{j,bq^2}^{-1}\boldsymbol{\omega}_{j,aq^3}^{-1}+\boldsymbol{\omega}_{i,bq^3}^{-1}\boldsymbol{\omega}_{j,aq^3}^{-1}.\end{aligned}$$

Notice that, if $a/b \neq q^{\pm 3}$, the only dominant ℓ-weight in $\mathrm{wt}_\ell(W_q(\omega_{i,a}\omega_{j,b}))$ is $\omega_{i,a}\omega_{j,b}$. Therefore, in that the case, $W_q(\omega_{i,a}\omega_{j,b}) \cong V_q(\omega_{i,a}) \otimes V_q(\omega_{j,b}) \cong V_q(\omega_{j,b}) \otimes V_q(\omega_{i,a})$ is irreducible.

Let us assume $b = aq^3$. It follows from Lemma 3.2.6 and Corollary 3.2.7 that

$$W_q(\omega_{i,a}\omega_{j,aq^3}) \cong V_q(\omega_{j,aq^3}) \otimes V_q(\omega_{i,a}).$$

Moreover, the above computation shows that $\mathbf{1} \in \mathrm{wt}_\ell(W_q(\omega_{i,a}\omega_{j,aq^3}))$. Although this is not sufficient to conclude that the trivial representation is an irreducible factor of $W_q(\omega_{i,a}\omega_{j,aq^3})$, let us show that this is indeed true. Notice that, as a $U_q(\mathfrak{sl}_3)$-module, we have

$$W_q(\omega_{i,a}\omega_{i,aq^3}) \cong V_q(\omega_i) \otimes V_q(\omega_j) \cong V_q(\omega_i + \omega_j) + V_q(0).$$

As mentioned earlier, if $\mathfrak{g} = \mathfrak{sl}_n$, there exist quantum analogues of evaluation maps and, therefore, there exist $U_q(\tilde{\mathfrak{sl}}_3)$-modules such that the underlying $U_q(\mathfrak{sl}_3)$-module is isomorphic to $V_q(\omega_i + \omega_j)$. The Drinfeld polynomial of such a module must be of the form $\omega_{i,a}\omega_{j,b}$ for some $a, b \in \mathbb{F}^\times$. As seen above, if $a/b \neq q^{\pm 3}$, the associated simple module is 9-dimensional and, hence, cannot be an evaluation module. Therefore, the Drinfeld polynomial must be either $\omega_{i,a}\omega_{j,aq^3}$ or $\omega_{i,a}\omega_{j,bq^{-3}}$. Noticing that $(\omega_{i,a}\omega_{j,aq^3})^* = \omega_{i,aq^6}\omega_{j,aq^3}$, it follows that both $V_q(\omega_{i,a}\omega_{j,aq^3})$ and $V_q(\omega_{i,a}\omega_{j,aq^{-3}})$ are isomorphic to $V_q(\omega_i + \omega_j)$ as $U_q(\mathfrak{sl}_3)$-modules. This shows that we have a short exact sequence

$$0 \to V_q(\mathbf{1}) \to W_q(\omega_{i,a}\omega_{i,aq^3}) \to V_q(\omega_{i,a}\omega_{i,aq^3}) \to 0$$

and, moreover,

$$\begin{aligned}\mathrm{qch}(V_q(\omega_{i,a}\omega_{j,aq^3})) = {} & \omega_{i,a}\omega_{j,aq^3} + \omega_{i,aq^2}^{-1}\omega_{j,aq}\omega_{j,aq^3} + \omega_{i,a}\omega_{i,aq^4}\omega_{j,aq^5}^{-1} \\ & + \omega_{i,aq^6}^{-1}\omega_{i,a} + \omega_{i,aq^4}\omega_{i,aq^2}^{-1}\omega_{j,aq}\omega_{j,aq^5}^{-1} \\ & + \omega_{i,aq^2}^{-1}\omega_{i,aq^6}^{-1}\omega_{j,aq} + \omega_{i,aq^4}\omega_{j,aq^5}^{-1}\omega_{j,aq^3}^{-1} + \omega_{i,aq^6}^{-1}\omega_{j,aq^3}^{-1}.\end{aligned}$$

Notice that $V_q(\omega_{i,a}\omega_{j,aq^3})$ is ℓ-minuscule and all ℓ-weight spaces are one-dimensional.

One can avoid the use of the existence of quantum evaluation maps in order to conclude the above. Namely, one can find an explicit expression for a highest-weight vector generating the trivial representation inside $V_q(\omega_i) \otimes V_q(\omega_j)$. Then, using Proposition 1.4.10 and (3.2.1), one can show that this highest-weight vector is also a highest-ℓ-weight vector. ◊

Example 3.2.10. Once more, let $\mathfrak{g} = \mathfrak{sl}_3$ and denote by i and j the two distinct elements of I. Proceeding as in the previous example, one can show that $W_q(\omega_{i,a}\omega_{i,b})$ is irreducible unless $b = aq^{\pm 2}$. Moreover, if $b = aq^2$, then $W_q(\omega_{i,a}\omega_{i,aq^2}) \cong V_q(\omega_{i,aq^2}) \otimes V_q(\omega_{i,a})$ and we have a short exact sequence

$$0 \to V_q(\omega_{j,aq}) \to W_q(\omega_{i,a}\omega_{i,aq^2}) \to V_q(\omega_{i,a}\omega_{i,aq^2}) \to 0.$$

In particular,

$$\begin{aligned}\mathrm{qch}(W_q(\omega_{i,a}\omega_{i,aq^2})) = {} & \omega_{i,a}\omega_{i,aq^2} + \omega_{j,aq} + \omega_{i,a}\omega_{i,aq^4}^{-1}\omega_{j,aq^3} \\ & + \omega_{i,aq^2}\omega_{j,aq^3}^{-1} + \omega_{i,a}\omega_{j,aq^5}^{-1} + \omega_{i,aq^2}^{-1}\omega_{i,aq^4}^{-1}\omega_{j,aq}\omega_{j,aq^3} \\ & + \omega_{i,aq^2}^{-1}\omega_{j,aq}\omega_{j,aq^5}^{-1} + \omega_{i,aq^4}^{-1} + \omega_{j,aq^3}^{-1}\omega_{j,aq^5}^{-1}\end{aligned}$$

and

$$\mathrm{qch}(V_q(\omega_{i,a}\omega_{i,aq^2})) = \omega_{i,a}\omega_{i,aq^2} + \omega_{i,a}\omega_{i,aq^4}^{-1}\omega_{j,aq^3} + \omega_{i,a}\omega_{j,aq^5}^{-1}$$
$$+ \omega_{i,aq^2}^{-1}\omega_{i,aq^4}^{-1}\omega_{j,aq}\omega_{j,aq^3} + \omega_{i,aq^2}^{-1}\omega_{j,aq}\omega_{j,aq^5}^{-1} + \omega_{j,aq^3}^{-1}\omega_{j,aq^5}^{-1}.$$

It follows that $\omega_{i,a}\omega_{i,aq^2} = \omega_{i,aq,2}$ is minuscule. ◊

Assume we have proved Theorem 3.2.5 in the $\mathfrak{sl}_2$-case and let us deduce the general case.

Proof of Theorem 3.2.5. We proceed by induction on m which clearly starts at $m = 1$. Let $v^j, j = 1, \ldots, m$ be highest-ℓ-weight vectors for $V_q(\lambda_j), V = V_q(\lambda_1), V' = V_q(\lambda_2) \otimes \cdots \otimes V_q(\lambda_m)$, and $v' = v^2 \otimes \cdots \otimes v^m$. By the induction hypothesis, $V' = U_q(\tilde{\mathfrak{g}})v'$. Choose a reduced expression $r_{i_l} \cdots r_{i_1}$ for w_0 satisfying the property for q-resonance in Definition 3.2.3. By Lemma 3.2.2, it suffices to show that

$$v^1_{r_{i_j}\cdots r_{i_1}} \otimes v' \in U_q(\tilde{\mathfrak{g}}_{i_j})(v^1_{r_{i_{j-1}}\cdots r_{i_1}} \otimes v') \qquad \text{for all} \qquad j = 1, \ldots, l. \tag{3.2.2}$$

By Remark 3.2.4, the $(m-1)$-tuple of polynomials $((\lambda_2)_{i_j}(u), \ldots, (\lambda_m)_{i_j}(u))$ is in q_{i_j}-resonant order. In particular, it follows from the $\mathfrak{sl}_2$-case that

$$U_q(\tilde{\mathfrak{g}}_{i_j})v' = (U_q(\tilde{\mathfrak{g}}_{i_j})v^2) \otimes \cdots \otimes (U_q(\tilde{\mathfrak{g}}_{i_j})v^m).$$

On the other hand, it follows from Proposition 3.1.2 that $U_q(\tilde{\mathfrak{g}}_{i_j})v^1_{r_{i_{j-1}}\cdots r_{i_1}}$ is a quotient of the local Weyl module for $U_q(\tilde{\mathfrak{g}}_{i_j}) \cong U_{q_{i_j}}(\tilde{\mathfrak{sl}}_2)$ with highest-ℓ-weight given by the polynomial $(T_{i_{j-1}} \cdots T_{i_1}\lambda_1)_{i_j}(u)$. By Corollary 3.2.7 (in the $\mathfrak{sl}_2$-case), this Weyl module is a tensor product of the form $V_{q_{i_j}}(\omega_{i_j,a_1}) \otimes \cdots \otimes V_{q_{i_j}}(\omega_{i_j,a_k})$ for some $a_1, \ldots, a_k \in \mathbb{F}^\times$, where $k = \mathrm{wt}(T_{i_{j-1}} \cdots T_{i_1}\lambda_1)(h_{i_j})$ and the order is so that $(\omega_{i_j,a_1}, \ldots, \omega_{i_j,a_k})$ is a tuple of polynomials in q_{i_j}-resonant order. By our choice of reduced expression for w_0, the tuple of polynomials $\big((T_{i_{j-1}} \cdots T_{i_1}\lambda_1)_{i_j}(u), (\lambda_2)_{i_j}(u), \ldots, (\lambda_m)_{i_j}(u)\big)$ is in q_{i_j}-resonant order. One easily checks that this implies that $(\omega_{i_j,a_1}, \ldots, \omega_{i_j,a_k}, (\lambda_2)_{i_j}, \ldots, (\lambda_m)_{i_j})$ is also a tuple of polynomials in q_{i_j}-resonant order. Since $U_q(\tilde{\mathfrak{g}}_{i_j})v^n$ is an irreducible $U_q(\tilde{\mathfrak{g}}_{i_j})$-module for all $n \geq 1$ by Lemma 2.2.13, it then follows from the $\mathfrak{sl}_2$-case that

$$U_q(\tilde{\mathfrak{g}}_{i_j})(v^1_{r_{i_{j-1}}\cdots r_{i_1}} \otimes v') = (U_q(\tilde{\mathfrak{g}}_{i_j})v^1_{r_{i_{j-1}}\cdots r_{i_1}}) \otimes (U_q(\tilde{\mathfrak{g}}_{i_j})v').$$

This clearly implies (3.2.2). □

It remains the prove Theorem 3.2.5 for $\mathfrak{g} = \mathfrak{sl}_2$. Thus for the remainder of the subsection we set $\mathfrak{g} = \mathfrak{sl}_2$ and, as usual, let i denote the unique element of I. Before continuing, let us remark that the claim made in Example 2.3.25 that $V_q(1, a) \otimes V_q(1, aq^2)$ is a highest-ℓ-weight module follows easily from Theorem 3.2.5.

Theorem 3.2.5 will follow from the following Proposition which will enable us to prove Theorem 2.3.20 as well.

Proposition 3.2.11. Let $k > 0$, $a_1, \ldots, a_k \in \mathbb{F}^\times$, and $r_1, \ldots, r_k \in \mathbb{Z}_{\geq 0}$. If $(\omega_{i,a_1,r_1}, \ldots, \omega_{i,a_k,r_k})$ is in q-resonant order, then $V_q(\omega_{i,a_1,r_1}) \otimes \cdots \otimes V_q(\omega_{i,a_k,r_k})$ is a highest-ℓ-weight module. Moreover, the same holds if $k = 2$ and $(\omega_{i,a_1,r_1}, \omega_{i,a_2,r_2})$ is in weak q-resonant order.

Proof of Theorem 2.3.20. We need to prove that, if $\boldsymbol{\lambda} = \prod_{j=1}^{m} \boldsymbol{\omega}_{i,a_j,r_j}$ is the q-factorization of $\boldsymbol{\lambda} \in \mathscr{P}^+$, then $V_q(\boldsymbol{\lambda}) \cong V$ where $V = V_q(\boldsymbol{\omega}_{i,a_1,r_1}) \otimes \cdots \otimes V_q(\boldsymbol{\omega}_{i,a_m,r_m})$. Notice that $(\boldsymbol{\omega}_{i,a_j,r_j}, \boldsymbol{\omega}_{i,a_k,r_k})$ and $(\boldsymbol{\omega}^*_{i,a_k,r_k}, \boldsymbol{\omega}^*_{i,a_j,r_j})$ are in weak q-resonant resonant order for every $j, k = 1, \ldots, m$. The second statement of Proposition 3.2.11 and together with Lemma 3.1.10 imply that

$$V_q(\boldsymbol{\omega}_{i,a_j,r_j}) \otimes V_q(\boldsymbol{\omega}_{i,a_k,r_k}) \cong V_q(\boldsymbol{\omega}_{i,a_j,r_j}\boldsymbol{\omega}_{i,a_k,r_k}) \cong V_q(\boldsymbol{\omega}_{i,a_k,r_k}) \otimes V_q(\boldsymbol{\omega}_{i,a_j,r_j}).$$

It follows that

$$V \cong V_q(\boldsymbol{\omega}_{i,a_{\sigma(1)},r_{\sigma(1)}}) \otimes \cdots \otimes V_q(\boldsymbol{\omega}_{i,a_{\sigma(m)},r_{\sigma(m)}}) \qquad \text{and} \qquad V^* \cong V_q(\boldsymbol{\omega}_{i,a_{\sigma(1)},r_{\sigma(1)}})^* \otimes \cdots \otimes V_q(\boldsymbol{\omega}_{i,a_{\sigma(m)},r_{\sigma(m)}})^*$$

for all $\sigma \in S_m$. It then follows from Lemma 3.2.6 that both V and V^* are highest-ℓ-weight modules and we are done by Lemma 3.1.10. $\square$

Proof of Theorem 3.2.5 for $\mathfrak{sl}_2$. Given $1 \leq l \leq m$, let $\boldsymbol{\lambda}_l = \prod_{j=1}^{g_l} \boldsymbol{\omega}_{i,a_{j,l},r_{j,l}}$ be the q-factorization of $\boldsymbol{\lambda}_l$. Notice that, by definition of q-resonant order, if $(\boldsymbol{\lambda}_l, \boldsymbol{\lambda}_n)$ is in q-resonant order, so is $(\boldsymbol{\omega}_{i,a_{j,l},r_{j,l}}, \boldsymbol{\omega}_{i,a_{k,n},r_{k,n}})$ for all $1 \leq j \leq g_l, 1 \leq k \leq g_n$. Therefore, by Theorem 2.3.20, in order to prove Theorem 3.2.5, it suffices to prove Proposition 3.2.11. $\square$

Remark 3.2.12. Notice that in the proof of Theorem 2.3.20 we use Proposition 3.2.11 with k being the number of q-factors of $\boldsymbol{\lambda}$. In the proof of Proposition 3.2.11 we will perform an induction on k and use Theorem 2.3.20 with $\boldsymbol{\lambda}$ having less than k q-factors which, therefore, does not characterize a circling argument. $\Diamond$

Proof of Proposition 3.2.11. We begin with the the second statement. To simplify notation, we set $a_1 = a, a_2 = b, r_1 = m$, and $r_2 = n$. Let v_m^j, $j = 0, \ldots, m$, be as in the proof of Theorem 2.3.14 and similarly define v_n^j, $j = 0, \ldots, n$. We first prove that $v_m^j \otimes v_n^0, v_m^{j-1} \otimes v_n^1 \in U_q(\tilde{\mathfrak{g}})(v_m^0 \otimes v_n^0)$ for $0 \leq j \leq \min\{m, n\}$ by induction on j which clearly starts when $j = 0$. Hence, assume $0 \leq j < \min\{m, n\}$ is such that $v_m^j \otimes v_n^0 \in U_q(\tilde{\mathfrak{g}})(v_m^0 \otimes v_n^0)$. By Proposition 1.4.10, given $r > 0$, we have

$$x_{i,r}^-(v_m^j \otimes v_n^0) = a_{r,1}\, v_m^{j+1} \otimes v_n^0 + a_{r,2}\, v_m^j \otimes v_n^1,$$

for some $a_{r,s} \in \mathbb{F}^\times$ with $s \in \{1, 2\}$. Consider the matrix $A = (a_{r,s})$ with $r, s \in \{1, 2\}$. It suffices to show that $\det(A) \neq 0$. Using (2.3.7) and (2.3.8) we can compute A precisely:

$$A = \begin{bmatrix} [j+1]_q a q^{m+n-2j} & b q^n \\ [j+1]_q a q^{m+n-2j}(a q^{m-2j} + b\,(q^n - q^{-n})) & (b q^n)^2 \end{bmatrix}.$$

Therefore, $\det(A) = [j+1]_q ab q^{m+2n-2j}\left(b q^{-n} - a q^{m-2j}\right) \neq 0$ since $j < \min\{m, n\}$ and $(\boldsymbol{\omega}_{i,a,m}, \boldsymbol{\omega}_{i,b,n})$ is in weak q-resonant order.

By Lemma 3.2.2, it suffices to show that $v_m^m \otimes v_n^0 \in U_q(\tilde{\mathfrak{g}})(v_m^0 \otimes v_n^0)$ which is immediate from the above computation if $m \leq n$. Otherwise, set $V = V_q(m, a) \otimes V_q(n, b)$ and observe that $(m+n)\omega_i - m\alpha_i = w_0((m+n)\omega_i - n\alpha_i)$. Since $v_m^m \otimes v_n^0 \in V_{(m+n)\omega_i - m\alpha_i}$, it follows from Proposition 1.3.7 that it suffices to show that

$$V_{(m+n)\omega_i - n\alpha_i} \subseteq U_q(\tilde{\mathfrak{g}})(v_m^0 \otimes v_n^0).$$

We now show that the above computations imply that

$$V_{(m+n)\omega_i - j\alpha_i} \subseteq U_q(\tilde{\mathfrak{g}})(v_m^0 \otimes v_n^0) \qquad \text{for all} \qquad 0 \le j \le n \tag{3.2.3}$$

which completes the proof.

Observe that the vectors $v_{j,l} := v_m^{j-l} \otimes v_n^l, l = 0, \ldots, j$, form a basis of $V_{(m+n)\omega_i - j\alpha_i}$. We prove (3.2.3) by induction on j which again clearly starts when $j = 0$. Thus, assume $0 \le j < n$ is such that (3.2.3) holds. This implies $v_{j,l} \in U_q(\tilde{\mathfrak{g}})(v_m^0 \otimes v_n^0)$ for all $0 \le l \le j$. Using Proposition 1.4.10 and (2.3.7) once more we get

$$\begin{aligned} x_i^- v_{j,l} &= [j-l+1]_q q^{n-2l}(v_m^{j-l+1} \otimes v_n^l) + [l+1]_q (v_m^{j-l} \otimes v_n^{l+1}) \\ &= [j-l+1]_q q^{n-2l} v_{j+1,l} + [l+1]_q v_{j+1,l+1} \ \in \ U_q(\tilde{\mathfrak{g}})(v_m^0 \otimes v_n^0). \end{aligned} \tag{3.2.4}$$

Setting $l = 0$ above and recalling that we have already shown that $v_{j+1,0} \in U_q(\tilde{\mathfrak{g}})(v_m^0 \otimes v_n^0)$, it follows that $v_{j+1,1} \in U_q(\tilde{\mathfrak{g}})(v_m^0 \otimes v_n^0)$. An obvious induction on $l = 0, \ldots, j$, using (3.2.4) shows that $v_{j+1,l+1} \in U_q(\tilde{\mathfrak{g}})(v_m^0 \otimes v_n^0)$ thus proving (3.2.3).

We now prove the first statement of Proposition 3.2.11 by induction on $k \ge 1$ which clearly starts when $k = 1$. Also, the case $k = 2$ follows from the second statement. The inductive step will follow from studying a generalization of the matrix A above. To simplify notation, set $V_j = V(r_j, a_j), j = 1, \ldots, k$. By induction hypothesis, $V_2 \otimes \cdots \otimes V_k$ is a highest-ℓ-weight module. Let v_j be a highest-ℓ-weight vector for V_j and $v_j^l = (x_i^-)^l v_j, 0 \le l \le r_j$ (we dropped the use of divided powers here so that we get rid of the terms $[j+1]_q$ appearing in the first column of matrix A above). Let also $w_1 = v_2 \otimes \cdots \otimes v_k$, and $V = U_q(\tilde{\mathfrak{g}})(v_1 \otimes w_1)$. By Lemma 3.2.2, it suffices to show that $v_1^{r_1} \otimes w_1 \in V$. As before, we prove this by showing that

$$v_1^l \otimes w_1 \in V \qquad \text{for all} \qquad 0 \le l \le r_1 \tag{3.2.5}$$

by a further induction on $0 \le l \le r_1$ which clearly starts when $l = 0$.

Assume $0 < l < r_1$ and, by induction hypothesis, that $v_1^{l-1} \otimes w_1 \in V$. By Proposition 1.4.10, given $s > 0$, we have

$$x_{i,s}^-(v_1^{l-1} \otimes w_1) = v_1^{l-1} \otimes (x_{i,s}^- w_1) + \sum_{t=0}^{s-1} (x_{i,s-t}^- v_1^{l-1}) \otimes (\psi_{i,t}^+ w_1). \tag{3.2.6}$$

Since w_1 is a highest-ℓ-weight vector, we must have $\psi_{i,t}^+ w_1 = \phi_{t,1} w_1$ for some $\phi_{t,1} \in \mathbb{F}^\times$. Using (2.3.7) we get

$$\sum_{t=0}^{s-1} (x_{i,s-t}^- v_1^{l-1}) \otimes (\psi_{i,t}^+ w_i) = \left(\sum_{t=0}^{s-1} \left(a_1 q^{r_1 - 2(l+1)} \right)^{s-t} \phi_{t,1} \right) v_1^l \otimes w_1. \tag{3.2.7}$$

More generally, set $w_j = v_{j+1} \otimes \ldots \otimes v_k$ and let $\phi_{t,j}$ be the eigenvalue of $\psi_{i,t}^+$ acting on w_j, $1 \le j \le k-1$. Given $1 \le s \le k$, set

$$a_{s,j} = \begin{cases} \sum\limits_{t=0}^{s-1} \left(a_j q^{r_j} \right)^{s-t} \phi_{t,j}, & \text{if} \quad 1 \le j < k \\ (a_k q^{r_k})^s, & \text{if} \quad j = k \end{cases}$$

Proceeding as above, we get

$$\begin{aligned}x_{i,s}^-(v_1^{l-1}\otimes w_1) = a_{s,1}(v_1^l\otimes w_1) + a_{s,2}(v_1^{l-1}\otimes v_2^1\otimes w_2) + \cdots + a_{s,j}(v_1^{l-1}\otimes v_2\otimes\cdots\otimes v_j\otimes v_j^1\otimes w_j) +\\ + \cdots + a_{s,k}(v_1^{l-1}\otimes v_2\otimes\cdots\otimes v_{k-1}\otimes v_k^1).\end{aligned}$$

Let $\mathbf{v}_j$ be the vector being multiplied by $a_{s,j}$ above so that $x_{i,s}^-(v_1^{l-1}\otimes w_1) = \sum_{j=1}^k a_{s,j}\mathbf{v}_j$. Consider the matrix $A = (a_{s,j})$ with $1 \le s, j \le k$. The set $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is clearly linearly independent and A is the matrix whose s-th row is formed by the coordinates of $(x_{i,s}^-)(v_1^{l-1}\otimes w_1)$ with respect to β. Thus, $\alpha := \{x_{i,s}^-(v_1^{l-1}\otimes w_1) : s = 1, \dots, k\}$ is linearly independent if and only if $\det(A) \neq 0$. In particular, if we show that $\det(A) \neq 0$, it follows that $\mathbf{v}_j$ is a linear combination of the elements of α. Since $\alpha \subseteq V$ by induction hypothesis on l, it follows that $\mathbf{v}_1 = v_1^l \otimes w_1 \in V$ which completes the proof of (3.2.5).

The remainder of the proof is dedicated to showing that

$$\det(A) = q^{r_2+2r_3+\ldots+(k-1)r_k}\left(\prod_{j=1}^k b_j\right)\left(\prod_{j>m}\left(b_j - b_m q^{2r_j}\right)\right) \tag{3.2.8}$$

where

$$b_j = \begin{cases} a_1 q^{r_1-2(l-1)}, & \text{if } j = 1\\ a_j q^{r_j}, & \text{otherwise.}\end{cases}$$

Using (3.2.8) one easily sees that $\det(A) \neq 0$ since $(\omega_{i,a_1,r_1}, \dots, \omega_{i,a_k,r_k})$ is in q-resonant order.

For proving (3.2.8), observe first that

$$A = \begin{bmatrix} \phi_{0,1}\, b_1 & \phi_{0,2}\, b_2 & \cdots & \phi_{0,k-1}\, b_{k-1} & b_k\\ \phi_{0,1}\, b_1^2 + \phi_{1,1}\, b_1 & \phi_{0,2}\, b_2^2 + \phi_{1,2}\, b_2 & \cdots & \phi_{0,k-1}\, b_{k-1}^2 + \phi_{1,k-1}\, b_{k-1} & b_k^2\\ \vdots & \vdots & & \vdots & \vdots\\ \sum\limits_{t=0}^{s-1}\phi_{t,1}\, b_1^{s-t} & \cdots & \cdots & \sum\limits_{t=0}^{s-1}\phi_{t,k-1}\, b_{k-1}^{s-t} & b_k^k\end{bmatrix}$$

and let

$$a'_{s,j} = \begin{cases} b_k^{s-1}, & \text{if } j = k,\\ \sum\limits_{t=0}^{s-1}\phi_{t,j}\, b_j^{s-t-1}, & \text{otherwise,}\end{cases}$$

so that $\det(A) = (\prod_{j=1}^k b_j)\det(A')$, where $A' = (a'_{s,j})$. Observe also that, since $\psi_{i,0} = k_i$, we have $\phi_{0,j} = q^{r_{j+1}+\cdots+r_m}$. We shall also need the following identities.

$$a'_{s+1,j} = b_j a'_{s,j} + \phi_{s,j} \qquad \text{for all} \qquad 1 \le j < k \tag{3.2.9}$$

and

$$\phi_{s,j} = q^{r_{j+1}}\phi_{s,j+1} + b_{j+1}(q^{r_{j+1}} - q^{-r_{j+1}})a'_{s,j+1} \qquad \text{for all} \qquad 1 \le j < k-1. \tag{3.2.10}$$

The former is easily obtained from the definition of $a'_{s,j}$ while latter is obtained as follows. By Proposition 1.4.10, we have

$$\begin{aligned}\phi_{s,j}w_j = \psi_{i,s}^+ w_j &= \sum_{t=0}^s(\psi_{i,t}v_{j+1})\otimes(\psi_{i,s-t}w_{j+1})\\ &= (k_i v_{j+1})\otimes(\phi_{s,j+1}w_{j+1}) + \sum_{t=1}^s(\psi_{i,t}^+ v_{j+1})\otimes(\phi_{s-t,j+1}w_{j+1}).\end{aligned}$$

Using (2.3.8), we get $\psi^+_{i,t} v_{j+1} = (b_{j+1})^t [r_{j+1}]_q (q - q^{-1}) v_{j+1} = (b_{j+1})^t (q^{r_{j+1}} - q^{-r_{j+1}}) v_{j+1}$. Therefore,

$$\begin{aligned}
\phi_{s,j} w_j &= \left(q^{r_{j+1}} \phi_{s,j+1} + (q^{r_{j+1}} - q^{-r_{j+1}}) \sum_{t=1}^{s} (b_{j+1})^t \phi_{s-t,j+1} \right) w_j \\
&= \left(q^{r_{j+1}} \phi_{s,j+1} + (q^{r_{j+1}} - q^{-r_{j+1}}) b_{j+1} \sum_{t=1}^{s} (b_{j+1})^{t-1} \phi_{s-t,j+1} \right) w_j \\
&= \left(q^{r_{j+1}} \phi_{s,j+1} + (q^{r_{j+1}} - q^{-r_{j+1}}) b_{j+1} \sum_{t'=0}^{s-1} \phi_{t',j+1} (b_{j+1})^{s-t'-1} \right) w_j \\
&= \left(q^{r_{j+1}} \phi_{s,j+1} + (q^{r_{j+1}} - q^{-r_{j+1}}) b_{j+1} a'_{s,j+1} \right) w_j.
\end{aligned}$$

It is clear from the above proof of (3.2.10) that $\phi_{s,j}$ is a polynomial on the elements b_m and, hence, so is $\det(A')$. By looking at the degree of $\det(A')$ as a polynomial on b_j for some fixed j, it follows that, if we show that $b_m q^{2r_j}$ is a root of $\det(A')$ for all $m < j$, than these must be all the roots. In other words, it follows that $\det(A')$ is a nonzero scalar multiple of $\prod_{j>m}(b_j - b_m q^{2r_j})$. This scalar is computed by looking at the coefficient of the monomial $b_2 b_3^2 \dots b_k^{k-1}$. Namely, the coefficient of $b_2 b_3^2 \dots b_k^{k-1}$ in $\prod_{j>m}(b_j - b_m q^{2r_j})$ is 1 while, in $\det(A')$, it is $\phi_{0,1}\ \phi_{0,2} \cdots \phi_{0,k-1} = q^{r_2 + 2r_3 + \cdots + (k-1) r_k}$.

Thus, it remains to show that $\det(A') = 0$ whenever $b_j = b_m q^{2r_j}$ for some $1 \le m < j \le k$. We first show that $\det(A') = 0$ if $b_{j+1} = b_j q^{2r_{j+1}}$ for some $1 \le j < k-1$. We will do this by showing that, in this case, the $(j+1)$-th column of A' is a scalar multiple of the j-th column. More precisely, we will show, by induction on $s \ge 1$, that

$$a'_{s,j} = q^{r_{j+1}} a'_{s,j+1}. \tag{3.2.11}$$

If $s = 1$, then $a'_{s,j} = \phi_{0,j}$ and $a'_{s,j+1} = \phi_{0,j+1}$ showing that induction starts. Assume (3.2.11) holds for $a'_{s,j}$ by induction hypothesis and notice that

$$\begin{aligned}
a'_{s+1,j} &\overset{(3.2.9)}{=} b_j a'_{s,j} + \phi_{s,j} = (b_{j+1} q^{-2r_{j+1}})(q^{r_{j+1}} a'_{s,j+1}) + \phi_{s,j} \\
&\overset{(3.2.10)}{=} b_{j+1} a'_{s,j+1} q^{-r_{j+1}} + q^{r_{j+1}} \phi_{s,j+1} + b_{j+1}(q^{r_{j+1}} - q^{-r_{j+1}}) a'_{s,j+1} \\
&= q^{r_{j+1}} b_{j+1} a'_{s,j+1} + q^{r_{j+1}} \phi_{s,j+1} = q^{r_{j+1}} (b_{j+1} a'_{s,j+1} + \phi_{s,j+1}) \overset{(3.2.9)}{=} q^{r_{j+1}} a'_{s+1,j+1}.
\end{aligned}$$

In the second equality above we used both the induction hypothesis and the hypothesis $b_{j+1} = b_j q^{2r_{j+1}}$. This shows that $\det(A') = 0$ if $b_{j+1} = b_j q^{2r_{j+1}}$ with $1 \le j < k-1$. Similarly one shows that $\det(A') = 0$ if $b_k = b_{k-1} q^{2r_k}$.

It remains to show that $\det(A') = 0$ if $b_j = b_m q^{2r_j}$ with $j - m > 1$. We proceed by induction on $k \ge 2$ which was shown to start when we proved the second statement of Proposition 3.2.11. Let $k > 2$ and that (3.2.8) holds for $(k-1)$-fold tensor products by induction hypothesis. Since $\det(A')$ is a polynomial on $b_1, \dots, b_k$, it depends continuously on these numbers. Therefore, for computing (3.2.8), we might as well assume that $b_1, ..., b_k$ are such that $\det(A') \neq 0$. Furthermore, we can also assume that $b_2, \dots, b_k$ are such that $(\omega_{i,a_2,r_2}, \dots, \omega_{i,a_k,r_k})$ is in general position. By the induction hypothesis on k and Theorem 2.3.20 with $\lambda = \prod_{j=2}^k \omega_{i,a_j,r_j}$ (which can be used by Remark 3.2.12), we have isomorphisms

$$V_1 \otimes V_2 \otimes \cdots \otimes V_k \xrightarrow{\varphi_\sigma} V_1 \otimes V_{\sigma(2)} \otimes \cdots \otimes V_{\sigma(k)} \quad \text{for all} \quad \sigma \in S_k, \sigma(1) = 1. \tag{3.2.12}$$

Moreover, we can assume $\varphi_\sigma(v_1\otimes\cdots\otimes v_k) = v_1\otimes v_{\sigma(2)}\otimes\cdots\otimes v_{\sigma(m)}$. It follows that $\alpha = \{x^-_{i,s}(v_1^{l-1}\otimes w_1) : s = 1,\dots,k\}$ is linearly independent if and only if $\alpha_\sigma = \{x^-_{i,s}(v_1^{l-1}\otimes w_1^\sigma) : s = 1,\dots,k\}$ is linearly independent, where $w_1^\sigma = v_{\sigma(2)}\otimes\cdots\otimes v_{\sigma(k)}$. Since we are assuming that $b_1,\dots,b_k$ are such that $\det(A')\neq 0$, both α and α' are linearly independent.

Let A_σ and A'_σ be the matrices defined as before using $V_{\sigma(2)}\otimes\cdots\otimes V_{\sigma(m)}$ in place of $V_2\otimes\cdots\otimes V_k$. By the previous argument, we know that, if $1\leq j,m\leq k$ are such that $\sigma(j) = \sigma(m)+1$, then $b_j - b_m q^{2r_j}$ is a factor of $\det(A'_\sigma)$. Recall that we defined $\beta = \{\mathbf{v}_1,\dots,\mathbf{v}_k\}$ and similarly define β_σ. By our assumptions $[\beta] = [\alpha]$ and $[\beta_\sigma] = [\alpha_\sigma]$. Notice also that $\varphi_\sigma([\beta])\subseteq[\beta_\sigma]$. By abuse of notation, let φ_σ denote also the induced linear map $[\beta]\to[\beta_\sigma]$ and observe that $[\varphi_\sigma]^\alpha_{\alpha_\sigma}$ is the identity matrix. Then,

$$\mathrm{Id} = [\varphi_\sigma]^\alpha_{\alpha_\sigma} = [\mathrm{Id}]^{\beta_\sigma}_{\alpha_\sigma}[\varphi_\sigma]^\beta_{\beta_\sigma}[\mathrm{Id}]^\alpha_\beta = (A^t_\sigma)^{-1}[\varphi_\sigma]^\beta_{\beta_\sigma}A^t$$

and it follows that

$$\det(A_\sigma) = \det(A)\det\left([\varphi_\sigma]^\beta_{\beta_\sigma}\right). \tag{3.2.13}$$

It is not difficult to see that $\det\left([\varphi_\sigma]^\beta_{\beta_\sigma}\right)$ is a rational function on $b_2,\dots,b_k$. Moreover, if $m<j$ are such that $\sigma(m)<\sigma(j)$, then, for any choice of $c\in\mathbb{F}^\times$, $(b_j - b_m c)$ is not a factor of $\det\left([\varphi_\sigma]^\beta_{\beta_\sigma}\right)$. Fix m and j such that $j-m>1$ and let σ be the permutation $(m, j-1)$. Since $\sigma(j) = j = \sigma(m)+1$, it follows that $b_j - b_m q^{2r_j}$ is a factor of $\det(A'_\sigma)$ which is not a factor of $\det\left([\varphi_\sigma]^\beta_{\beta_\sigma}\right)$. By (3.2.13), $b_j - b_m q^{2r_j}$ must be a factor of $\det(A')$ as well. □

Remark 3.2.13. It should be instructive for the reader to compare the above proof with that of Theorem 2.1.3. In particular, compare (3.2.8) with the determinant of the Vandermonde matrix associated to $a_1,\dots,a_k$. ◊

3.3. Blocks

Let us recall the basics about the block decomposition of Jordan-Hölder categories. Let $\mathscr{C}$ be such a category and recall that every object has a unique splitting into a direct sum of indecomposable sub-objects.

Definition 3.3.1. Two nonzero indecomposable objects $V_1, V_2\in\mathscr{C}$ are said to be linked if there is no splitting of $\mathscr{C}$ in a direct sum of two full abelian subcategories, say $\mathscr{C} = \mathscr{C}_1\oplus\mathscr{C}_2$, such that $V_1\in\mathscr{C}_1$ and $V_2\in\mathscr{C}_2$. The category $\mathscr{C}$ is said to be indecomposable if every two of its indecomposable objects are linked. ◊

It is easy to see that linkage is an equivalence relation on the subclass of indecomposable objects of $\mathscr{C}$. The next lemma is a straightforward consequence the Jordan-Hölder Theorem.

Lemma 3.3.2. Let $V, W\in\mathscr{C}$ be indecomposable. The following are sufficient conditions for V and W to be linked.

(a) V and W are simple factors of the same indecomposable object.

(b) V and W have a common simple factor. □

Proposition 3.3.3. There exists a unique splitting of $\mathscr{C}$ into a direct sum of full indecomposable abelian subcategoies.

Proof. Let $\mathscr{L}$ be the set of equivalence classes of linked indecomposable objects, and given $\gamma \in \mathscr{L}$, let $\mathscr{C}_\gamma$ be the full subcategory of $\mathscr{C}$ whose objects are direct sums of objects in γ. By the uniqueness of the decomposition of objects into direct sum of indecomposable sub-objects, we have

$$\mathscr{C} = \bigoplus_{\gamma \in \mathscr{L}} \mathscr{C}_\gamma. \tag{3.3.1}$$

Furthermore, the categories $\mathscr{C}_\gamma$ are indecomposable. Indeed, if that was not the case, let $\mathscr{C}_\delta = \mathscr{C}_1 \oplus \mathscr{C}_2$ be a nontrivial decomposition of $\mathscr{C}_\delta$ for some $\delta \in \mathscr{L}$. Then we can consider the following splitting of $\mathscr{C}$:

$$\mathscr{C} = \mathscr{C}_1 \oplus \left(\mathscr{C}_2 \oplus \bigoplus_{\gamma \neq \delta} \mathscr{C}_\gamma \right).$$

It follows that if V_j is an indecomposable object in $\mathscr{C}_j$, $j = 1, 2$, then V_1 and V_2 are not linked. This contradicts the definition of $\mathscr{C}_\delta$ since both $V_j \in \mathscr{C}_\delta$. This proves the existence of such splitting of $\mathscr{C}$. The uniqueness is now obvious. □

Let $\mathscr{L}$ be as in (3.3.1). Then, given $V \in \mathscr{C}$, there exists unique sub-objects $V_\gamma, \gamma \in \mathscr{L}$, such that

$$V = \bigoplus_{\gamma \in \mathscr{L}} V_\gamma. \tag{3.3.2}$$

Definition 3.3.4. The decomposition (3.3.1) is called the block decomposition of $\mathscr{C}$ and the subcategories $\mathscr{C}_\gamma$ are called the blocks of $\mathscr{C}$. Similarly, given $V \in \mathscr{C}$, the decomposition (3.3.2) is called the block decomposition of V and the sub-objects V_γ are called the blocks of V. ◊

Example 3.3.5. If $\mathscr{C}$ is semi-simple, $\mathscr{L}$ coincides with the set of isomorphism classes of simple objects. ◊

The main goal of this subsection is to give a concrete description, in terms of ℓ-weights, of the set $\mathscr{L}$ and the objects lying in each block in the case that $\mathscr{C}$ is the category of finite-dimensional (type 1) $U_q(\tilde{\mathfrak{g}})$-modules (which is obviously a Jordan-Hölder category).

Definition 3.3.6. The group $\mathscr{E}_q = \mathscr{P}/\mathscr{Q}_q$ is called the group of elliptic characters of $U_q(\tilde{\mathfrak{g}})$. Let $\epsilon_q : \mathscr{P} \to \mathscr{E}_q$ be the canonical projection. An object $V \in \mathscr{C}$ is said to have elliptic character $\gamma \in \mathscr{E}_q$ if $\mathrm{wt}_\ell(V) \subseteq \epsilon_q^{-1}(\gamma)$. Given $\gamma \in \mathscr{E}_q$, let $\mathscr{C}_\gamma$ be the full subcategory of $\mathscr{C}$ all of whose objects have elliptic character γ. ◊

Remark 3.3.7. We shall describe generators and relations for the abelian group $\mathscr{E}_q$ later on. It will then become clear that $\epsilon_q^{-1}(\gamma) \cap \mathscr{P}^+ \neq \emptyset$ for all $\gamma \in \mathscr{E}_q$. In particular, the subcategories $\mathscr{C}_\gamma$ are all nonempty. ◊

Notice that Proposition 2.3.3 implies:

Corollary 3.3.8. Let $\gamma, \delta \in \mathscr{E}_q$. If $V \in \mathscr{C}_\gamma$ and $W \in \mathscr{C}_\delta$, then $V \otimes W \in \mathscr{C}_{\gamma\delta}$. □

The following proposition is straightforward.

Proposition 3.3.9. The category $\mathscr{C}_\gamma$ is an abelian subcategory of $\mathscr{C}$ for every $\gamma \in \mathscr{E}_q$. □

The next theorem gives the promised description of the set of equivalence classes of linked indecomposable objects and the modules lying in each block in terms of ℓ-weights.

Theorem 3.3.10. The categories $\mathscr{C}_\gamma, \gamma \in \mathscr{E}_q$, are the blocks of $\mathscr{C}$.

Remark 3.3.11. Recall that if V and W are finite-dimensional simple $\mathfrak{g}$-modules (or $U_q(\mathfrak{g})$-modules) both not isomorphic to the trivial representation, then $V \otimes W$ is reducible and completely reducible. Therefore, by Example, 3.3.5, even though V and W belong to a block of the category of finite-dimensional $\mathfrak{g}$-modules, $V \otimes W$ does not belong to a block. In fact, examples of categories with the property described in Corollary 3.3.8 appear to be very rare. ◊

Theorem 3.3.10 is a consequence of the following proposition.

Proposition 3.3.12. Let V and W be objects in $\mathscr{C}$.

(i) If V is indecomposable, then $V \in \mathscr{C}_\gamma$ for some $\gamma \in \mathscr{E}_q$.

(ii) If V and W are simple and have the same elliptic character, then they are linked.

Proof of Theorem 3.3.10. Part (i) clearly implies that $\mathscr{C} = \bigoplus_{\gamma \in \mathscr{E}_q} \mathscr{C}_\gamma$. Therefore, we have found a splitting of $\mathscr{C}$ into a direct sum of full abelian subcategories. On the other hand, part (ii) implies that, for all $\gamma \in \mathscr{E}_q$, the subcategory $\mathscr{C}_\gamma$ is a full abelian subcategory of a block and, hence, must be indecomposable. By the uniqueness part of Proposition 3.3.3, this must be the block decomposition of $\mathscr{C}$. □

In order to prove Proposition 3.3.12(i) we will need the following theorem.

Theorem 3.3.13. If V is a finite-dimensional highest-ℓ-weight module of highest-ℓ-weight $\boldsymbol{\lambda} \in \mathscr{P}^+$ and $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V)$, then $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$.

By Proposition 2.3.3 and Corollary 3.2.7, it suffices to prove Theorem 3.3.13 in the case that V is a fundamental module. If $\mathfrak{g} = \mathfrak{sl}_n$, this follows from Example 3.1.6. For general $\mathfrak{g}$ the proof will be given in Subsection 4.2. However, since the proof is available for $\mathfrak{sl}_n$, we can use Theorem 3.3.13 in this case and, in fact, along the proof of its general case, we will make use of the $\mathfrak{sl}_2$ case. Notice that the Theorem 3.3.13 immediately implies:

Corollary 3.3.14. If V is a finite-dimensional highest-ℓ-weight module of highest-ℓ-weight $\boldsymbol{\lambda} \in \mathscr{P}^+$, then $V \in \mathscr{C}_{\epsilon_q(\boldsymbol{\lambda})}$. □

Let us show that Theorem 3.3.13 implies part (i) of Proposition 3.3.12.

Proof of Proposition 3.3.12(i). We claim that it suffices to show that if U and W have different elliptic characters, then every short exact sequence of the form $0 \to U \to V \to W \to 0$ splits. Indeed, assuming this, the proof of Proposition 3.3.12(i) is completed as follows. Let $V \in \mathscr{C}$ be indecomposable and consider a decomposition series for V: $\{0\} = V_0 \hookrightarrow V_1 \hookrightarrow \cdots \hookrightarrow V_l = V$. Let us prove by induction on l that V_j has elliptic character for all $1 \le j \le l$. If $l = 1$, then V is simple and we are done by Corollary 3.3.14. Thus, assume $l > 1$ and, by induction hypothesis, that V_{l-1} has elliptic character. Set $U = V_{l-1}$ and $W = V/U$ so that we have a short exact sequence as above. Since W is simple, it has elliptic character. Since V is indecomposable, the short exact sequence $0 \to U \to V \to W \to 0$ does not split and it follows from our claim the elliptic characters of U and W must be the same. This implies V has this same elliptic character.

To prove the claim, let us first consider the case that both U and W are simple. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ denote their highest ℓ-weights, respectively. By turning to the dual sequence if necessary, we can assume that $\mathrm{wt}(\boldsymbol{\mu})$ is not strictly smaller than $\mathrm{wt}(\boldsymbol{\lambda})$. Notice that this implies that $x_{i,r}^+ V_{\boldsymbol{\mu}} = \{0\}$ for all $i \in I$ and $r \in \mathbb{Z}$. In particular, there must be a highest-ℓ-weight vector in $V_{\boldsymbol{\mu}}$. Since $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$ (otherwise the elliptic characters of U and W would be the same), it follows that this vector must be in W. If the sequence did not split, then the submodule of V generated by W would be all of V and, hence, V would be a highest-ℓ-weight module. But then, Corollary 3.3.14 would contradict the assumption that the elliptic characters of U and W are different.

The inductive step for proving the claim is actually generic, i.e., it does not depend on which Jordan-Hödler category we are working at. Namely, suppose we know that $\mathscr{C}_1$ and $\mathscr{C}_2$ are full abelian subcategories of Jordan-Hölder category $\mathscr{C}$ satisfying: for every two simple objects $U \in \mathscr{C}_1$ and $W \in \mathscr{C}_2$, a short exact sequence of the form $0 \to U \to V \to W \to 0$ splits. Then, every short exact sequence of the form $0 \to U \to V \to W \to 0$ with $U \in \mathscr{C}_1$ and $W \in \mathscr{C}_2$ splits. The proof goes by induction on the length of V. Let $\pi : V \to W$ be the projection having U as kernel. We again use induction on the length of V. Let us first show that the induction hypothesis indeed brings us to the analysis of modules of length 2 and only then show that induction starts when the length is 2. We begin showing that we can reduce to the case that W is simple. If that was not the case, let W' be a proper nonzero submodule of W and consider $V' = \pi^{-1}(W')$. Then, U is a proper submodule of V' and we have a short exact sequence $0 \to U \to V' \to W' \to 0$. Since, W has elliptic character, all of its submodules have the same elliptic character, the sequence $0 \to U \to V' \to W' \to 0$ splits by the induction hypothesis. Therefore, $V' \cong U \oplus W'$ and we obtain a short exact sequence $0 \to U \oplus W' \to V \to W/W' \to 0$. This in turn gives rise the sequence $0 \to U \to V/W' \to W/W' \to 0$ which splits by induction hypothesis. But this implies that the sequence $0 \to U \to V \to W \to 0$. An obvious sub-induction on the length of W shows that we can assume W is simple. Similarly, we can use the induction hypothesis to reduce to the case that U is also simple. $\square$

Even for $\mathfrak{sl}_n$, with $n > 3$, there are still a few extra results needed to prove Proposition 3.3.12(ii) which we have not developed so far. Therefore, we restrict ourselves to $n = 2$ for the time being. This is all that we will need for developing the results of the next section which, in particular, will imply the general case of Theorem 3.3.13. Thus, for the remainder of this subsection, we assume $\mathfrak{g} = \mathfrak{sl}_2$ and let i denote the unique element of I. Recall that, in this case, $\boldsymbol{\alpha}_{i,a} = \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{i,aq^2} \in \mathscr{P}^+$.

Lemma 3.3.15. Let $\boldsymbol{\lambda} \in \mathscr{P}^+$ and $a \in \mathbb{F}^\times$. Then, $V_q(\boldsymbol{\lambda})$ is a simple factor of both $W_q(\boldsymbol{\lambda})$ and $W_q(\boldsymbol{\lambda}\boldsymbol{\alpha}_{i,a})$.

Proof. In Example 2.3.25 we have seen that there exists a short exact sequence $0 \to V_q(\mathbf{1}) \to$

$W_q(\boldsymbol{\alpha}_{i,a}) \to V_q(\boldsymbol{\alpha}_{i,a}) \to 0$. Therefore, $V_q(\boldsymbol{\lambda})$ is a common simple factor of $W_q(\boldsymbol{\lambda})$ and $W_q(\boldsymbol{\lambda}) \otimes W_q(\boldsymbol{\alpha}_{i,a})$. On the other hand, we know that $W_q(\boldsymbol{\alpha}_{i,a}) \cong V_q(\boldsymbol{\omega}_{i,aq^2}) \otimes V_q(\boldsymbol{\omega}_{i,a})$. Moreover, by Corollary 3.2.7,

$$W_q(\boldsymbol{\lambda}) \cong V_q(\boldsymbol{\omega}_{i,a_1}) \otimes \cdots \otimes V_q(\boldsymbol{\omega}_{i,a_m})$$

for some $m \geq 0$ and $a_1, \ldots, a_m \in \mathbb{F}^\times$. Therefore,

$$W_q(\boldsymbol{\lambda}) \otimes W_q(\boldsymbol{\alpha}_{i,a}) \cong V_q(\boldsymbol{\omega}_{i,a_1}) \otimes \cdots \otimes V_q(\boldsymbol{\omega}_{i,a_m}) \otimes V_q(\boldsymbol{\omega}_{i,aq^2}) \otimes V_q(\boldsymbol{\omega}_{i,a}). \tag{3.3.3}$$

Again, by Corollary 3.2.7, $W_q(\boldsymbol{\lambda}\boldsymbol{\alpha}_{i,a})$ is a tensor product of fundamental modules and, hence, it is isomorphic to a re-ordering of the tensor product on the right-hand-side of (3.3.3). Hence, $\mathrm{qch}(W_q(\boldsymbol{\lambda}\boldsymbol{\alpha}_{i,a})) = \mathrm{qch}(W_q(\boldsymbol{\lambda}) \otimes W_q(\boldsymbol{\alpha}_{i,a}))$ and the lemma follows. □

Proof of Proposition 3.3.12(ii). Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^+$ determine the same element of $\mathscr{E}_q$. Then, there exist $m, n \geq 0, a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{F}^\times$ such that

$$\boldsymbol{\lambda} \prod_{j=1}^{m} \boldsymbol{\alpha}_{i,a_j} = \boldsymbol{\mu} \prod_{j=1}^{n} \boldsymbol{\alpha}_{i,b_j}.$$

Consider the following two sequences of elements of $\mathscr{P}^+$:

$$\boldsymbol{\lambda}_r = \boldsymbol{\lambda} \prod_{j=1}^{r} \boldsymbol{\alpha}_{i,a_j} \quad \text{and} \quad \boldsymbol{\mu}_s = \boldsymbol{\mu} \prod_{j=1}^{s} \boldsymbol{\alpha}_{i,b_j} \quad \text{for} \quad 0 \leq r \leq m, 0 \leq s \leq n.$$

Clearly, $V_q(\boldsymbol{\lambda})$ is a simple factor of $W_q(\boldsymbol{\lambda}_0)$, $V_q(\boldsymbol{\mu})$ is a simple factor of $W_q(\boldsymbol{\mu}_0)$, and $W_q(\boldsymbol{\lambda}_m) = W_q(\boldsymbol{\mu}_n)$. Then, since the local Weyl modules are indecomposable, by Lemma 3.3.2, it suffices to show that $W_q(\boldsymbol{\lambda}_r)$ and $W_q(\boldsymbol{\lambda}_{r-1})$ have a common simple factor for every $1 \leq r \leq m$ and similarly for the sequence $\boldsymbol{\mu}_s$. But this follows from Lemma 3.3.15. □

3.4. Bibliographical notes

1. Duality

Proposition 3.1.2 was proved originally by V. Chari in [12]. The connection with the theory of general ℓ-weights and, in particular, with the theory of qcharacters was not considered in that paper and, therefore, Corollaries 3.1.3 and 3.1.4 were not announced there. Evidently, they easily follow from that work as seen above (both Corollaries were announced in [19]). An alternate proof of Corollary 3.1.3 was given in [15, §2.10]. Corollary 3.1.7 is also immediate from the results of [12]. However, the formula for the Drinfeld polynomial of the dual modules was known before the action of the braid group on the ℓ-weight lattice was considered. Namely, it was proved in [26] that there existed $m \in \mathbb{Z}$ such that $\lambda_i^*(u) = \lambda_{w_0(i)}(q^m u)$ for all $i \in I$. On the other hand, the Drinfeld polynomial of the double dual $(V_q(\boldsymbol{\lambda})^*)^*$ was already known from an explicit formula for S^2 on an enlargement of $U_q(\hat{\mathfrak{g}})$ due to Drinfeld [30]. Combining these results one gets the formula for $\boldsymbol{\lambda}^*$ given in Remark 3.1.8 without using the braid group at all. The same formula was recovered in [36] using consequences of the fact that the Frenkel-Mukhin algorithm can be used to compute the qcharacter of fundamental representations.

2. Tensor Products

Theorem 3.2.5 was the main result of V. Chari's work [12] where Lemma 3.2.2 was also proved. The condition of q-resonant ordering can be made into a very explicit description of the finite set for the the values of a/b for which the tensor product $V_q(\omega_{i,a,m}) \otimes V_q(\omega_{i,b,n})$ may fail to be a highest-ℓ-weight module (see [12, Section 6]). Therefore, outside this set and the one for $V_q(\omega_{i,b,n}) \otimes V_q(\omega_{i,a,m})$, these tensor products are isomorphic and, hence, a simple module. Using this, Chari obtained proofs of conjectures made in [1, 41, 63] on the irreducibility of tensor products of Kirillov-Reshetikhin modules. The same results were also proved by E. Frenkel and E. Mukhin in [36] using the Frenkel-Mukhin algorithm for qcharacters which we will study in Section 4.

As seen above, the proof of the general case of Theorem 3.2.5 uses its $\mathfrak{sl}_2$-version – the first statement of Proposition 3.2.11 – which was essentially obtained by Chari and Pressley in [24]. The statement from [24] had the additional hypothesis that $r_1 \leq r_2 \leq \cdots \leq r_k$. Notice that, in this case, q-resonant ordering is the same as weak q-resonant ordering. This statement was needed in [24] in order to prove Theorem 2.3.20. The proof of Theorem 2.3.20 we presented above has a point where it differs from the original one since we used the second statement of Proposition 3.2.11 which was not stated in [24]. We used this in the proof of Theorem 2.3.20 in order to be able to change the order of the tensor products. The proof of the second statement of Proposition 3.2.11 was taken from an earlier version of [58] which is available in the ArXiv.

Some remarks regarding the terminologies "general position" and "q-resonant order" are due. The first appearance of the term "general position" was in [24] meaning exactly what we mean here. However, the same term was used in [12] to mean what we are calling q-resonant order. Since "general position" gives the idea of something which does not depend on ordering, we kept this term for its original meaning. The choice of the term q-resonant order was made in our joint work with D. Jakelić [58] where the term weak q-resonant order was created. The choice is based on the term "resonant order" used in [31] for a related, but not exactly the same, concept. Namely, in [31], a tuple of fundamental ℓ-weights $(\omega_{i_1,a_1}, \ldots, \omega_{i_m,a_m})$ was said to be in resonant order if the corresponding tensor product of fundamental representations is a highest-ℓ-weight module. Therefore, Theorem 3.2.5 says that the present definition of q-resonant order is a sufficient condition for the tuple $(\omega_{i_1,a_1}, \ldots, \omega_{i_m,a_m})$ to be in resonant order in the sense of [31]. The definition of resonant ordering was extended to the root of unity setting in [58] and the corresponding version Theorem 3.2.5 was also obtained.

Corollary 3.2.7 which is essentially a consequence of Theorem 3.2.5 and Corollary 2.4.1 was first announced in [19]. Recall from Subsection 2.4 that Corollary 2.4.1 is related to the study of the character or dimension of the Weyl modules. Therefore, several results related to this study are indirectly being used in the proof of Corollary 3.2.7.

In this section we were mostly concerned with the question "when is the tensor product of simple modules a highest-ℓ-weight module". Another interesting question is "when is the tensor product of simple modules a simple module as well?". In the $\mathfrak{sl}_2$ case, this is answered by Theorem 2.3.20 and Corollary 2.3.21. The answer can be stated as the following corollary.

Corollary 3.4.1. Let $\mathfrak{g} = \mathfrak{sl}_2$ and suppose $V_1, \cdots, V_m$ are simple $U_q(\tilde{\mathfrak{g}})$-modules. Then, $V_1 \otimes \cdots \otimes V_m$ is simple if and only if $V_r \otimes V_s$ is simple for all $r \neq s$. In particular, the tensor powers of a simple module is simple. $\square$

It has been recently proved by D. Hernandez in [50] that we can remove the assumption $\mathfrak{g} = \mathfrak{sl}_2$

from the first statement of the above corollary (but not from the second!). The works [50, 51] have drawn attention to a very important class of representations which was not much studied until then – that of prime representations. A prime representation is a simple representation which is not isomorphic to a tensor product of other two nontrivial representations. Evidently, every simple representation must be a tensor product of prime representations. The classification of prime representations and the description of their qcharacters is one of the main questions to be answered in the direction of understanding the tensor structure of the category. It follows from Theorem 2.3.20 and Corollary 2.3.21 that, if $\mathfrak{g} = \mathfrak{sl}_2$, a representation is prime if and only if it is an evaluation module. It is known that all minimal affinizations in the sense defined by Chari in [10] are prime (see [15, §7.1]), but there are examples of prime representations which are not minimal affinizations. As we mentioned earlier, there exists quantum analogues of evaluation maps when $\mathfrak{g} = \mathfrak{sl}_n$ and, therefore, all minimal affinizations are evaluation modules in this case. One may then expect that every simple finite-dimensional $U_q(\tilde{\mathfrak{sl}}_n)$-module is a tensor product of evaluation representations just as in the $\mathfrak{sl}_2$ case (Theorem 2.3.20). However, this is not true. Using the relation with cluster algebras, Hernandez and Leclerc gave an example of a prime representation which is not an evaluation module in the case that $\mathfrak{g} = \mathfrak{sl}_3$ – its Drinfeld polynomial is $\boldsymbol{\omega}_{i,aq,2}\boldsymbol{\omega}_{j,aq^4,2}$ (where i and j are the two distinct elements of I) and its simple factors as $U_q(\mathfrak{g})$-module are $V_q(2\theta)$ and $V_q(\theta)$ both with multiplicity one. The existence of such example is another reason for the theory of qcharacter to be so much more intricate in the quantum setting in comparison with its almost trivial description in the classical context. By Proposition 2.3.3, the description of the qcharacters of the prime representations completes one possible way of describing the qcharacters of all simple modules. As we remarked in [58], Theorem 3.2.5 provides an algorithm for deciding if the simple module associated to a given dominant ℓ-weight $\boldsymbol{\lambda}$ is not prime. Namely, if $\boldsymbol{\lambda} = \boldsymbol{\mu}\boldsymbol{\nu}$ for some nontrivial $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathscr{P}^+$ such that both $(\boldsymbol{\mu}, \boldsymbol{\nu})$ and $(\boldsymbol{\nu}, \boldsymbol{\mu})$ are in q-resonant order, then $V_q(\boldsymbol{\lambda})$ is isomorphic to $V_q(\boldsymbol{\mu}) \otimes V_q(\boldsymbol{\nu})$ and, hence, is not prime. This was also remarked by Chari in a talk whose video is available online [13]. At some point, it should be interesting to try to turn such algorithm into some explicit sufficient condition for non-primality.

3. Blocks

Theorem 3.3.13 implies that the general quantum version of Corollary 2.3.16 holds. Recall that the corresponding classic result saying that the weights of highest-weight module are smaller than the highest weight is trivially established. The reader is already noticing that the proof of the ℓ-analogue of this (Theorem 3.3.13) requires a great deal of more work. So far we have proved it only when $\mathfrak{g} = \mathfrak{sl}_n$. The general case will follow from the results of Subsection 4.2 which require some results on the combinatorial aspects of the theory of qcharacters to be developed in Subsection 4.1. It turns out that the results needed to complete the proof of Theorem 3.3.13 have a strong relation with the block decomposition of the category of finite-dimensional $U_q(\tilde{\mathfrak{sl}}_2)$-modules. We will leave the bibliographical notes regarding the proof of Theorem 3.3.13 to be given in Subsection 4.4.

The description of the blocks of the category of finite-dimensional $U_q(\tilde{\mathfrak{g}})$-modules was originally obtained in a joint work with by P. Etingof [31]. However, the definition of elliptic characters was very different than the one presented here. The original definition (under the name of elliptic central characters by analogy with the fact that the blocks of category $\mathscr{O}$ are given by central characters) was in terms of the action of the R-matrix of $U_q(\tilde{\mathfrak{g}})$ on certain tensor products. Because of this, the original proof of Theorem 3.3.10 actually required the assumption that $\mathbb{F} \subseteq \mathbb{C}$ and q is not in the unity circle! On the other hand, it had the advantage that Proposition 3.3.12(i) followed essentially from the definition of elliptic characters. In particular, there was no need of the knowledge of Weyl modules which were not mentioned at all in [31]. The version presented here is taken from a joint work with V. Chari [19], where the connection of elliptic characters and qcharacters was established (the term “central” was dropped from the terminology due to the lack of an actual meaning and qcharacters were renamed ℓ-characters in [19] – a terminology abandoned due to historical reasons as mentioned previously). In a joint work with D. Jakelić [58], we showed that the theory of limits $q \to \xi, \xi \in \mathbb{F}^\times$, can be used to extend Theorem 3.3.10 to the root-of-unity setting (in Lusztig's sense). This also works for the limit $q \to 1$ which then describes the blocks of the category of finite-dimensional $\tilde{\mathfrak{g}}$-modules which was previously obtained in [18]. The elliptic characters were renamed spectral characters in the $q \to 1$ limit since their elliptic nature degenerates into a spectral nature (depends only on the spectral parameters of the dominant ℓ-weights of the module). Let us explain what this elliptic nature is in the $\mathfrak{sl}_2$ case. To make things clearer, we need the following lemma.

Lemma 3.4.2. Let $\mathfrak{g} = \mathfrak{sl}_2$. The group $\mathscr{E}_q$ is isomorphic to the abelian group with generators $\gamma_a, a \in \mathbb{F}^\times$, satisfying the defining relations $\gamma_a\gamma_{aq^2} = 1$ for all $a \in \mathbb{F}^\times$.

Proof. Denote by $\mathscr{G}$ the group defined by the aforementioned generators and relations. Notice that there exists a unique surjective homomorphism $\mathscr{G} \to \mathscr{E}_q$ such that $\gamma \mapsto \epsilon_q(\omega_{i,a})$. This is clear from the fact that $\alpha_{i,a} = \omega_{i,a}\omega_{i,aq^2}$. Injectivity follows easily from the fact that $\mathbb{F}[u]$ is a unique factorization domain. □

In particular, we have $\gamma_{aq^4} = \gamma_{aq^4}(\gamma_{aq^2}\gamma_a) = (\gamma_{aq^4}\gamma_{aq^2})\gamma_a = \gamma_a$. It follows that the parameterizing set of distinct generators of the group of elliptic characters is the elliptic curve $\mathbb{F}^\times/q^{4\mathbb{Z}}$. Notice that when $q \to 1$ the elliptic curve degenerates to $\mathbb{F}^\times$ and the group of elliptic characters degenerates to the group of functions $\gamma : \mathbb{F}^\times \to P/Q$ with finite-support. We shall obtain the general version of Lemma 3.4.2 below, from where we will see that the parameterizing set of distinct generators of the group of elliptic characters is the elliptic curve $\mathbb{F}^\times/q^{2r^\vee h^\vee \mathbb{Z}}$, except in the case of algebras of type D_n with n even which is a little more complicated (the complication degenerates in the $q \to 1$ limit to the fact that $P/Q \cong \mathbb{Z}_4$ if n is odd and $P/Q \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ if n is even). Finally, observe that $\mathbb{F}^\times$ is in bijection with the set of maximal ideals of $\mathbb{F}[t, t^{-1}]$. Therefore, in the context of the algebras of the form $\mathfrak{g} \otimes A$, one should expect, after what we commented in Subsection 2.4, that the blocks are parameterized by functions γ : specm$(A) \to P/Q$ with finite-support. This agrees with the results of R. Kodera [66] where the first Ext groups between simple modules was obtained. The results of Kodera were generalized to the context of equivariant map algebras very recently by E. Neher and A. Savage in [80]. It is interesting to notice that [80] avoids the use of Weyl modules in the proof of the analogue of Theorem 3.3.10. It should be interesting to transport this method to the quantum setting making the argument free of Corollary 3.2.7 and, therefore, free of the use of global crystal basis results (recall that the proof of Corollary 3.2.7 depends on Corollary 2.4.1 whose available proofs for general $\mathfrak{g}$ depend on crystal arguments). We also remark that the theory of limits $q \to \xi$ can be used alongside the theory of reduction modulo p to obtain results in positive characteristic. We shall describe this in a more precise manner in a joint work with D. Jakelić which is ongoing (see also [55] for the case of hyper loop algebras).

4. Algorithms for qcharacters

We now return to the theory of qcharacters and study a few algorithms designed to compute the qcharacter of certain classes of simple $U_q(\tilde{\mathfrak{g}})$-modules. We begin with the first to be proposed, the Frenkel-Mukhin algorithm, and present the proof that it works for ℓ-minuscule modules. We then present the class of Kirillov-Reshetikhin modules which are all ℓ-minuscule and give a few other examples where the algorithm works and also one for which it does not. We also present an algorithm for computing the qcharacter of fundamental representations in terms of the braid group action on the ℓ-weight lattice which works in the case that the underlying simple Lie algebra is not of exceptional type.

4.1. The Frenkel-Mukhin algorithm

The goal of this subsection is to describe the first algorithm proposed for computing the qcharacter of simple modules $V_q(\boldsymbol{\lambda})$ - the Frenkel-Mukhin algorithm. It is essentially an ℓ-analogue of a classic algorithm for computing the character of minuscule modules. The main result of the subsection is a sufficient condition for the algorithm to actually return the correct desired qcharacter. In particular, this condition is satisfied by all minuscule ℓ-weights. Our description of the algorithm makes use of the block decomposition of the category of finite-dimensional $U_q(\tilde{\mathfrak{sl}}_2)$-modules described in Subsection 3.3.

Definition 4.1.1. Let $J \subseteq I$. Denote by $\mathscr{P}_J$ the subgroup of $\mathscr{P}^+$ generated by $\boldsymbol{\omega}_{j,a}, j \in J, a \in \mathbb{F}^\times$ and set $\mathscr{P}_J^+ = \mathscr{P}^+ \cap \mathscr{P}_J$. Similarly, let $\mathscr{Q}_J$ be the subgroup generated by $\boldsymbol{\alpha}_{j,a}, j \in J, a \in \mathbb{F}^\times$ and $\mathscr{Q}_J^\pm = \mathscr{Q}_q^\pm \cap \mathscr{Q}_J$. The unique group homomorphism $\rho_J : \mathscr{P} \to \mathscr{P}_J$ which is the identity on $\mathscr{P}_J$ and has $\mathscr{P}_{I\setminus J}$ as its kernel is called the J-restriction homomorphism. An element $\boldsymbol{\mu}$ is called J-dominant in $\rho_J(\boldsymbol{\mu}) \in \mathscr{P}_J^+$. Denote by $\mathscr{P}_{J,+}$ the set of all J-dominant ℓ-weights. Set also $\mathscr{E}_J = \mathscr{P}/\mathscr{Q}_J$ and let $\epsilon_J : \mathscr{P} \to \mathscr{E}_J$ be the canonical projection. Let $\tau_J : \mathscr{P} \to \mathscr{P}_J \times \mathscr{E}_J$ be the homomorphism $\tau_J(\boldsymbol{\mu}) = (\rho_J(\boldsymbol{\mu}), \epsilon_J(\boldsymbol{\mu}))$. We shall refer to $\mathscr{E}_J$ as the group of J-elliptic characters of $U_q(\tilde{\mathfrak{g}})$. ◊

Remark 4.1.2. The subgroup $\mathscr{P}_J$ can be naturally identified with the ℓ-weight lattice of $U_q(\tilde{\mathfrak{g}}_J)$ and similarly for the subgroup $\mathscr{Q}_J$. Under the aforementioned identification, the group $\mathscr{P}_J/\mathscr{Q}_J$ gets identified with the group of elliptic characters of $U_q(\tilde{\mathfrak{g}}_J)$. Notice that this is not the same as the group of J-elliptic characters $\mathscr{E}_J$, unless $J = I$. When $J = \{i\}$ for some $i \in I$, we shall use the shortened notations $\mathscr{P}_i, \mathscr{Q}_i, \tau_i$, and so on. Notice that, under the identification of $\mathscr{P}_i$ with the ℓ-weight lattice of $U_{q_i}(\tilde{\mathfrak{sl}}_2)$, $\rho_i(\boldsymbol{\mu})$ is identified with the rational function $\boldsymbol{\mu}_i(u)$. Thus, we shall write $\boldsymbol{\mu}_i(u)$ or simply $\boldsymbol{\mu}_i$ instead of $\rho_i(\boldsymbol{\mu})$ when convenient. ◊

Lemma 4.1.3. Let $J \subseteq I$ and $\gamma \in \mathscr{E}_J$. Then, τ_J and the restriction of ρ_J to $\epsilon_J^{-1}(\gamma)$ are injective.

Proof. Suppose that $\boldsymbol{\mu} \in \ker(\tau_J)$. In particular, $\boldsymbol{\mu} \in \ker(\epsilon_J)$ and, therefore, $\boldsymbol{\mu} \in \mathscr{Q}_J$. Identifying $\mathscr{P}_J$ and $\mathscr{Q}_J$ with the corresponding ℓ-lattices of $U_q(\tilde{\mathfrak{g}}_J)$, it follows from Lemma 2.3.8 that $\mathscr{Q}_J$ is freely generated by $\rho_J(\boldsymbol{\alpha}_{j,a}), j \in J, a \in \mathbb{F}^\times$. Therefore, since $\boldsymbol{\mu} \in \ker(\rho_J)$ and ρ_J is the identity on $\mathscr{P}_J$, it follows that $\boldsymbol{\mu} = \mathbf{1}$.

Now suppose $\boldsymbol{\mu}, \boldsymbol{\nu} \in \epsilon_J^{-1}(\gamma)$ and $\rho_J(\boldsymbol{\mu}) = \rho_J(\boldsymbol{\nu})$. It follows that $\tau_J(\boldsymbol{\mu}) = \tau_J(\boldsymbol{\nu})$ and, hence, $\boldsymbol{\mu} = \boldsymbol{\nu}$. □

Proposition 4.1.4. Let V be a finite-dimensional $U_q(\tilde{\mathfrak{g}})$-module, $i \in I$, and $\gamma \in \mathscr{E}_i$. Then,

$$V_\gamma = \bigoplus_{\mu \in \epsilon_i^{-1}(\gamma)} V_{\boldsymbol{\mu}}$$

is a $U_q(\tilde{\mathfrak{g}}_i)$-submodule of V. In particular, $V = \bigoplus_{\delta \in \mathscr{E}_i} V_\delta$.

Proof. Recall the relation, $[h_{j,s}, x_{i,r}^{\pm}] = \pm\frac{1}{s}\,[sc_{ji}]_{q_j} x_{i,r+s}^{\pm}, i, j \in I, r, s \in \mathbb{Z}, s \neq 0$. Given $j \in I$, consider $\tilde{h}_{j,s} = h_{j,s} - \frac{[sc_{ji}]_{q_j}}{[2s]_{q_i}}\, h_{i,s}$, and notice $[\tilde{h}_{j,s}, x_{i,r}^{\pm}] = 0$ for all $j \in I, r, s \in \mathbb{Z}, s \neq 0$. Consider also the subalgebra $U_i^{\perp}$ of $U_q(\tilde{\mathfrak{h}})$ generated by $\tilde{h}_{j,s}, j \neq i, s \neq 0$, and notice that $U_q(\tilde{\mathfrak{h}})$ is generated by $U_i^{\perp}$ together with $U_q(\tilde{\mathfrak{h}}_i) = U_q(\tilde{\mathfrak{h}}) \cap U_q(\tilde{\mathfrak{g}}_i)$ and $U_q(\mathfrak{h})$. Moreover, the elements of $U_i^{\perp}$ commute with those of $U_q(\tilde{\mathfrak{g}}_i)$ by definition.

Since V is finite-dimensional, we can write $V = \bigoplus_{n=1}^{m} V_n$ where each V_n is a generalized eigenspace for the action of the elements of $U_i^{\perp}$. Notice that, since the elements of $U_i^{\perp}$ commute with those of $U_q(\tilde{\mathfrak{g}}_i)$, each of the subspaces V_n is a $U_q(\tilde{\mathfrak{g}}_i)$-submodule of V. Let

$$V_n = \bigoplus_{r=1}^{l_n} V_{n,l}$$

be the block decomposition of the $U_q(\tilde{\mathfrak{g}}_i)$-module V_n. Notice that, since $U_q(\tilde{\mathfrak{h}})$ is commutative, we have

$$V_{n,l} = \bigoplus_{\mu \in \mathscr{P}} V_{n,l} \cap V_{\boldsymbol{\mu}} \qquad \text{for all} \qquad n, l.$$

Set $\mathrm{wt}_\ell(V_{n,l}) = \{\boldsymbol{\mu} \in \mathscr{P} : V_{n,l} \cap V_{\boldsymbol{\mu}} \neq \{0\}\}$. By the $\mathfrak{sl}_2$ case of Theorem 3.3.10, the i-th rational functions of the ℓ-weights $\boldsymbol{\mu}$ lying in $\mathrm{wt}_\ell(V_{n,l})$ can be obtained from each other by successive multiplication by elements of the form $\rho_i(\boldsymbol{\alpha}_{i,a}^{\pm 1})$ with $a \in \mathbb{F}^{\times}$. Let $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathrm{wt}_\ell(V_{n,l})$ for some n, l, be such that $\rho_i(\boldsymbol{\mu}) = \rho_i(\boldsymbol{\nu})\rho_i(\boldsymbol{\alpha}_{i,a})$ for some $a \in \mathbb{F}^{\times}$. We claim that, in this case, we must have $\boldsymbol{\mu} = \boldsymbol{\nu}\boldsymbol{\alpha}_{i,a}$. Assuming this, it follows that $V_{n,l} \subseteq V_\gamma$ where $\gamma = \epsilon_i(\boldsymbol{\mu})$, which completes the proof.

Let v be an eigenvector for the action of $U_q(\tilde{\mathfrak{h}})$ associated to $\boldsymbol{\nu}$ and w be one associated to $\boldsymbol{\mu}$. We now perform a computation similar to that at end of the proof of Proposition 3.1.2. Let $\mu_{j,s}$ and $\nu_{j,s}$ be the eigenvalues of the actions of $h_{j,s}$ on w and v, respectively. Then,

$$\left(\sum_{r\geq 0} \Lambda_{j,r} u^r\right) w = \exp\left(-\sum_{s>0} \frac{h_{j,s}}{[s]_{q_j}}\, u^s\right) w = \exp\left(-\sum_{s>0} \frac{\mu_{j,s}}{[s]_{q_j}}\, u^s\right) w = \boldsymbol{\mu}_j(u)\, w$$

and

$$\left(\sum_{r\geq 0} \Lambda_{j,r} u^r\right) v = \exp\left(-\sum_{s>0} \frac{h_{j,s}}{[s]_{q_j}}\, u^s\right) v = \exp\left(-\sum_{s>0} \frac{\nu_{j,s}}{[s]_{q_j}}\, u^s\right) v = \boldsymbol{\nu}_j(u)\, v.$$

For $j = i$, we know that $\boldsymbol{\mu}_i(u) = \boldsymbol{\nu}_i(u)(1 - au)(1 - aq_i^2 u)$ (since $\rho_i(\boldsymbol{\alpha}_{i,a}) = \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{i,aq_i^2}$). Therefore,

$$\begin{aligned}\exp\left(-\sum_{s>0} \frac{\mu_{i,s}}{[s]_{q_i}}\, u^s\right) &= \exp\left(-\sum_{s>0} \frac{\nu_{i,s}}{[s]_{q_i}}\, u^s\right) \exp\left(\ln\left((1 - au)(1 - aq_i^2 u)\right)\right) \\ &= \exp\left(-\sum_{s>0} \left(\frac{\nu_{i,s}}{[s]_{q_i}} + \frac{a^s(1 + q_i^{2s})}{s}\right) u^s\right).\end{aligned}$$

In other words,

$$\mu_{i,s} = \nu_{i,s} + a^s \frac{[s]_{q_i}}{s}(1+q_i^{2s}) = \nu_{i,s} + (aq_i)^s \frac{[2s]_{q_i}}{s} \qquad \text{for all} \qquad s \in \mathbb{Z}, s \neq 0. \tag{4.1.1}$$

Recall that V_n is a generalized eigenspace for the action of $U_i^{\perp}$. Therefore, v and w must be eigenvectors for the action $\tilde{h}_{j,s}$ associated to the same eigenvalue, say $\lambda_{j,s}$. This implies,

$$h_{j,s}w = \left(\tilde{h}_{j,s} + \frac{[sc_{ji}]_{q_j}}{[2s]_{q_i}} h_{i,s}\right) w = \left(\lambda_{j,s} + \frac{[sc_{ji}]_{q_j}}{[2s]_{q_i}}\mu_{i,s}\right) w = \mu_{j,s}\, w$$

and

$$h_{j,s}v = \left(\tilde{h}_{j,s} + \frac{[sc_{ji}]_{q_j}}{[2s]_{q_i}} h_{i,s}\right) v = \left(\lambda_{j,s} + \frac{[sc_{ji}]_{q_j}}{[2s]_{q_i}}\nu_{i,s}\right) v = \nu_{j,s}\, w.$$

Using (4.1.1) we get,

$$\begin{aligned}
\boldsymbol{\mu}_j(u) &= \exp\left(-\sum_{s>0}\left(\frac{\lambda_{j,s}}{[s]_{q_j}} + \frac{[sc_{ji}]_{q_j}}{[s]_{q_j}[2s]_{q_i}}\mu_{i,s}\right)u^s\right) \\
&= \exp\left(-\sum_{s>0}\left(\frac{\lambda_{j,s}}{[s]_{q_j}} + \frac{[sc_{ji}]_{q_j}}{[s]_{q_j}[2s]_{q_i}}\left(\nu_{i,s} + (aq_i)^s \frac{[2s]_{q_i}}{s}\right)\right)u^s\right) \\
&= \boldsymbol{\nu}_j(u)\, \exp\left(-\sum_{s>0}\left((aq_i)^s \frac{[sc_{ji}]_{q_j}}{s[s]_{q_j}}\right)u^s\right) = \boldsymbol{\nu}_j(u)\, \exp\left(\sum_{r=0}^{-1-c_{ji}}\sum_{s>0}\frac{1}{s}\left(aq_iq_j^{-c_{ji}-1-2r}u\right)^s\right) \\
&= \boldsymbol{\nu}_j(u) \prod_{r=0}^{-1-c_{ji}}\left(1 - aq^{s_i - s_j(c_{ji}+1+2r)}u\right)^{-1} = (\boldsymbol{\nu}\boldsymbol{\omega}^{-1}_{j,aq_i,-c_{ji}})_j(u).
\end{aligned}$$

Finally, a quick glance at Definition 2.3.6 and we see that $\boldsymbol{\mu} = \boldsymbol{\nu}\boldsymbol{\alpha}_{i,a}$. □

Definition 4.1.5. Let V be a finite-dimensional $U_q(\tilde{\mathfrak{g}})$-module and $i \in I$. The decomposition $V = \bigoplus_{\gamma \in \mathscr{E}_i} V_\gamma$ will be referred to as the *i*-th elliptic decomposition of V. ◊

Remark 4.1.6. Assuming Theorem 3.3.10, the above proposition and definition can be easily generalized for any subset J of I. Notice that the i-th elliptic decomposition of V is not the same as the i-th block decomposition of V (the block decomposition of V when regarded as a $U_q(\tilde{\mathfrak{g}}_i)$-module). Indeed, identifying the group of elliptic characters of $U_q(\tilde{\mathfrak{g}}_i)$ with the subgroup $\mathscr{E}^i = \mathscr{P}_i/\mathscr{Q}_i$ of $\mathscr{E}_i$ (the image of $\mathscr{P}_i$ in $\mathscr{E}_i$ under ϵ_i), the i-th block decomposition of V can be described as follows. Given $\delta \in \mathscr{E}^i$, the block of V associated to δ is

$$V_\delta = \bigoplus_{\boldsymbol{\mu} \in \epsilon_i^{-1}(\delta)} V_{\boldsymbol{\mu}}.$$

Notice that if $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathscr{P}$ are such that $\epsilon_i(\boldsymbol{\mu}) = \epsilon_i(\boldsymbol{\nu})$, then $V_{\boldsymbol{\mu}}$ and $V_{\boldsymbol{\nu}}$ are subspaces of the same block of the $U_q(\tilde{\mathfrak{g}}_i)$-module V. In other words, given $\gamma \in \mathscr{E}_i$, V_γ is contained in a block. However, one can easily produce an example of $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathrm{wt}_\ell(V)$ for some $U_q(\tilde{\mathfrak{g}})$-module V such that $V_{\boldsymbol{\mu}}$ and $V_{\boldsymbol{\nu}}$ are in the same block of the $U_q(\tilde{\mathfrak{g}}_i)$-module V, but $\epsilon_i(\boldsymbol{\mu}) \neq \epsilon_i(\boldsymbol{\nu})$. Therefore, the i-th elliptic decomposition of V is finer than its i-th block decomposition. ◊

Let $i \in I$. By Theorem 3.3.13, given $\boldsymbol{\mu} \in \mathscr{P}_{i,+}$, the qcharacter of $V_q(\boldsymbol{\mu}_i)$ is of the form

$$\text{(4.1.2)} \qquad \mathrm{qch}(V_q(\boldsymbol{\mu}_i)) = \boldsymbol{\mu}_i \left(1 + \sum_{j=1}^{m} \rho_i(\eta_j)\right) \qquad \text{for some unique} \qquad m \geq 0, \eta_j \in \mathscr{Q}_i^-, j = 1, \ldots, m.$$

Set

$$\text{(4.1.3)} \qquad \zeta_{\boldsymbol{\mu}}^i = \boldsymbol{\mu}\left(1 + \sum_{j=1}^{m} \eta_j\right)$$

where m and η_j are given by (4.1.2). Let $\gamma \in \mathscr{E}_i$. Since the restriction of ρ_i to $\epsilon_i^{-1}(\gamma)$ is injective by Lemma 4.1.3, given a $U_q(\tilde{\mathfrak{g}})$-module V, there exist unique $m > 0, \boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m \in \mathrm{wt}_\ell(V) \cap \epsilon_i^{-1}(\gamma)$ such that

$$\text{(4.1.4)} \qquad \mathrm{qch}(V_\gamma) = \sum_{j=1}^{m} [V_\gamma : \rho_i(\boldsymbol{\mu}_j)]\ \mathrm{qch}(V_q(\rho_i(\boldsymbol{\mu}_j))).$$

Moreover, if we set

$$\text{(4.1.5)} \qquad \zeta(V_\gamma) = \sum_{j=1}^{m} \zeta_{\boldsymbol{\mu}_j}^i,$$

where m and $\boldsymbol{\mu}_j$ are as in (4.1.4), it follows from Proposition 4.1.4 that

$$\text{(4.1.6)} \qquad \mathrm{qch}(V) = \sum_{\gamma \in \mathscr{E}_i} \zeta(V_\gamma) \qquad \text{and} \qquad \mathrm{wt}_\ell(V) = \bigsqcup_{\gamma\, \in\, \mathscr{E}_i} \mathrm{wt}_\ell(\zeta(V_\gamma)).$$

This proves:

Proposition 4.1.7. Let V be a $U_q(\tilde{\mathfrak{g}})$-module and $i \in I$. Then, there exist unique $m > 0, \boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m \in \mathrm{wt}_\ell(V) \cap \mathscr{P}_{i,+}$ such that

$$\text{(4.1.7)} \qquad \mathrm{qch}(V) = \sum_{j=1}^{m} \zeta_{\boldsymbol{\mu}_j}^i, \qquad \mathrm{wt}_\ell(V) = \bigsqcup_{j\,=\,1}^{m} \mathrm{wt}_\ell(\zeta_{\boldsymbol{\mu}_j}^i),$$

and $V_\gamma = \bigoplus_{j\,:\,\epsilon_i(\boldsymbol{\mu}_j) = \gamma} V(\boldsymbol{\mu}_j)$ for all $\gamma \in \mathscr{E}_i$. □

For notational convenience, we set $\zeta_{\boldsymbol{\mu}}^i = \boldsymbol{\mu}$ if $\boldsymbol{\mu}$ is not i-dominant.

Definition 4.1.8. The element $\zeta_{\boldsymbol{\mu}}^i$ is called the i-th expansion of $\boldsymbol{\mu}$. The decompositions (4.1.7) are called the i-th $\mathfrak{sl}_2$ decompositions of $\mathrm{qch}(V)$ and $\mathrm{wt}_\ell(V)$, respectively. We will refer to each subset of the form $\mathrm{wt}_\ell(\zeta_{\boldsymbol{\mu}}^i)$ appearing in (4.1.7) as an i-stratum of $\mathrm{wt}_\ell(V)$. We say that $\boldsymbol{\mu} \in \mathscr{P}$ is an i-root of $\mathrm{wt}_\ell(V)$ if $\mathrm{wt}_\ell(\zeta_{\boldsymbol{\mu}}^i)$ is an i-stratum of $\mathrm{wt}_\ell(V)$. Let $[V_{\epsilon_i(\boldsymbol{\mu})} : \boldsymbol{\mu}_i]$ denote the multiplicity of $\boldsymbol{\mu}$ as an i-root of $\mathrm{wt}_\ell(V)$. ◊

The set $\mathbb{Z}[\mathscr{P}]$ can be equipped with the following partial order: $\chi \le \chi'$ if $\chi(\boldsymbol{\mu}) \le \chi'(\boldsymbol{\mu})$ for all $\boldsymbol{\mu} \in \mathscr{P}$. In particular, we write $\chi \ge 0$ if $\chi \in \mathbb{Z}_{\ge 0}[\mathscr{P}]$. Notice that (4.1.7) implies

$$\sum_{\boldsymbol{\mu}\in\mathscr{P}} [V_{\epsilon_i(\boldsymbol{\mu})} : \boldsymbol{\mu}_i]\, \zeta^i_{\boldsymbol{\mu}}(\boldsymbol{\nu}) = \mathrm{qch}(V)(\boldsymbol{\nu}) \qquad \text{for all} \qquad \boldsymbol{\nu} \in \mathscr{P}. \tag{4.1.8}$$

Proposition 4.1.7 enables us to equip $\mathrm{wt}_\ell(V)$ with a structure of I-colored quiver with multiplicities. Let us first establish some terminology about such objects.

Definition 4.1.9. A quiver Γ is said to be I-colored if it is equipped with a function from its set of arrows to I. Γ is said to be a quiver with multiplicities if is equipped with a function from its set of vertices to $\mathbb{Z}_{>0}$. Given a quiver Γ, a vertex v which is connected to any other vertex of Γ by an oriented path is called a root of Γ. If Γ has a unique root and the root has no incoming arrow, we shall say that Γ is a tree. $\Diamond$

The structure of I-colored quiver with multiplicities on $\mathrm{wt}_\ell(V)$ is constructed as follows. The underlying set of vertices is $\mathrm{wt}_\ell(V)$ itself and the multiplicity of $\boldsymbol{\mu}$ is $\mathrm{qch}(V)(\boldsymbol{\mu})$. Given $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathrm{wt}_\ell(V)$, there is an arrow of color i from $\boldsymbol{\mu}$ to $\boldsymbol{\nu}$ only if $\boldsymbol{\mu} = \boldsymbol{\nu}\boldsymbol{\alpha}_{i,a}$ for some $a \in \mathbb{F}^\times$. In that case, the number of I-colored arrows from $\boldsymbol{\mu}$ to $\boldsymbol{\nu}$ is the number of i-strata of $\mathrm{wt}_\ell(V)$ containing both $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$.

Definition 4.1.10. We refer to the above constructed quiver as the Frenkel-Mukhin quiver of V. Set $c^i_V(\boldsymbol{\mu}) = \mathrm{qch}(V)(\boldsymbol{\mu}) - [V_{\epsilon_i(\boldsymbol{\mu})} : \boldsymbol{\mu}_i]$. $\Diamond$

Remark 4.1.11. By definition, if $\boldsymbol{\mu}$ is not an i-root (in particular, if $\boldsymbol{\mu}$ is not i-dominant), then $c^i_V(\boldsymbol{\mu}) = \mathrm{qch}(V)(\boldsymbol{\mu})$. Since, when there is an i-colored arrow $\mu \xrightarrow{i} \nu$ it implies that $\boldsymbol{\mu} = \boldsymbol{\nu}\boldsymbol{\alpha}_{i,a}$ for some $a \in \mathbb{F}^\times$, we shall actually record the information about a in the quiver by drawing $\mu \xrightarrow{(i,a)} \nu$. Also, if $\mathrm{qch}(V)(\boldsymbol{\mu}) = m$ and $\mathrm{qch}(V)(\boldsymbol{\nu}) = n$, we draw $m\mu \xrightarrow{(i,a)} n\nu$. $\Diamond$

Example 4.1.12. Let $\mathfrak{g} = \mathfrak{sl}_2$ and i be the unique element of I. The Frenkel-Mukhin quiver of $V_q(m,a)$ is

$$\mu_0 \xrightarrow{(i,aq^{m-1})} \mu_1 \xrightarrow{(i,aq^{m-3})} \mu_2 \xrightarrow{(i,aq^{m-5})} \cdots \xrightarrow{(i,aq^{1-m})} \mu_m$$

where $\boldsymbol{\mu}_0 = \boldsymbol{\omega}_{i,a,m}$. If $a/b \notin \{1, q^{\pm 2}\}$, then the Frenkel-Mukhin quiver of $V_q(\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{i,b}) \cong V_q(1,a) \otimes V_q(1,b)$ is

$$\begin{tikzcd} & \mu_1 \arrow[dr, "{(i,b)}"] & \\ \lambda \arrow[ur, "{(i,a)}"] \arrow[dr, "{(i,b)}"'] & & \mu_3 \\ & \mu_2 \arrow[ur, "{(i,a)}"'] & \end{tikzcd}$$

where $\boldsymbol{\lambda} = \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{i,b}$. If $a = b$, then $V_q(\boldsymbol{\lambda}) \cong V(1,a)^{\otimes 2}$ and the quiver is

$$\lambda \xrightarrow{(i,a)} 2\mu_1 \xrightarrow{(i,a)} \mu_2 \ .$$

If $b = aq^{\pm 2}$, then $V_q(\boldsymbol{\lambda}) \cong V_q(2, aq)$ and the quiver was given above. Notice that, in this case, the quiver of the Weyl module $W_q(\boldsymbol{\lambda})$ is the disjoint union of the quivers of $V_q(\boldsymbol{\lambda})$ and $V_q(\mathbf{1})$. We draw the case $b = aq^2$:

$$\begin{tikzcd}
\lambda \arrow[r, "(i{,}aq^2)"] \arrow[dr, dotted, "(i{,}a)"'] & \mu_1 \arrow[r, "(i{,}a)"] & \mu_2 \\
& \mathbf{1} \arrow[ur, dotted, "(i{,}aq^2)"'] &
\end{tikzcd}$$

The two dashed arrows above are not in the Frenkel-Mukhin quiver since the trivial representation is a submodule of $W_q(\boldsymbol{\lambda})$ and, hence, $\mathbf{1}$ is in a different stratum than the other ℓ-weights. However, $\mathbf{1} \in \mathrm{wt}_\ell(W_q(\boldsymbol{\lambda}))$. The reader is invited to compare this fact with Example 4.2.14 below. $\Diamond$

Example 4.1.13. Let $\mathfrak{g} = \mathfrak{sl}_2$ and i be the unique element of I. Let also V have simple factors $V_q(\boldsymbol{\omega}_{i,a})$ and $V_q(\boldsymbol{\omega}_{i,a}^2\boldsymbol{\omega}_{i,aq^2})$. Set $\boldsymbol{\mu}_0 = \boldsymbol{\omega}_{i,a}^2\boldsymbol{\omega}_{i,aq^2}$ and $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_0\boldsymbol{\alpha}_{i,a}^{-1} = \boldsymbol{\omega}_{i,a}$. Notice that $V_q(\boldsymbol{\mu}_0) \cong V_q(1,a) \otimes V_q(2,aq)$. Then, the Frenkel-Mukhin quiver quiver of V is

$$\begin{tikzcd}
& & \mu_0 \arrow[dl, "(i{,}a)"'] \arrow[dr, "(i{,}aq^2)"] & \\
& 2\mu_1 \arrow[dl, "(i{,}a)"'] \arrow[dr, "(i{,}aq^2)"'] & & \mu_2 \arrow[dl, "(i{,}a)"] \\
\mu_3 & & 2\mu_4 \arrow[d, "(i{,}a)"] & \\
& & \mu_5 &
\end{tikzcd}$$

The above quiver is obtained from the quivers of $V_q(\boldsymbol{\mu}_1)$ and $V_q(\boldsymbol{\mu}_0)$ which can be computed using the previous example and are given by:

$$\begin{tikzcd}
& \mu_1 \arrow[dl, "(i{,}a)"'] \\
\mu_3 &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& \mu_0 \arrow[dl, "(i{,}a)"'] \arrow[dr, "(i{,}aq^2)"] & \\
\mu_1 \arrow[dr, "(i{,}aq^2)"'] & & \mu_2 \arrow[dl, "(i{,}a)"] \\
& 2\mu_4 \arrow[d, "(i{,}a)"] & \\
& \mu_5 &
\end{tikzcd}$$

respectively. $\Diamond$

Proposition 4.1.14. The Frenkel-Mukhin quiver of a simple finite-dimensional $U_q(\tilde{\mathfrak{sl}}_2)$-module is a tree where the root is the highest ℓ-weight.

Proof. For evaluation modules this is easily seen from the formulas of Theorem 2.3.14. The general case is easily deduced from Theorem 2.3.20. $\square$

Corollary 4.1.15. Let V be a finite-dimensional $U_q(\tilde{\mathfrak{g}})$-module, $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V)$, and $i \in I$. If $\boldsymbol{\mu}$ has no incoming arrow of color i, then $\boldsymbol{\mu}$ is an i-root and $\mathrm{qch}(V)(\boldsymbol{\mu}) = [V_{\epsilon_i(\boldsymbol{\mu})} : \boldsymbol{\mu}_i]$. $\square$

Definition 4.1.16. Let $\chi \in \mathbb{Z}[\mathscr{P}], \chi \geq 0$. A coloring of χ is a function from $\mathscr{P}$ to $\mathbb{Z}_{\geq 0}^I, \boldsymbol{\mu} \mapsto (c_\chi^i(\boldsymbol{\mu}))_{i\in I}$, such that $c_\chi^i(\boldsymbol{\mu}) \leq \chi(\boldsymbol{\mu})$ for all $i \in I$. The trivial coloring is the function given by $c_\chi^i(\boldsymbol{\mu}) = 0$ for all $i \in I$ and $\boldsymbol{\mu} \in \mathscr{P}$. An element $\boldsymbol{\mu} \in \mathrm{wt}_\ell(\chi)$ is said to be admissible with respect to a given coloring of χ if $\boldsymbol{\mu}$ is i-dominant for all $i \in I$ such that $c_\chi^i(\boldsymbol{\mu}) < \chi(\boldsymbol{\mu})$. We shall simply say that $\boldsymbol{\mu}$ is admissible when it is clear which coloring has been chosen. A coloring of χ is said to be an admissible coloring if $\boldsymbol{\mu}$ is admissible for every $\boldsymbol{\mu} \in \mathscr{P}$. $\diamondsuit$

Suppose $\chi \in \mathbb{Z}_{\geq 0}[\mathscr{P}]$ has been colored and let $i \in I$. Given $\boldsymbol{\mu} \in \mathrm{wt}_\ell(\chi)$, define a new colored element $\chi_{\boldsymbol{\mu}}^i \in \mathbb{Z}_{\geq 0}[\mathscr{P}]$ by

$$\chi_{\boldsymbol{\mu}}^i(\boldsymbol{\nu}) = \max\left\{\chi(\boldsymbol{\nu}),\ c_\chi^i(\boldsymbol{\nu}) + (\chi(\boldsymbol{\mu}) - c_\chi^i(\boldsymbol{\mu}))\, \zeta_{\boldsymbol{\mu}}^i(\boldsymbol{\nu})\right\} \tag{4.1.9}$$

and

$$c^j_{\chi_{\boldsymbol{\mu}}^i}(\boldsymbol{\nu}) = \begin{cases} c_\chi^j(\boldsymbol{\nu}) + \delta_{ij}\,(\chi(\boldsymbol{\mu}) - c_\chi^i(\boldsymbol{\mu}))\zeta_{\boldsymbol{\mu}}^i(\boldsymbol{\nu}), & \text{if} \quad \boldsymbol{\nu} \in \mathrm{wt}_\ell(\chi), \\ \delta_{ij}\, \chi_{\boldsymbol{\mu}}^i(\boldsymbol{\nu}), & \text{otherwise.} \end{cases} \tag{4.1.10}$$

One easily sees that $\chi_{\boldsymbol{\mu}}^i$ is a well-defined colored element and we have equality of colored elements:

$$(\chi_{\boldsymbol{\mu}}^i)_{\boldsymbol{\mu}}^j = (\chi_{\boldsymbol{\mu}}^j)_{\boldsymbol{\mu}}^i \qquad \text{for all} \qquad i, j \in I$$

Notice also that $\chi_{\boldsymbol{\mu}}^i = \chi$ if $\boldsymbol{\mu}$ is not i-dominant and

$$\chi_{\boldsymbol{\mu}}^i(\boldsymbol{\nu}) = (\chi(\boldsymbol{\mu}) - c_\chi^i(\boldsymbol{\mu}))\, \zeta_{\boldsymbol{\mu}}^i(\boldsymbol{\nu}) \qquad \text{if} \qquad \boldsymbol{\nu} \notin \mathrm{wt}_\ell(\chi).$$

Definition 4.1.17. Let $\chi \in \mathbb{Z}_{\geq 0}[\mathscr{P}]$ be colored. The colored element $\chi_{\boldsymbol{\mu}}^i$ is called the i-th expansion of χ with respect to $\boldsymbol{\mu}$. The expansion of χ with respect to $\boldsymbol{\mu}$ is the element $\chi_{\boldsymbol{\mu}} = (\cdots(\chi_{\boldsymbol{\mu}}^{i_1})_{\boldsymbol{\mu}}^{i_2}\cdots)_{\boldsymbol{\mu}}^{i_n}$ where $n = |I|$ and $i_1, \ldots, i_n$ is a choice of ordering on I. Given $\mu \in P$, an expansion of χ with respect to μ is an element of the form $\chi_\mu = (\cdots(\chi_{\boldsymbol{\mu}_1})_{\boldsymbol{\mu}_2}\cdots)_{\boldsymbol{\mu}_j}$ for some choice of ordering $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m$ of the set $\{\boldsymbol{\mu} \in \mathrm{wt}_\ell(\chi) : \mathrm{wt}(\boldsymbol{\mu}) = \mu\}$. $\diamondsuit$

The Frenkel-Mukhin algorithm, which we now describe, is recursive procedure for approximating the qcharacter of the simple $U_q(\tilde{\mathfrak{g}})$-modules by successive expansion beginning from the highest ℓ-weight. It essentially tries to "guess" what the i-roots are (counted with multiplicities). Notice that, if all the i-roots are found for some i, then the qcharacter can be reconstructed using (4.1.7).

Definition 4.1.18. Let $\boldsymbol{\lambda} \in \mathscr{P}^+, \lambda = \mathrm{wt}(\boldsymbol{\lambda})$, and choose a total ordering $\mu_0, \mu_1, \ldots, \mu_m$ of the finite set $\mathrm{wt}(\lambda)$ such that $\mu_r < \mu_s$ only if $r > s$ (in particular, $\mu_0 = \mathrm{wt}(\boldsymbol{\lambda})$ and $\mu_m = w_0(\lambda)$). Define colored elements $\chi_r, r \geq 0$, recursively by letting $\chi_0 = \boldsymbol{\lambda}$ be trivially colored and, for $r \geq 0, \chi_{r+1} = (\chi_r)_{\mu_r}$ for some choice of expansion with respect to μ_r. Set

$$\mathrm{FM}(\boldsymbol{\lambda}) = \chi_m = ((\cdots((\chi_0)_{\mu_1})_{\mu_2})\cdots)_{\mu_{m-1}}.$$

The algorithm is said to have failed at step $r < m$ if χ_r is not admissibly colored. $\diamondsuit$

A priori, $\mathrm{FM}(\boldsymbol{\lambda})$ depends on the choices of orderings made, but we shall not incorporate this dependence in the notation. It easily follows from the definition of expansions that

$$\boldsymbol{\mu} \in \mathrm{wt}_\ell(\mathrm{FM}(\boldsymbol{\lambda})) \qquad \text{only if} \qquad \boldsymbol{\mu} \leq \boldsymbol{\lambda} \tag{4.1.11}$$

regardless of the choices of orderings made. The following is the main theorem of this subsection.

Theorem 4.1.19. Let $\boldsymbol{\lambda} \in \mathscr{P}^+$ and $V = V_q(\boldsymbol{\lambda})$. If $\mathrm{FM}(\boldsymbol{\lambda})(\boldsymbol{\mu}) = \mathrm{qch}(V)(\boldsymbol{\mu})$ for all $\boldsymbol{\mu} \in \mathscr{P}^+$, then $\mathrm{FM}(\boldsymbol{\lambda}) = \mathrm{qch}(V)$. In particular, this is the case if $\boldsymbol{\lambda}$ is minuscule.

Proof. Since the condition $\mathrm{FM}(\boldsymbol{\lambda})(\boldsymbol{\mu}) = \mathrm{qch}(V)(\boldsymbol{\mu})$ for all $\boldsymbol{\mu} \in \mathscr{P}^+$ is obviously satisfied for minuscule ℓ-weights, the second statement is immediate from the first. The first statement clearly follows if we prove that the following hold for all $0 \leq r < m$:

(a_r) The algorithm does not fail at step r.

(b_r) $\chi_r(\boldsymbol{\nu}) \leq \mathrm{qch}(V)(\boldsymbol{\nu})$ for all $\boldsymbol{\nu} \in \mathscr{P}$.

(c_r) If $\boldsymbol{\nu} \in \mathrm{wt}_\ell(V)$ is such that $\mathrm{wt}(\boldsymbol{\nu}) = \mu_s$ for some $s \leq r$, then $\chi_r(\boldsymbol{\nu}) = \mathrm{qch}(V)(\boldsymbol{\nu})$. In particular, all arrows of the Frenkel-Mukhin quiver outgoing from $\boldsymbol{\nu}$ reach elements in $\mathrm{wt}_\ell(\chi_{r+1})$.

(d_r) $c^j_{\chi_r}(\boldsymbol{\nu}) = c^j_V(\boldsymbol{\nu})$ for all $j \in I$ and $\boldsymbol{\nu} \in \mathrm{wt}_\ell(\chi_r)$.

The above statements will be be proved by induction on $r \geq 0$ which clearly starts when $r = 0$. Thus, assume $r \geq 0$ and that ($\mathrm{x}_{r'}$) holds for x=a,b,c,d and $r' \leq r$.

Let $\boldsymbol{\mu} \in \mathrm{wt}_\ell(\chi_r)$ be such that $\mathrm{wt}(\boldsymbol{\mu}) = \mu_r$ and suppose $c^i_{\chi_r}(\boldsymbol{\mu}) < \chi_r(\boldsymbol{\mu})$ for some $i \in I$. By (c_r) and (d_r), we have $c^i_V(\boldsymbol{\mu}) < \mathrm{qch}(V)(\boldsymbol{\mu})$ and, hence, $\boldsymbol{\mu}$ is i-dominant by Remark 4.1.11. This proves (a_{r+1}).

Fix $\boldsymbol{\nu} \in \mathscr{P}$ and let us prove (x_{r+1}) for $\boldsymbol{\nu}$ with x=b,c,d. Suppose first that $\boldsymbol{\nu} \notin \mathrm{wt}_\ell(\zeta^i_{\boldsymbol{\mu}})$ for any $i \in I, \boldsymbol{\mu} \in \mathrm{wt}_\ell(\chi_r)$ such that $\mathrm{wt}(\boldsymbol{\mu}) = \mu_r$. In this case $\chi_{r+1}(\boldsymbol{\nu}) = \chi_r(\boldsymbol{\nu})$ and $c^i_{\chi_{r+1}}(\boldsymbol{\nu}) = c^i_{\chi_r}(\boldsymbol{\nu})$ for all $i \in I$. In particular, (b_{r+1}) and (d_{r+1}) follow immediately while (c_{r+1}) follows in case $\mathrm{wt}(\boldsymbol{\nu}) = \mu_s$ with $s \leq r$. Suppose $\boldsymbol{\nu}$ is as in (c_{r+1}) with $\mathrm{wt}(\boldsymbol{\nu}) = \mu_{r+1}$. If $\boldsymbol{\nu} \in \mathscr{P}^+$, then $\mathrm{FM}(\boldsymbol{\lambda})(\boldsymbol{\nu}) = \mathrm{qch}(V)(\boldsymbol{\nu})$ by hypothesis. Since no ℓ-weight of weight μ_{r+1} is obtained by the expansions performed after step r, we must have $\chi_{r+1}(\boldsymbol{\nu}) = \mathrm{FM}(\boldsymbol{\lambda})(\boldsymbol{\nu})$. Otherwise, let $i \in I$ be such that $\boldsymbol{\mu}$ is not i-dominant. By Remark 4.1.11 and (d_{r+1}), we have $\mathrm{qch}(V)(\boldsymbol{\nu}) = c^i_V(\boldsymbol{\nu}) = c^i_{\chi_{r+1}}(\boldsymbol{\nu})$. Since $c^i_{\chi_{r+1}}(\boldsymbol{\nu}) \leq \chi_{r+1}(\boldsymbol{\nu})$ by definition of coloring of an element, we conclude $\mathrm{qch}(V)(\boldsymbol{\nu}) \leq \chi_{r+1}(\boldsymbol{\nu})$. Finally, by ($\mathrm{b}_{r+1}$) we have $\mathrm{qch}(V)(\boldsymbol{\nu}) \geq \chi_{r+1}(\boldsymbol{\nu})$ and (c_{r+1}) follows.

It remains to consider the case $\boldsymbol{\nu} \in \mathrm{wt}_\ell(\zeta^i_{\boldsymbol{\mu}})$ for some $i \in I, \boldsymbol{\mu} \in \mathrm{wt}_\ell(\chi_r)$ such that $\mathrm{wt}(\boldsymbol{\mu}) = \mu_r$. In particular, $\mu_r - \mathrm{wt}(\boldsymbol{\nu})$ is a nonnegative multiple of α_i and, hence, if $j \neq i$, the j-expansions with respect to μ_r do not affect $\chi_{r+1}(\boldsymbol{\nu})$ nor the coloring of χ_{r+1} at $\boldsymbol{\nu}$. This implies (d_{r+1}) with $j \neq i$. Also, we can assume $\boldsymbol{\nu} \neq \boldsymbol{\mu}$, otherwise all statements follow by induction hypothesis on r. Moreover, $\boldsymbol{\mu}$ is i-dominant and $\mathrm{qch}(V)(\boldsymbol{\mu}) - c^i_V(\boldsymbol{\mu}) = [V_{\epsilon_i(\boldsymbol{\mu})} : \boldsymbol{\mu}_i]$. By ($\mathrm{c}_r$) and ($\mathrm{d}_r$) this is equal to $\chi_r(\boldsymbol{\mu}) - c^i_{\chi_r}(\boldsymbol{\mu})$. We now study separately the cases $\boldsymbol{\nu} \in \mathrm{wt}_\ell(\chi_r)$ and $\boldsymbol{\nu} \notin \mathrm{wt}_\ell(\chi_r)$.

1) Assume $\boldsymbol{\nu} \in \mathrm{wt}_\ell(\chi_r)$. Then, by ($\mathrm{d}_r$), $c^i_{\chi_r}(\boldsymbol{\nu}) = c^i_V(\boldsymbol{\nu})$. This implies that all the i-strata containing $\boldsymbol{\nu}$ were already obtained at the step $r-1$. In particular, $\boldsymbol{\nu}$ cannot be an i-root and, hence, $\mathrm{qch}(V)(\boldsymbol{\nu}) = c^i_V(\boldsymbol{\nu})$. Also, we must have $[V_{\epsilon_i(\boldsymbol{\mu})} : \boldsymbol{\mu}_i] = 0$ for all $\boldsymbol{\mu}$ such that $\mathrm{wt}(\boldsymbol{\mu}) = \mu_r$. This implies $\chi_{r+1}(\boldsymbol{\nu}) = \chi_r(\boldsymbol{\nu})$ and $c^i_{\chi_{r+1}}(\boldsymbol{\nu}) = c^i_{\chi_r}(\boldsymbol{\nu}) = c^i_V(\boldsymbol{\nu})$ which immediately implies (b_{r+1}) and (d_{r+1}). But then, $\mathrm{qch}(V)(\boldsymbol{\nu}) = c^i_V(\boldsymbol{\nu}) = c^i_{\chi_{r+1}}(\boldsymbol{\nu}) \leq \chi_{r+1}(\boldsymbol{\nu}) \leq \mathrm{qch}(V)(\boldsymbol{\nu})$, where we used ($\mathrm{b}_{r+1}$) to obtain the second inequality and the fact that χ_{r+1} has well-defined coloring to obtain the first inequality. This implies (c_{r+1}).

2) Assume $\boldsymbol{\nu} \notin \mathrm{wt}_\ell(\chi_r)$. Then, by ($\mathrm{c}_{r-1}$), all arrows incoming to $\boldsymbol{\nu}$ must come from a j-root $\boldsymbol{\mu}$ such that

$\mathrm{wt}(\boldsymbol{\mu}) = \mu_s$ with $s \geq r$ for some $j \in I$. This and (4.1.8) together imply

$$\mathrm{qch}(V)(\boldsymbol{\nu}) = \sum_{\substack{\boldsymbol{\mu}\in\mathscr{P}_{i,+}\\ \mathrm{wt}(\boldsymbol{\mu})=\mu_s, s\geq r}} [V_{\epsilon_i}(\boldsymbol{\mu}) : \boldsymbol{\mu}_i]\, \zeta^i_{\boldsymbol{\mu}}(\boldsymbol{\nu}). \tag{4.1.12}$$

Since, we have already proved (d_{r+1}) for $j \neq i$ and, if $s = r$, no j-colored arrow incomes to $\boldsymbol{\nu}$, all arrows incoming to $\boldsymbol{\nu}$ must be i-colored. Moreover, $\boldsymbol{\nu} \notin \mathrm{wt}_\ell(\chi_r)$ also implies $c^i_{\chi_r}(V) = 0$ and one easily computes, using (4.1.9), that

$$\chi_{r+1}(\boldsymbol{\nu}) = \sum_{\substack{\boldsymbol{\mu}\in\mathscr{P}_{i,+}\\ \mathrm{wt}(\boldsymbol{\mu})=\mu_r}} \left(\chi_r(\boldsymbol{\mu}) - c^i_{\chi_r}(\boldsymbol{\mu})\right) \zeta^i_{\boldsymbol{\mu}}(\boldsymbol{\nu}) = \sum_{\substack{\boldsymbol{\mu}\in\mathscr{P}\\ \mathrm{wt}(\boldsymbol{\mu})=\mu_r}} [V_{\epsilon_i}(\boldsymbol{\mu}) : \boldsymbol{\mu}_i]\, \zeta^i_{\boldsymbol{\mu}}(\boldsymbol{\nu}). \tag{4.1.13}$$

Together with (4.1.12), this implies (b_{r+1}). Furthermore, by (4.1.10), we have $c^i_{\chi_{r+1}}(\boldsymbol{\nu}) = \chi_{r+1}(\boldsymbol{\nu})$. Let us prove ($\mathrm{c}_{r+1}$). Thus, assume $\mathrm{wt}(\boldsymbol{\nu}) = \mu_{r+1}$ which implies $\mu_{r+1} = \mu_r - \alpha_i$. Since there are no j-colored arrows incoming to $\boldsymbol{\nu}$, $\boldsymbol{\nu}$ must be j-dominant by Corollary 4.1.15. In particular, if $\boldsymbol{\nu}$ is i-dominant, it is dominant and, by hypothesis, $\mathrm{FM}(\boldsymbol{\lambda})(\boldsymbol{\nu}) = \mathrm{qch}(V)(\boldsymbol{\nu})$. The condition $\mu_{r+1} = \mu_r - \alpha_i$ implies that $\mathrm{FM}(\boldsymbol{\lambda})(\boldsymbol{\nu}) = \chi_{r+1}(\boldsymbol{\nu})$ and (c_{r+1}) follows in this case. Notice also that $\boldsymbol{\nu}$ cannot be an i-root in this case and, hence, we have $c^i_V(\boldsymbol{\nu}) = \mathrm{qch}(V)(\boldsymbol{\nu}) = \chi_{r+1}(\boldsymbol{\nu}) = c^i_{\chi_{r+1}}(\boldsymbol{\nu})$ proving (d_{r+1}) as well. Now assume $\boldsymbol{\nu}$ is not i-dominant which, as we have seen, implies $\mathrm{FM}(\boldsymbol{\lambda})(\boldsymbol{\nu}) = \chi_{r+1}(\boldsymbol{\nu}) = c^i_{\chi_{r+1}}(\boldsymbol{\nu})$ and $\mathrm{qch}(V)(\boldsymbol{\nu}) = c^i_V(\boldsymbol{\nu})$. In particular, if we show (c_{r+1}), (d_{r+1}) also follows. The condition $\mu_{r+1} = \mu_r - \alpha_i$ together with $\boldsymbol{\nu} \notin \mathscr{P}_{i,+}$ clearly $\zeta^i_{\boldsymbol{\mu}}(\boldsymbol{\nu}) = 0$ for all $\boldsymbol{\mu}$ such that $\mathrm{wt}(\boldsymbol{\mu}) = \mu_s$ with $s > r$. Therefore, (4.1.12) becomes identical to (4.1.13) showing that $\mathrm{qch}(V)(\boldsymbol{\nu}) = \chi_{r+1}(\boldsymbol{\nu})$. Observe that the above argument actually shows that if $\mu_r - \mathrm{wt}(\boldsymbol{\nu}) = \alpha_i$, then $\chi_{r+1}(\boldsymbol{\nu}) = \mathrm{FM}(\boldsymbol{\lambda})(\boldsymbol{\nu} = \mathrm{qch}(V)(\boldsymbol{\nu})$ and (d_{r+1}) holds. Therefore, it remains to prove (d_{r+1}) with $j = i$ and $\boldsymbol{\nu}$ such that $\mu_r - \mathrm{wt}(\boldsymbol{\nu}) \neq \alpha_i$. Since $\mu_r - \mathrm{wt}(\boldsymbol{\nu}) = k\alpha_i$ for some $k > 0$, one easily checks that if $\boldsymbol{\mu}$ is such that $\mathrm{wt}(\boldsymbol{\nu}) = \mu_s$ with $s > r$ and $\mu_s \neq \mu_r - l\alpha_i$ for all $l \geq 0$, then there are no arrows from $\boldsymbol{\mu}$ to $\boldsymbol{\nu}$ (just right $\mu_r - \mu_s$ as a linear combination of simple roots). This implies that the coloring and multiplicity of $\boldsymbol{\nu}$ in $\mathrm{FM}(\boldsymbol{\lambda})$ can only be changed by performing i-expansions at μ_s with s such that $\mu_s = \mu_r - l\alpha_i$ with $l < k$. Repeating part of the arguments, one shows by induction on $l \geq 1$ that no ℓ-weight can be an i-root and, therefore, no change will be made by such expansions as well (notice that the case $l = 1$ has been proved above). This completes the proof. □

4.2. Applications of the Frenkel-Mukhin Algorithm and a Counter-example

In this subsection we give a few examples to illustrate both successful and failed applications of the Frenkel-Mukhin algorithm.

Example 4.2.1. Let $\mathfrak{g} = \mathfrak{sl}_3$, denote by i and j the distinct elements of I, and consider $\boldsymbol{\lambda} = \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,b}$ for some $a, b \in \mathbb{F}^\times$. In Example 3.2.9 we have seen that the modules $V_q(\boldsymbol{\lambda})$ is ℓ-minuscule for any choice of a, b. In particular, $\mathrm{FM}(\boldsymbol{\lambda}) = \mathrm{qch}(V_q(\boldsymbol{\lambda}))$. Although we have already computed $\mathrm{qch}(V_q(\boldsymbol{\lambda}))$ in Example 3.2.9 without using the algorithm, let us use this example to illustrate the several steps of the algorithm and draw the Frenkel-Mukhin quiver. Recall that $\omega_i + \omega_j = \theta$ and, hence, $\mathrm{wt}(\theta) = \{\theta, \alpha_j, \alpha_i, 0, -\alpha_i, -\alpha_j, -\theta\}$. We choose the ordering $\mu_0, \ldots, \mu_6$ of $\mathrm{wt}(\theta)$ as written in the above list and set $\chi_0 = \boldsymbol{\lambda}$ and $\chi_{r+1} = (\chi_r)_{\mu_r}$ as in the description of the algorithm.

Since $\zeta^i_{\boldsymbol{\lambda}} = \boldsymbol{\lambda} + \boldsymbol{\lambda}\boldsymbol{\alpha}^{-1}_{i,a}$ and $\zeta^j_{\boldsymbol{\lambda}} = \boldsymbol{\lambda} + \boldsymbol{\lambda}\boldsymbol{\alpha}^{-1}_{j,b}$, we get

$$\chi_1 = \boldsymbol{\lambda} + \boldsymbol{\lambda}\boldsymbol{\alpha}^{-1}_{i,a} + \boldsymbol{\lambda}\boldsymbol{\alpha}^{-1}_{j,b} = \boldsymbol{\lambda} + \boldsymbol{\omega}^{-1}_{i,aq^2}\boldsymbol{\omega}_{j,aq}\boldsymbol{\omega}_{j,b} + \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{i,bq}\boldsymbol{\omega}^{-1}_{j,bq^2}.$$

Set $\boldsymbol{\mu}_1 = \boldsymbol{\lambda}\boldsymbol{\alpha}^{-1}_{i,a}$ and $\boldsymbol{\mu}_2 = \boldsymbol{\lambda}\boldsymbol{\alpha}^{-1}_{j,b}$. Then $\chi_2 = (\chi_1)_{\boldsymbol{\mu}_1} = (\chi_1)^j_{\boldsymbol{\mu}_1}$ and $\chi_3 = (\chi_2)_{\boldsymbol{\mu}_2} = (\chi_2)^i_{\boldsymbol{\mu}_2}$. The actual computation will now depend whether $a/b \in \{q^{\pm 1}, q^{\pm 3}\}$ or not.

Let us first consider the case $b = aq^3$ (which is similar to the case $b = aq^{-3}$). In this case, $\boldsymbol{\mu}_1 = \boldsymbol{\omega}^{-1}_{i,aq^2}\boldsymbol{\omega}_{j,aq^2,2}$ and $\zeta^j_{\boldsymbol{\mu}_1} = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_1\boldsymbol{\alpha}^{-1}_{j,aq^3} + \boldsymbol{\mu}_1\boldsymbol{\alpha}^{-1}_{j,aq^3}\boldsymbol{\alpha}^{-1}_{j,aq}$. Therefore,

$$\chi_2 = \chi_1 + \boldsymbol{\omega}^{-1}_{i,aq^2}\boldsymbol{\omega}_{i,aq^4}\boldsymbol{\omega}_{j,aq}\boldsymbol{\omega}^{-1}_{j,aq^5} + \boldsymbol{\omega}_{i,aq^4}\boldsymbol{\omega}^{-1}_{j,aq^4,2}.$$

Set $\boldsymbol{\mu}_3 = \boldsymbol{\omega}^{-1}_{i,aq^2}\boldsymbol{\omega}_{i,aq^4}\boldsymbol{\omega}_{j,aq}\boldsymbol{\omega}^{-1}_{j,aq^5}$ and $\boldsymbol{\mu}_4 = \boldsymbol{\omega}_{i,aq^4}\boldsymbol{\omega}^{-1}_{j,aq^4,2}$. Proceeding, we have $\zeta^i_{\boldsymbol{\mu}_2} = \boldsymbol{\mu}_2 + \boldsymbol{\mu}_2\boldsymbol{\alpha}^{-1}_{i,a} + \boldsymbol{\mu}_2\boldsymbol{\alpha}^{-1}_{i,aq^4} + \boldsymbol{\mu}_2\boldsymbol{\alpha}^{-1}_{i,a}\boldsymbol{\alpha}^{-1}_{i,aq^4}$. Notice that $\boldsymbol{\mu}_2\boldsymbol{\alpha}^{-1}_{i,a} = \boldsymbol{\mu}_3$ and that, by definition of $(\chi_2)^i_{\boldsymbol{\mu}_2}$, the multiplicity of $\boldsymbol{\mu}_3$ in χ_3 is the same as in χ_2. Therefore,

$$\chi_3 = \chi_2 + \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}^{-1}_{i,aq^6} + \boldsymbol{\omega}^{-1}_{i,aq^2}\boldsymbol{\omega}^{-1}_{i,aq^6}\boldsymbol{\omega}_{j,aq}.$$

Set $\boldsymbol{\mu}_5 = \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}^{-1}_{i,aq^6}$ and $\boldsymbol{\mu}_6 = \boldsymbol{\omega}^{-1}_{i,aq^2}\boldsymbol{\omega}^{-1}_{i,aq^6}\boldsymbol{\omega}_{j,aq}$. Next, we have $\chi_4 = (\chi_3)_{\mu_3}$. But the ℓ-weights of χ_3 with underlying classic weight $\mu_3 = 0$ are $\boldsymbol{\mu}_3$ and $\boldsymbol{\mu}_5$. Hence, $\chi_4 = \chi_3$. The only ℓ-weight of χ_4 with underlying classic weight equal to μ_4 is $\boldsymbol{\mu}_6$ and, therefore, $\chi_5 = (\chi_4)^j_{\boldsymbol{\mu}_6}$. Since $\zeta^j_{\boldsymbol{\mu}_6} = \boldsymbol{\mu}_6 + \boldsymbol{\mu}_6\boldsymbol{\alpha}^{-1}_{j,aq}$, we get

$$\chi_5 = \chi_4 + \boldsymbol{\omega}^{-1}_{i,aq^6}\boldsymbol{\omega}^{-1}_{j,aq^3}.$$

Set $\boldsymbol{\mu}_7 = \boldsymbol{\omega}^{-1}_{i,aq^6}\boldsymbol{\omega}^{-1}_{j,aq^3}$. Finally, the only ℓ-weight of χ_4 with classic weight μ_5 is $\boldsymbol{\mu}_4$ and $\chi_6 = (\chi_5)^i_{\boldsymbol{\mu}_4}$. Since $\zeta^i_{\boldsymbol{\mu}_4} = \boldsymbol{\mu}_4 + \boldsymbol{\mu}_7$, we get $\chi_6 = \chi_5$. This implies

$$\mathrm{FM}(\boldsymbol{\lambda}) = \boldsymbol{\lambda} + \sum_{r=1}^{7} \boldsymbol{\mu}_r$$

which coincides with the formula obtained in Example 3.2.9. Here is the associated Frenkel-Mukhin quiver:

$$\begin{tikzcd}
 & \boldsymbol{\lambda} \arrow[dl, "{(i,a)}"'] \arrow[dr, "{(j,aq^3)}"] & \\
\boldsymbol{\mu}_1 \arrow[d, "{(j,aq^3)}"'] & & \boldsymbol{\mu}_2 \arrow[dll, "{(i,a)}"'] \arrow[d, "{(i,aq^4)}"] \\
\boldsymbol{\mu}_3 \arrow[d, "{(j,aq)}"'] \arrow[drr, "{(i,aq^4)}"'] & & \boldsymbol{\mu}_5 \arrow[d, "{(i,a)}"] \\
\boldsymbol{\mu}_4 \arrow[dr, "{(i,aq^4)}"'] & & \boldsymbol{\mu}_6 \arrow[dl, "{(j,aq)}"] \\
 & \boldsymbol{\mu}_7 &
\end{tikzcd}$$

Notice that the i-roots are $\boldsymbol{\lambda}, \boldsymbol{\mu}_2$, and $\boldsymbol{\mu}_4$ while the j-roots are $\boldsymbol{\lambda}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_5$, and $\boldsymbol{\mu}_6$.

The other cases are obtained similarly. We list the quivers for the case $b = aq$ and $a/b \notin \{q^{\pm 1}, q^{\pm 3}\}$:

$$\begin{tikzcd}
 & \lambda \arrow[dl, "{(i,a)}"'] \arrow[dr, "{(j,aq)}"] & \\
\mu_1 \arrow[d, "{(j,aq)}"'] & & \mu_2 \arrow[d, "{(i,aq^2)}"] \\
2\mu_3 \arrow[d, "{(j,aq)}"'] & & \mu_5 \arrow[d, "{(i,a)}"] \\
\mu_4 \arrow[dr, "{(i,aq^2)}"'] & & \mu_6 \arrow[dl, "{(j,aq)}"] \\
 & \mu_7 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \lambda \arrow[dl, "{(i,a)}"'] \arrow[dr, "{(j,b)}"] & \\
\mu_1 \arrow[d, "{(j,aq)}"'] \arrow[dr, "{(j,b)}"] & & \mu_2 \arrow[dl, "{(i,a)}"'] \arrow[d, "{(i,bq)}"] \\
\mu_3 \arrow[d, "{(j,b)}"'] & \mu_4 \arrow[dl, "{(j,aq)}"] \arrow[dr, "{(i,bq)}"'] & \mu_6 \arrow[d, "{(i,a)}"] \\
\mu_5 \arrow[dr, "{(i,bq)}"'] & & \mu_7 \arrow[dl, "{(j,aq)}"] \\
 & \mu_8 &
\end{tikzcd}$$

In the left quiver, the i-roots are $\boldsymbol{\lambda}, \boldsymbol{\mu}_2, \boldsymbol{\mu}_3$ (with multiplicity 2), and $\boldsymbol{\mu}_4$ and the j-roots are $\boldsymbol{\lambda}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_5$, and $\boldsymbol{\mu}_6$. In the right quiver, the i-roots are $\boldsymbol{\lambda}, \boldsymbol{\mu}_2, \boldsymbol{\mu}_3$, and $\boldsymbol{\mu}_5$ and the j-roots are $\boldsymbol{\lambda}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_6$, and $\boldsymbol{\mu}_7$. $\Diamond$

In the above example, the use of the algorithm was purely illustrative since we already knew the qcharacter. We now exhibit a large and important class of ℓ-minuscule modules for which we do not have another general tool for computing the qcharacter other than the algorithm.

Definition 4.2.2. A Kirillov-Reshetikihin module is a $U_q(\tilde{\mathfrak{g}})$-module of the form $V_q(\boldsymbol{\omega}_{i,a,m})$ for some $i \in I, a \in \mathbb{F}^\times$ and $m \geq 0$. $\Diamond$

We shall say more about the importance of the Kirillov-Reshetikhin modules in Subsection 4.4. For the moment, the importance we mention is the following theorem.

Theorem 4.2.3. The Kirillov-Reshetikhin modules are ℓ-minuscule. In particular, their qcharacters are given by the Frenkel-Mukhin algorithm.

Before proving Theorem 4.2.3, let us use it to prove Theorem 3.3.13.

Proof of Theorem 3.3.13. Recall that by Proposition 2.3.3 and Corollary 3.2.7, it suffices to prove Theorem 3.3.13 for the fundamental modules which are Kirillov-Reshetikhin modules. In this case the result follows immediately from Theorem 4.2.3 and (4.1.11). $\square$

Remark 4.2.4. Notice that, if $\mathfrak{g} = \mathfrak{sl}_2$, the Kirillov-Reshetikhin module $V_q(\boldsymbol{\omega}_{i,a,m})$ is nothing but the evaluation module $V(m,a)$. Therefore, Theorem 4.2.3 follows from the explicit formulas for the qcharacters of such modules given in Theorem 2.3.14. We will use the $\mathfrak{sl}_2$ case of Theorem 3.3.10 during the proof of Theorem 4.2.3. Since the proof of the $\mathfrak{sl}_2$ case of Theorem 3.3.10 uses only the $\mathfrak{sl}_2$ case of Theorem 3.3.13 whose proof then uses only Theorem 2.3.14, our argument for proving the general case of Theorem 4.2.3 is not circular. $\Diamond$

We need some more terminology and a couple of preliminary technical lemmas.

Lemma 4.2.5. Let $i \in I, a \in \mathbb{F}^\times$, and consider the group homomorphism $f : \mathscr{P}_i \times \mathscr{E}_i \to \mathscr{P}_i \times \mathscr{E}_i$ given by $f(x,y) = (\rho_i(\boldsymbol{\alpha}_{i,a}^{-1})x, y)$. Then, there exists a unique group homomorphism $F : \mathscr{P} \to \mathscr{P}$ such that $\tau_i \circ F = f \circ \tau_i$ and $F(\boldsymbol{\mu}) = \boldsymbol{\alpha}_{i,a}^{-1}\boldsymbol{\mu}$ for all $\boldsymbol{\mu} \in \mathscr{P}$.

Proof. It is clear that the group homomorphism $F : \mathscr{P} \to \mathscr{P}$ given by $F(\boldsymbol{\mu}) = \boldsymbol{\alpha}_{i,a}^{-1}\boldsymbol{\mu}$ satisfies $\tau_i \circ F = f \circ \tau_i$. Since f is injective by Lemma 2.3.8, the uniqueness follows from Lemma 4.1.3. □

Lemma 4.2.6. Let V be a $U_q(\tilde{\mathfrak{g}})$-module and W be a $U_q(\tilde{\mathfrak{h}})$-submodule of V. Then, for all $j \in I$, $W^j = \sum\limits_{r \in \mathbb{Z}} x_{j,r}^- W$ is a $U_q(\tilde{\mathfrak{h}})$-submodule of V. Moreover, if V is finite-dimensional, then $W^j = \bigoplus\limits_{\boldsymbol{\mu} \in \mathscr{P}} W^j \cap V_{\boldsymbol{\mu}}$.

Proof. The first statement follows immediately from the defining relations $k_i x_{j,r}^- k_i^{-1} = q_i^{-c_{ij}} x_{j,r}^-$ and $[h_{i,s}, x_{j,r}^-] = -\frac{1}{s}\,[sc_{ij}]_{q_i} x_{j,r+s}^-$ with $i,j \in I, r,s \in \mathbb{Z}, s \neq 0$ of $U_q(\tilde{\mathfrak{g}})$. The second is standard linear algebra. □

Lemma 4.2.7. Suppose $\mathfrak{g} = \mathfrak{sl}_2$ and let $I = \{i\}$. Given a finite-dimensional $U_q(\tilde{\mathfrak{g}})$-module V and $m \geq 0$, consider the $U_q(\tilde{\mathfrak{h}})$-submodules of V: $V_{\geq m} = \bigoplus\limits_{n \geq m} V_{n\omega_i}$ and $V'_{\geq m} = \sum\limits_{r \in \mathbb{Z}} x_{i,r}^- V_{\geq m}$. Then, if $\boldsymbol{\mu} \in \mathscr{P}$ is such that $V'_{\geq m} \cap V_{\boldsymbol{\mu}} \neq \{0\}$, there exists $\boldsymbol{\lambda} \in \mathrm{wt}_\ell(V)$ such that $V_{\geq m} \cap V_{\boldsymbol{\lambda}} \neq \{0\}$ and $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$.

Proof. If $V_{\boldsymbol{\mu}} \cap V_{\geq m} \neq \{0\}$ take $\boldsymbol{\lambda} = \boldsymbol{\mu}$. Otherwise, we must have $\mathrm{wt}(\boldsymbol{\mu}) = k\omega_i$ for some $k < m$ and we will show that there exists $\boldsymbol{\lambda} \in \mathrm{wt}_\ell(V)$ such that $V_{\geq m} \cap V_{\boldsymbol{\lambda}} \neq \{0\}$ and $\boldsymbol{\mu} < \boldsymbol{\lambda}$. In that case, let $v \in V'_{\geq m} \cap V_{\boldsymbol{\mu}}$ and observe that, by definition of $V'_{\geq m}$, there must exist $v_r \in V_{k\omega_i + \alpha_i}$ and $c_r \in \mathbb{F}$ (all but finitely many nonzero) such that $v = \sum_r c_r\, x_{i,r}^- v_r$.

Proceed by induction on the dimension of $V_{\geq m}$ which clearly starts when $V_{\geq m} = \{0\}$. Choose a highest-ℓ-weight vector $w \in V \cap V_{\geq m}$ and consider the submodule W of V generated by v. If $W_{\boldsymbol{\mu}} \neq 0$ we are done by Theorem 3.3.13. Otherwise, notice that the image of $w' = \sum_r c_r v_r$ in $U = V/W$ is nonzero. Indeed, if $w' \in W$ then $v \in W$ as well. Therefore, the image $U_{\geq m}$ of $V_{\geq m}$ in U is nonzero and $U_{\geq m} = \sum_{n \geq m} U_{n\omega_i}$. Moreover, if we set $U'_{\geq m} = \sum_r x_{i,r}^- U_{\geq m}$, then the image of v in U belongs to $U'_{\geq m}$ and, hence, $U'_{\geq m} \cap U_{\boldsymbol{\mu}} \neq \{0\}$. Since $\dim(U) < \dim(V)$, the induction hypothesis completes the proof. □

Definition 4.2.8. Let $\boldsymbol{\mu} = \prod\limits_{i \in I, a \in \mathbb{F}^\times} \boldsymbol{\omega}_{i,a}^{p_{i,a}} \in \mathscr{P}$. The support of $\boldsymbol{\mu}$ is the set $\mathrm{supp}(\boldsymbol{\mu}) = \{(i,a) \in I \times \mathbb{F}^\times : p_{i,a} \neq 0\}$. An element $a \in \mathbb{F}^\times$ is said to be a spectral parameter of $\boldsymbol{\mu}$ if $(i,a) \in \mathrm{supp}(\boldsymbol{\mu})$ for some $i \in I$. A (quantum) spectral parameter base for $\mathrm{supp}(\boldsymbol{\mu})$ is a subset $\{a_1, \ldots, a_m\}$ of the set of spectral parameters of $\boldsymbol{\mu}$ such that a_j/a_k is not a power of q for all j,k and for every other spectral parameter a of $\boldsymbol{\mu}$ there exists j such that a_j/a is a power of q. A spectral parameter a of $\boldsymbol{\mu}$ is said to be a right-most spectral parameter of $\boldsymbol{\mu}$ if and a/b is not a negative power of q for every other spectral parameter b of $\boldsymbol{\mu}$. If $p_{i,a} < 0$ for every right-most spectral parameter a of $\boldsymbol{\mu}$, then $\boldsymbol{\mu}$ is said to be right-negative. ◊

Remark 4.2.9. The choice for the terminology right-most is the following. Suppose $\mathrm{supp}(\boldsymbol{\mu})$ has a spectral parameter basis with a single element and let $a_1, \ldots, a_m$ be the distinct spectral parameters. Suppose they are ordered in such away that a_j/a_k is a negative power of q if and only if $j < k$. Then, a_m is the right-most spectral parameter. Observe that the product of two right-negative ℓ-weights is also right-negative and that the inverse of the simple ℓ-roots are right-negative. In particular, if $\boldsymbol{\mu}$ is right-negative and $\boldsymbol{\nu} \le \boldsymbol{\mu}$, then $\boldsymbol{\nu}$ is also right-negative. Evidently, a right-negative ℓ-weight is not dominant. Observe also that $\boldsymbol{\omega}_{i,a,m}\boldsymbol{\alpha}^{-1}_{i,aq^{m-1}}$ is right-negative for every $i \in I, a \in \mathbb{F}^\times$, and $m > 0$. ◊

Using the above remark, Theorem 4.2.3 is an immediate corollary of the following lemma.

Lemma 4.2.10. Let $i \in I, a \in \mathbb{F}^\times$, and $m > 0$. If $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}_{i,a,m})) \setminus \{\boldsymbol{\omega}_{i,a,m}\}$, then $\boldsymbol{\mu} \le \boldsymbol{\omega}_{i,a,m}\boldsymbol{\alpha}^{-1}_{i,aq^{m-1}}$.

Proof. Let v be a highest-ℓ-weight vector for $V = V_q(\boldsymbol{\omega}_{i,a,m})$ and let V^i be $U_q(\tilde{\mathfrak{g}}_i)$-submodule of V generated by v which is irreducible by Lemma 2.2.13. In fact, it is isomorphic to the $U_q(\tilde{\mathfrak{g}}_i)$-evaluation module $V_{q_i}(a,m)$. It follows from Theorem 2.3.14 that

$$\boldsymbol{\omega}_{i,a,m}\left(\prod_{r=1}^{s}\boldsymbol{\alpha}^{-1}_{i,aq^{m+1-2r}}\right) \in \mathrm{wt}_\ell(V) \qquad \text{for all} \qquad 0 \le s \le m.$$

Notice that these are exactly the elements of $\mathrm{wt}_\ell(\zeta^i_{\boldsymbol{\omega}_{i,a,m}})$.

We will prove by induction on the height h of $m\omega_i - \mu$ that, if $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}_{i,a,m})) \setminus \{\boldsymbol{\omega}_{i,a,m}\}$ is such that $\mathrm{wt}(\boldsymbol{\mu}) = \mu$, then $\boldsymbol{\mu} \le \boldsymbol{\omega}_{i,a,m}\boldsymbol{\alpha}^{-1}_{i,aq^{m-1}}$. This clearly proves the proposition. Notice that if $h = 1$, then $V_\mu \ne 0$ if and only if $\mu = m\omega_i - \alpha_i$ and, moreover, V_μ is one-dimensional. Therefore, we must have $\boldsymbol{\mu} \le \boldsymbol{\omega}_{i,a,m}\boldsymbol{\alpha}^{-1}_{i,aq^{m-1}}$ proving that induction starts. Thus, assume $h > 1$ and, by induction hypothesis, that $\boldsymbol{\mu} \le \boldsymbol{\omega}_{i,a,m}\boldsymbol{\alpha}^{-1}_{i,aq^{m-1}}$ for all $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V)$ such that $|m\omega_i - \mathrm{wt}(\boldsymbol{\mu})| < h$. Consider

$$W = \bigoplus_{\nu\,:\,|m\omega_i - \nu| < h} V_\nu$$

which is a $U_q(\tilde{\mathfrak{h}})$-submodule of V by (2.3.1). Given $j \in I$, let W^j be as in Lemma 4.2.6. In particular,

$$W^j = \bigoplus_{\boldsymbol{\mu} \in \mathscr{P}} W^j \cap V_{\boldsymbol{\mu}} \qquad \text{and} \qquad \bigoplus_{\mu\,:\,|m\omega_i - \mu| = h} V_\mu \subseteq \sum_{j \in I} W^j. \tag{4.2.1}$$

Set $W^j_{\boldsymbol{\mu}} = W^j \cap V_{\boldsymbol{\mu}}$. Let $V = \bigoplus_{\gamma \in \mathscr{E}_j} V_\gamma$ be the j-th elliptic decomposition of V and, given $\gamma \in \mathscr{E}_j$, let

$$W^j_\gamma = \bigoplus_{\boldsymbol{\mu}\,:\,\epsilon_j(\boldsymbol{\mu}) = \gamma} W^j_{\boldsymbol{\mu}} = \bigoplus_{\boldsymbol{\mu}\,:\,\epsilon_j(\boldsymbol{\mu}) = \gamma} W^j \cap V_\gamma.$$

In particular, we have $W^j = \bigoplus_{\gamma \in \mathscr{E}_j} W^j_\gamma$.

Let $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V)$ be such that $|m\omega_i - \mathrm{wt}(\boldsymbol{\mu})| = h$. It follows from the second part of (4.2.1) that there exists $j \in I$ such that $W^j_{\boldsymbol{\mu}} \ne \{0\}$. Set $\gamma = \epsilon_j(\boldsymbol{\mu})$ and $p = \mathrm{wt}(\boldsymbol{\mu})(h_j) + 1$. Consider $(V_\gamma)_{\ge p}$ and $(V_\gamma)'_{\ge p}$ as defined in Lemma 4.2.7. Notice that $(V_\gamma)_{\ge p} \cap V_{\boldsymbol{\mu}} = \{0\}$ since ρ_j is injective on $\epsilon_j^{-1}(\gamma)$ by Lemma 4.1.3. We claim that $(V_\gamma)_{\ge p} \ne \{0\}$. Assuming this, it follows that $(V_\gamma)'_{\ge p} \cap W^j_{\boldsymbol{\mu}} \ne \{0\}$ and Lemma 4.2.7 implies that $\rho_j(\boldsymbol{\mu}) = \boldsymbol{\lambda}\rho_j(\boldsymbol{\alpha}^{-1}_{j,b})$ for some $b \in \mathbb{F}^\times$ and some $\boldsymbol{\lambda} \in \mathscr{P}_j \cap \mathrm{wt}_\ell(V_\gamma)$. Using that ρ_j is injective on $\epsilon_j^{-1}(\gamma)$ once more, it follows that there exists unique $\boldsymbol{\nu} \in \mathscr{P}$ such that $\rho_j(\boldsymbol{\nu}) = \boldsymbol{\lambda}$ and $\epsilon_j(\boldsymbol{\nu}) = \gamma$ (more precisely, $\boldsymbol{\nu} = \tau_j^{-1}(\boldsymbol{\lambda}, \gamma)$). We must have $V_{\boldsymbol{\nu}} \ne \{0\}$ since, otherwise, $(V_\gamma)_{\boldsymbol{\lambda}}$ would be $\{0\}$. By Lemma 4.2.5, we must have $\boldsymbol{\mu} = \boldsymbol{\nu}\boldsymbol{\alpha}^{-1}_{j,b}$. Evidently, the induction hypothesis applies to $\boldsymbol{\nu}$ and the proof is complete.

It remains to prove that $(V_\gamma)_{\geq p} \neq \{0\}$. Suppose that was not the case, which is equivalent to saying that $\mathrm{wt}(\boldsymbol{\mu})(h_j)\omega_j$ is a maximal weight of V_γ. Then, there must exist a vector $w \in W^j_\gamma$ which is a highest-ℓ-weight vector of ℓ-weight $\rho_j(\boldsymbol{\mu})$ for the action of $U_q(\tilde{\mathfrak{g}}_j)$. Let N be the $U_q(\tilde{\mathfrak{g}}_j)$-submodule of V generated by w which is a submodule of V_γ. On the other hand, since $w \in W^j$, there exist ℓ-weight vectors $w_r \in V$ (all but finitely many nonzero) such that $w = \sum_r x^-_{j,r} w_r$. Let M be the $U_q(\tilde{\mathfrak{g}}_j)$-submodule of V generated by the vectors w_r which contains N. However, the ℓ-weight $\boldsymbol{\mu}_r$ of w_r satisfies $\gamma_r := \epsilon_j(\boldsymbol{\mu}_r) \neq \gamma$ since otherwise we would have $w_r \in (V_\gamma)_{\geq p}$. This implies $w_r \in V_{\gamma_r}$ and, therefore, $M \subseteq \bigoplus_r V_{\gamma_r}$. In particular, we have $\{0\} = M \cap V_\gamma \supseteq N$ which yields a contradiction. □

Example 4.2.11. Let $\mathfrak{g} = \mathfrak{sl}_3$ and denote by i and j the distinct elements of I. In Example 3.1.6 we computed the qcharacter of the fundamental modules $V_q(\boldsymbol{\omega}_{i,a})$ while in Example 3.2.10 we computed $\mathrm{qch}(V_q(\boldsymbol{\omega}_{i,a,2}))$. Now we use the algorithm to compute $\mathrm{qch}(V_q(\boldsymbol{\omega}_{i,a,3}))$. It will be more convenient to work with $\boldsymbol{\lambda} = \boldsymbol{\omega}_{i,aq^2,3} = \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{i,aq^2}\boldsymbol{\omega}_{i,aq^4}$. Setting $\chi_0 = \boldsymbol{\lambda}$, we have

$$\begin{aligned}\chi_1 = (\chi_0)^i_{\boldsymbol{\lambda}} &= \boldsymbol{\lambda} + \boldsymbol{\lambda}\boldsymbol{\alpha}^{-1}_{i,aq^4} + \boldsymbol{\lambda}\boldsymbol{\alpha}^{-1}_{i,aq^4}\boldsymbol{\alpha}^{-1}_{i,aq^2} + \boldsymbol{\lambda}\boldsymbol{\alpha}^{-1}_{i,aq^4}\boldsymbol{\alpha}^{-1}_{i,aq^2}\boldsymbol{\alpha}^{-1}_{i,aq}\\ &= \boldsymbol{\lambda} + \boldsymbol{\omega}_{i,aq,2}\boldsymbol{\omega}^{-1}_{i,aq^6}\boldsymbol{\omega}_{j,aq^5} + \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}^{-1}_{i,aq^5,2}\boldsymbol{\omega}_{j,aq^4,2} + \boldsymbol{\omega}^{-1}_{i,aq^4,3}\boldsymbol{\omega}_{j,aq^3,3} = \boldsymbol{\lambda} + \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + \boldsymbol{\mu}_3.\end{aligned}$$

Let us choose $\mu_r = 3\omega_i - r\alpha_i$ for $r = 1,2,3$. Then, $\chi_{r+1} = (\chi_r)^j_{\boldsymbol{\mu}_r}$, where $\boldsymbol{\mu}_r$ is defined above. Now, $\zeta^j_{\boldsymbol{\mu}_1} = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_1\boldsymbol{\alpha}^{-1}_{j,aq^5}$ and $\chi_2 = \chi_1 + \boldsymbol{\mu}_1\boldsymbol{\alpha}^{-1}_{j,aq^5}$. Also, $\zeta^j_{\boldsymbol{\mu}_2} = \boldsymbol{\mu}_2 + \boldsymbol{\mu}_2\boldsymbol{\alpha}^{-1}_{j,aq^5} + \boldsymbol{\mu}_2\boldsymbol{\alpha}^{-1}_{j,aq^5}\boldsymbol{\alpha}^{-1}_{j,aq^3}$ and $\chi_3 = \chi_2 + \boldsymbol{\mu}_2\boldsymbol{\alpha}^{-1}_{j,aq^5} + \boldsymbol{\mu}_2\boldsymbol{\alpha}^{-1}_{j,aq^5}\boldsymbol{\alpha}^{-1}_{j,aq^3}$. Finally, $\zeta^j_{\boldsymbol{\mu}_3} = \boldsymbol{\mu}_3 + \boldsymbol{\mu}_3\boldsymbol{\alpha}^{-1}_{j,aq^5} + \boldsymbol{\mu}_3\boldsymbol{\alpha}^{-1}_{j,aq^5}\boldsymbol{\alpha}^{-1}_{j,aq^3} + \boldsymbol{\mu}_3\boldsymbol{\alpha}^{-1}_{j,aq^5}\boldsymbol{\alpha}^{-1}_{j,aq}$ and $\chi_4 = \chi_3 + \boldsymbol{\mu}_3\boldsymbol{\alpha}^{-1}_{j,aq^5} + \boldsymbol{\mu}_3\boldsymbol{\alpha}^{-1}_{j,aq^5}\boldsymbol{\alpha}^{-1}_{j,aq^3} + \boldsymbol{\mu}_3\boldsymbol{\alpha}^{-1}_{j,aq^5}\boldsymbol{\alpha}^{-1}_{j,aq}$. One can now check that for any choice of ordering of the remaining elements of $\mathrm{wt}(3\omega_i)$, the algorithm will not produce any further changes to χ_4. Therefore $\mathrm{qch}(V_q(\boldsymbol{\lambda})) = \chi_4$. The Frenkel-Mukhin quiver is given by:

$$\begin{tikzcd}
\boldsymbol{\lambda} \arrow[r, "{(i,aq^4)}"] & \boldsymbol{\mu}_1 \arrow[r, "{(i,aq^2)}"] \arrow[d, "{(j,aq^5)}"'] & \boldsymbol{\mu}_2 \arrow[r, "{(i,a)}"] \arrow[d, "{(j,aq^5)}"'] & \boldsymbol{\mu}_3 \arrow[d, "{(j,aq^5)}"'] \\
 & \boldsymbol{\mu}_4 \arrow[r, "{(i,aq^2)}"] & \boldsymbol{\mu}_5 \arrow[r, "{(i,a)}"] \arrow[d, "{(j,aq^3)}"'] & \boldsymbol{\mu}_7 \arrow[d, "{(j,aq^3)}"'] \\
 & & \boldsymbol{\mu}_6 \arrow[r, "{(i,a)}"] & \boldsymbol{\mu}_8 \arrow[d, "{(j,aq)}"'] \\
 & & & \boldsymbol{\mu}_9
\end{tikzcd}$$

The i-roots are $\boldsymbol{\lambda}, \boldsymbol{\mu}_4, \boldsymbol{\mu}_6$, and $\boldsymbol{\mu}_9$ while the j-roots are $\boldsymbol{\lambda}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2$, and $\boldsymbol{\mu}_3$. The reader should have no difficult generalizing the above to obtain a formula for $\mathrm{qch}(V_q(\boldsymbol{\omega}_{i,a,m}))$ for all $m \geq 0$. ◊

Example 4.2.12. Let us consider $\mathfrak{g}$ of type B_2 ($\mathfrak{so}_5$). In this case we have a short simple root which denote by α_i and a long simple root which we denote by α_j. Then $q_i = q^2, q_j = q, \boldsymbol{\alpha}_{i,a} = \boldsymbol{\omega}_{i,aq_i,2}\boldsymbol{\omega}^{-1}_{j,aq^2,2}$, and $\boldsymbol{\alpha}_{j,a} = \boldsymbol{\omega}_{j,aq,2}\boldsymbol{\omega}^{-1}_{i,aq}$. The fundamental weight ω_j is minuscule and, therefore, $\mathrm{qch}(V_q(\boldsymbol{\omega}_{j,a}))$ is given by 3.1.4. The fundamental weight ω_i is not minuscule and we will use the algorithm to compute $\mathrm{qch}(V_q(\boldsymbol{\omega}_{i,a}))$. We have $\zeta^i_{\boldsymbol{\omega}_{i,a}} = \boldsymbol{\omega}_{i,a} + \boldsymbol{\omega}_{i,a}\boldsymbol{\alpha}^{-1}_{i,a}$ and $\chi_1 = \zeta^i_{\boldsymbol{\omega}_{i,a}}$. Setting $\boldsymbol{\mu}_1 = \boldsymbol{\omega}_{i,a}\boldsymbol{\alpha}^{-1}_{i,a} = \boldsymbol{\omega}^{-1}_{i,aq_i^2}\boldsymbol{\omega}_{j,aq^2,2}$

we proceed by computing $\chi_2 = (\chi_1)^j_{\mu_1}$. We have $\zeta^j_{\mu_1} = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_1\boldsymbol{\alpha}^{-1}_{j,aq^3} + \boldsymbol{\mu}_1\boldsymbol{\alpha}^{-1}_{j,aq^3}\boldsymbol{\alpha}^{-1}_{j,aq}$ and $\chi_2 = \chi_1 + \boldsymbol{\mu}_1\boldsymbol{\alpha}^{-1}_{j,aq^3} + \boldsymbol{\mu}_1\boldsymbol{\alpha}^{-1}_{j,aq^3}\boldsymbol{\alpha}^{-1}_{j,aq}$. The only ℓ-weight of χ_2 whose expansion will change χ_2 is $\boldsymbol{\mu}_3 := \boldsymbol{\mu}_1\boldsymbol{\alpha}^{-1}_{j,aq^3}\boldsymbol{\alpha}^{-1}_{j,aq} = \boldsymbol{\omega}_{i,aq_i}\boldsymbol{\omega}^{-1}_{j,aq^4,2}$. The corresponding expansion is similar to the first step above and we get $\chi_3 = \chi_2 + \boldsymbol{\mu}_3\boldsymbol{\alpha}^{-1}_{i,aq_i}$. This completes the algorithm and the qcharacter can be read off the quiver:

$$\begin{tikzcd}
\omega_{i,a} \arrow[r, "(i{,}a)"] & \mu_1 \arrow[d, "(j{,}aq^3)"'] & \\
& \mu_2 \arrow[d, "(j{,}aq)"'] & \\
& \mu_3 \arrow[r, "(i{,}aq_i)"] & \mu_4
\end{tikzcd}$$

$\Diamond$

Example 4.2.13. All the above examples concerned ℓ-minuscule modules. We now give an example which is not ℓ-minuscule and the algorithm returns the correct qcharacter. Let $\mathfrak{g} = \mathfrak{sl}_3$, denote by i and j the distinct elements of I, let $a \in \mathbb{F}^\times$, and consider $\boldsymbol{\lambda} = \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{i,aq,2} = \boldsymbol{\omega}^2_{i,a}\boldsymbol{\omega}_{i,aq^2}$. We will show that $V_q(\boldsymbol{\lambda}) \cong V_q(\boldsymbol{\omega}_{i,a}) \otimes V_q(\boldsymbol{\omega}_{i,aq,2})$.

Recall in Examples 3.1.6 and 3.2.10 that $\mathrm{qch}(V_q(\boldsymbol{\omega}_{i,a})) = \boldsymbol{\omega}_{i,a} + \boldsymbol{\omega}^{-1}_{i,aq^2}\boldsymbol{\omega}_{j,aq} + \boldsymbol{\omega}^{-1}_{j,aq^3}$ and

$$\mathrm{qch}(V_q(\boldsymbol{\omega}_{i,aq,2})) = \boldsymbol{\omega}_{i,aq,2} + \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}^{-1}_{i,aq^4}\boldsymbol{\omega}_{j,aq^3} + \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}^{-1}_{j,aq^5} + \boldsymbol{\omega}^{-1}_{i,aq^3,2}\boldsymbol{\omega}_{j,aq^2,2} + \boldsymbol{\omega}^{-1}_{i,aq^2}\boldsymbol{\omega}_{j,aq}\boldsymbol{\omega}^{-1}_{j,aq^5} + \boldsymbol{\omega}^{-1}_{j,aq^4,2}.$$

One quickly checks that $\mathrm{wt}_\ell(V_q(\boldsymbol{\omega}_{i,a})\otimes V_q(\boldsymbol{\omega}_{i,aq,2}))\cap\mathscr{P}^+ = \{\boldsymbol{\lambda}, \boldsymbol{\mu}\}$ where $\boldsymbol{\mu} = \boldsymbol{\omega}^{-1}_{i,aq^2}\boldsymbol{\omega}_{j,aq}\boldsymbol{\omega}_{i,aq,2} = \boldsymbol{\lambda}\boldsymbol{\alpha}^{-1}_{i,a} = \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,aq}$. Hence, if $V_q(\boldsymbol{\omega}_{i,a}) \otimes V_q(\boldsymbol{\omega}_{i,aq,2})$ were reducible it would have $V_q(\boldsymbol{\mu})$ as a simple factor. From Example 3.2.9 we know that $V_q(\boldsymbol{\mu}) \cong V_q(\boldsymbol{\omega}_{i,a}) \otimes V_q(\boldsymbol{\omega}_{j,aq})$ and, therefore, $\boldsymbol{\mu}\boldsymbol{\alpha}^{-1}_{i,a} = \boldsymbol{\omega}^{-1}_{i,aq^2}\boldsymbol{\omega}^2_{j,aq}$ would be an ℓ-weight of $V_q(\boldsymbol{\omega}_{i,a}) \otimes V_q(\boldsymbol{\omega}_{i,aq,2})$. A quick check above shows that this is not the case.

Since $\mathrm{qch}(V_q(\boldsymbol{\lambda}))(\boldsymbol{\mu}) = 1$, Theorem 4.1.19 implies that in order to check that $\mathrm{FM}(\boldsymbol{\lambda}) = \mathrm{qch}(V_q(\boldsymbol{\lambda}))$ it suffices to check that $\boldsymbol{\mu} \in \mathrm{wt}_\ell(\mathrm{FM}(\boldsymbol{\lambda}))$. It suffices to see that $\boldsymbol{\mu} \in \mathrm{wt}_\ell(\zeta^i_{\boldsymbol{\lambda}})$ which is clear since the $U_q(\tilde{\mathfrak{g}}_i)$-module $V_q(\boldsymbol{\lambda}_i)$ is isomorphic to $V_q(1, a) \otimes V_q(2, aq)$. $\Diamond$

Next, we give an example for which the algorithm fails to return the correct qcharacter.

Example 4.2.14. Let $\mathfrak{g} = \mathfrak{sl}_3$ and denote by i and j the distinct elements of I. Consider $\boldsymbol{\lambda} = \boldsymbol{\omega}^2_{i,a}\boldsymbol{\omega}_{j,aq^3}$. Let us show that $\mathrm{FM}(\boldsymbol{\lambda}) \neq \mathrm{qch}(V_q(\boldsymbol{\lambda}))$. We claim that $V_q(\boldsymbol{\lambda}) \cong V_q(\boldsymbol{\omega}_{i,a}) \otimes V_q(\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,aq^3})$. Indeed, recall from Example 3.2.9 that $\mathrm{qch}(V_q(\boldsymbol{\omega}_{i,a})) = \boldsymbol{\omega}_{i,a} + \boldsymbol{\omega}^{-1}_{i,aq^2}\boldsymbol{\omega}_{j,aq} + \boldsymbol{\omega}^{-1}_{j,aq^3}$ and

$$\begin{aligned}
\mathrm{qch}(V_q(\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,aq^3})) = {} & \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,aq^3} + \boldsymbol{\omega}^{-1}_{i,aq^2}\boldsymbol{\omega}_{j,aq}\boldsymbol{\omega}_{j,aq^3} + \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{i,aq^4}\boldsymbol{\omega}^{-1}_{j,aq^5} \\
& + \boldsymbol{\omega}^{-1}_{i,aq^6}\boldsymbol{\omega}_{i,a} + \boldsymbol{\omega}_{i,aq^4}\boldsymbol{\omega}^{-1}_{i,aq^2}\boldsymbol{\omega}_{j,aq}\boldsymbol{\omega}^{-1}_{j,aq^5} \\
& + \boldsymbol{\omega}^{-1}_{i,aq^2}\boldsymbol{\omega}^{-1}_{i,aq^6}\boldsymbol{\omega}_{j,aq} + \boldsymbol{\omega}_{i,aq^4}\boldsymbol{\omega}^{-1}_{j,aq^5}\boldsymbol{\omega}^{-1}_{j,aq^3} + \boldsymbol{\omega}^{-1}_{i,aq^6}\boldsymbol{\omega}^{-1}_{j,aq^3}.
\end{aligned}$$

One easily checks that

$$\mathrm{wt}_\ell\left(V_q(\boldsymbol{\omega}_{i,a}) \otimes V_q(\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,aq^3})\right) \cap \mathscr{P}^+ = \{\boldsymbol{\lambda}, \boldsymbol{\omega}_{i,a}\}.$$

Therefore, if $V_q(\boldsymbol{\omega}_{i,a}) \otimes V_q(\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,aq^3})$ were not simple, it would have $V_q(\boldsymbol{\omega}_{i,a})$ as a simple factor and, hence, there would be an inclusion $\mathrm{wt}_\ell(V_q(\boldsymbol{\omega}_{i,a})) \subseteq \mathrm{wt}_\ell\left(V_q(\boldsymbol{\omega}_{i,a}) \otimes V_q(\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,aq^3})\right)$. However, one easily checks that $\mathrm{wt}_\ell(V_q(\boldsymbol{\omega}_{i,a})) \cap \mathrm{wt}_\ell\left(V_q(\boldsymbol{\omega}_{i,a}) \otimes V_q(\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,aq^3})\right) = \{\boldsymbol{\omega}_{i,a}\}$. This proves the claim and, hence,

$$\mathrm{qch}(V_q(\boldsymbol{\lambda})) = \mathrm{qch}(V_q(\boldsymbol{\omega}_{i,a}))\,\mathrm{qch}(V_q(\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,aq^3})).$$

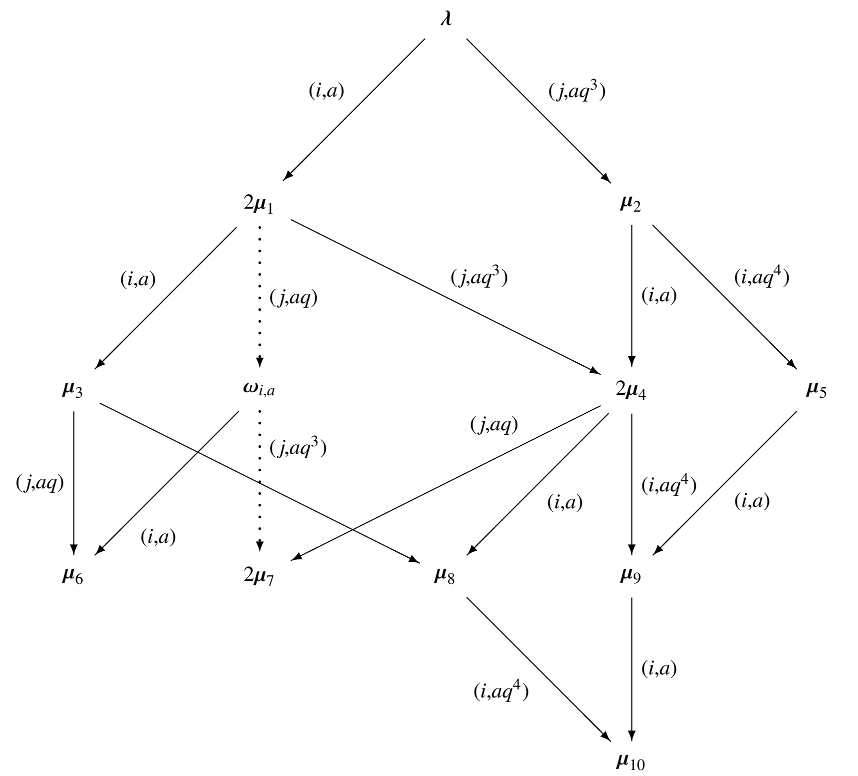
In particular, $\boldsymbol{\omega}_{i,a} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\lambda}))$.

Let $\mu_0 = \mathrm{wt}(\boldsymbol{\lambda}) = 2\omega_i + \omega_j, \mu_1 = \mu_0 - \alpha_i = 2\omega_j, \mu_2 = \mu_0 - \alpha_j = 3\omega_i - \omega_j$, and $\mu_3 = \mu_0 - \alpha_i - \alpha_j = \omega_i$. Notice that if $\mu \in \mathrm{wt}(\mu_0) \setminus \{\mu_0\}$, then $\mu \le \mu_r$ for some $1 \le r \le 4$. Therefore, we can use the sequence $\mu_0, \mu_1, \mu_2, \mu_3$, to compute the first four steps of the algorithm. In particular, if $\boldsymbol{\omega}_{i,a}$ does not appear after the expansion with respect to μ_2, it follows that $\boldsymbol{\omega}_{i,a} \notin \mathrm{wt}_\ell(\mathrm{FM}(\boldsymbol{\lambda}))$ showing that $\mathrm{FM}(\boldsymbol{\lambda}) \neq \mathrm{qch}(V_q(\boldsymbol{\lambda}))$. Thus, set $\chi_0 = \boldsymbol{\lambda}$ and notice that

$$\begin{aligned}\chi_1 = (\chi_0)_{\mu_0} &= \boldsymbol{\lambda} + 2\boldsymbol{\lambda}\boldsymbol{\alpha}_{i,a}^{-1} + \boldsymbol{\lambda}\boldsymbol{\alpha}_{i,a}^{-2} + \boldsymbol{\lambda}\boldsymbol{\alpha}_{j,aq^3}^{-1}\\ &= \boldsymbol{\lambda} + 2\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{i,aq^2}^{-1}\boldsymbol{\omega}_{j,aq^2,2} + \boldsymbol{\omega}_{i,aq^2}^{-2}\boldsymbol{\omega}_{j,aq}\boldsymbol{\omega}_{j,aq^2,2} + \boldsymbol{\omega}_{i,a}^{2}\boldsymbol{\omega}_{i,aq^4}\boldsymbol{\omega}_{j,aq^5}^{-1}.\end{aligned}$$

The only ℓ-weight of classic weight μ_1 in χ_1 is $\boldsymbol{\mu}_1 := \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{i,aq^2}^{-1}\boldsymbol{\omega}_{j,aq^2,2}$ which is not i-dominant. Therefore, $\chi_2 = (\chi_1)^j_{\boldsymbol{\mu}_1}$. One easily checks that $\boldsymbol{\omega}_{i,a} \notin \mathrm{wt}_\ell(\zeta^j_{\boldsymbol{\mu}_1})$ and, hence, $\boldsymbol{\omega}_{i,a} \notin \mathrm{wt}_\ell(\chi_2)$. Notice also that no new ℓ-weight of classic weight μ_2 is obtained in step 1 and, therefore, the only ℓ-weight of classic weight μ_2 in χ_2 is $\boldsymbol{\mu}_2 = \boldsymbol{\omega}_{i,a}^2\boldsymbol{\omega}_{i,aq^4}\boldsymbol{\omega}_{j,aq^5}^{-1}$ which is not j-dominant. Since $\boldsymbol{\mu}_2 = \boldsymbol{\lambda}\boldsymbol{\alpha}_{j,aq^3}^{-1}$ and $\boldsymbol{\omega}_{i,a} = \boldsymbol{\lambda}\boldsymbol{\alpha}_{j,aq}^{-1}\boldsymbol{\alpha}_{i,a}^{-1}$, $\boldsymbol{\omega}_{i,a} \notin \mathrm{wt}_\ell(\zeta^i_{\boldsymbol{\mu}_2})$ and, therefore, $\boldsymbol{\omega}_{i,a} \notin \mathrm{wt}_\ell(\chi_3)$ as claimed.

Let us draw a piece of the Frenkel-Mukhin quiver to illustrate the reason for the above failure.



Above we drew the arrows corresponding to the expansions of $\boldsymbol{\lambda}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\mu}_3$, and $\boldsymbol{\omega}_{i,a}$. Comparing with Example 4.1.12, we see that the failure arose because there exists a sub-quotient of the $U_q(\tilde{\mathfrak{g}}_j)$-module generated by $V_q(\boldsymbol{\lambda})_{\mu_1}$ which is a local Weyl module instead of an irreducible module. Notice that the ℓ-weight coming from the expansion of $\boldsymbol{\omega}_{i,a}$ is $\boldsymbol{\mu}_6$ which is also obtained from the expansion of $\boldsymbol{\mu}_3$ and, hence, $\boldsymbol{\mu}_6 \in \mathrm{wt}_\ell(\mathrm{FM}(\boldsymbol{\lambda}))$. One can carry out the algorithm and show that $\mathrm{FM}(\boldsymbol{\lambda}) = \mathrm{qch}(V_q(\boldsymbol{\lambda})) - \boldsymbol{\omega}_{i,a}$. However, it should be noticed that the coloring of the element χ obtained at the moment we need to expand with respect to $\boldsymbol{\mu}_6$ is not admissible. Indeed, $c^i_\chi(\boldsymbol{\mu}_6) = 0$ (since no prior expansion with respect to color i will generate $\boldsymbol{\mu}_6$) and $\boldsymbol{\mu}_6 = \boldsymbol{\omega}^{-1}_{i,aq^2}\boldsymbol{\omega}_{j,aq}$ is not i-dominant. $\Diamond$

4.3. Braid group and fundamental modules

We have seen that the qcharacters are not invariant under the braid group action. In this subsection, we show that if $\mathfrak{g}$ is not of exceptional type, the fundamental representations satisfy a weaker type of invariance. In particular, we will obtain an algorithm for computing their qcharacters as an expression involving the braid group action. We begin with the following lemma.

Lemma 4.3.1. Let V be a finite-dimensional $U_q(\tilde{\mathfrak{g}})$-module and $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V)$. Suppose there exist a nonzero $v \in V_{\boldsymbol{\mu}}$ and $j \in I$ such that $x^+_{j,s}v = 0$ for all $s \in \mathbb{Z}$. Then, $\boldsymbol{\mu}_j(u)$ is a polynomial of degree $\mathrm{wt}(\boldsymbol{\mu})(h_j)$ and $(x^-_j)^{\mathrm{wt}(\boldsymbol{\mu})(h_j)}v \in V_{T_j(\boldsymbol{\mu})}\setminus\{0\}$. Also, if the q_j-factorization of $\boldsymbol{\mu}_j$ is given by

$$\boldsymbol{\mu}_j = \prod_{r=1}^{k} \boldsymbol{\omega}_{j,a_r,m_r},$$

then $\boldsymbol{\mu}\boldsymbol{\alpha}^{-1}_{j,a_rq_j^{m_r-1}} \in \mathrm{wt}_\ell(V)$ for all $1 \le r \le k$. Furthermore, for all $s \in \mathbb{Z}$, we have

$$x^-_{j,s}v \in \sum_{r=1}^{k}\sum_{p=0}^{m_r-1} V_{\boldsymbol{\mu}\boldsymbol{\alpha}^{-1}_{j,a_rq_j^{m_r-1-2p}}},$$

and

$$\dim(V_{\boldsymbol{\mu}\boldsymbol{\alpha}^{-1}_{j,a_rq_j^{m_r-1}}}) \ge |\{1 \le s \le k : a_rq_j^{m_r} = a_sq_j^{m_s}\}|.$$

Similar statements (mutatis-mutandis) hold if we replace the hypothesis $x^+_{j,s}v = 0$ for all $s \in \mathbb{Z}$ by $x^-_{j,s}v = 0$ for all $s \in \mathbb{Z}$.

Proof. Assume first that v is an eigenvector for the action of $U_q(\tilde{\mathfrak{h}})$. The first statement is then clear since v will generate a highest-ℓ-weight module for $U_q(\tilde{\mathfrak{g}}_j)$ with highest ℓ-weight $\boldsymbol{\mu}_j$. The second statement is proved as in Proposition 3.1.2. To prove the third statement, let V' be the $U_q(\tilde{\mathfrak{g}}_j)$-module generated by v. Then, V' is isomorphic to a sub-quotient of the Weyl module for $U_{q_j}(\tilde{\mathfrak{sl}}_2)$ with highest-ℓ-weight $\boldsymbol{\mu}_j$. I particular, the ℓ-weights of V' of underlying classic weight $(\mathrm{wt}(\boldsymbol{\mu})(h_j) - 2)\omega_j$ are of the form $\rho_j(\boldsymbol{\mu}\boldsymbol{\alpha}^{-1}_{j,a_rq_j^{m_r-1-2p}})$ with $0 \le p < m_r$. Since $V_{\boldsymbol{\mu}} \subseteq V_\gamma$ where $\gamma = \epsilon_j(\boldsymbol{\mu}) \in \mathscr{E}_j$, a computation analogous to that closing the proof of Proposition 4.1.4 completes the proof of the third statement. Notice that, if $a_rq_j^{m_r} = a_sq_j^{m_s}$, then $\rho_j(\boldsymbol{\omega}_{j,a_r,m_r}\boldsymbol{\alpha}^{-1}_{j,a_rq_j^{m_r-1}})\boldsymbol{\omega}_{j,a_s,m_s} = \boldsymbol{\omega}_{j,a_r,m_r}\rho_j(\boldsymbol{\omega}_{j,a_s,m_s}\boldsymbol{\alpha}^{-1}_{j,a_sq_j^{m_s-1}})$ and, hence,

$$\mathrm{qch}(V_{q_j}(m_r,a_r) \otimes V_{q_j}(m_s,a_s))(\rho_j(\boldsymbol{\omega}_{j,a_r,m_r}\boldsymbol{\alpha}^{-1}_{j,a_rq_j^{m_r-1}})\boldsymbol{\omega}_{j,a_s,m_s}) = 2.$$

The last statement is easily deduced from this. The passage from an eigenvector to a generalized eigenvector is standard linear algebra. $\square$

For the remainder of the subsection we suppose $\mathfrak{g}$ is not of exceptional type. Given $\lambda \in P$ and a $U_q(\tilde{\mathfrak{g}})$-module V, set

$$\mathrm{wt}_\ell(V_\lambda) = \{\boldsymbol{\lambda} \in \mathrm{wt}_\ell(V) : \mathrm{wt}(\boldsymbol{\lambda}) = \lambda\}.$$

Theorem 4.3.2. Let $i \in I$, $a \in \mathbb{F}^\times$ and assume that $\boldsymbol{\lambda} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}_{i,a}))$ is such that $\mathrm{wt}(\boldsymbol{\lambda}) = \lambda \in P^+$. Then,

$$\dim(V_q(\boldsymbol{\omega}_{i,a})_\lambda) = \dim(V_q(\boldsymbol{\omega}_{i,a})_{T_w(\lambda)}) \qquad \text{and} \qquad T_w(\mathrm{wt}_\ell(V_q(\boldsymbol{\omega}_{i,a})_\lambda)) = \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}_{i,a})_{w(\lambda)})$$

for all $w \in \mathscr{W}_\lambda$. Suppose further that $\boldsymbol{\lambda} \neq \boldsymbol{\omega}_{i,a}$. Then, there exist $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}_{i,a})), b, c \in \mathbb{F}^\times$, and $j \in I$ such that $\boldsymbol{\mu}_j(u) = (1 - bu)(1 - cu)$, and

$$\boldsymbol{\lambda} = \boldsymbol{\mu}\boldsymbol{\alpha}_{j,b}^{-1}. \tag{4.3.1}$$

Moreover, if $c \neq bq_j^{-2}$, then $\boldsymbol{\mu}\boldsymbol{\alpha}_{j,c}^{-1} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}_{i,a}))$ and, if $c = b$, then $\dim(V_q(\boldsymbol{\omega}_{i,a})_\lambda) \geq 2$.

Proof. To simplify notation, set $V = V_q(\boldsymbol{\omega}_{i,a})$. We prove the first two statements by induction on $\ell(w)$, which clearly starts if $\ell(w) = 0$. Thus, assume that

$$\dim(V_\lambda) = \dim(V_{T_w(\lambda)}) \qquad \text{and} \qquad T_w(\mathrm{wt}_\ell(V_\lambda)) = \mathrm{wt}_\ell(V_{w(\lambda)})$$

and let $w' = r_j w$ where j is such that $\ell(w') = \ell(w) + 1$. In particular, $w^{-1}(\alpha_j) \in Q^+$ by Proposition 1.1.21(iii). By Lemma 1.1.24, either $\lambda = 0$ or $\lambda = \omega_r$ for some $r \in I$. Since $\mathscr{W}_0$ contains only the identity, we can assume $\lambda = \omega_r$. Since $w^{-1}(\alpha_j) \in Q^+$, we have $(\omega_r, w^{-1}(\alpha_j)) > 0$ which implies $w^{-1}(\alpha_j) - \alpha_r \in Q^+$ which, together with Lemma 1.1.24(ii), implies $\omega_i - (\omega_r + w^{-1}(\alpha_j)) \notin Q^+$. Therefore, $\omega_r + w^{-1}(\alpha_j) \notin \mathrm{wt}(\omega_i)$ or, equivalently, $w(\omega_r) + \alpha_j \notin \mathrm{wt}(\omega_i)$. Thus,

$$x_{j,s}^+ V(\boldsymbol{\omega}_{i,a})_{w(\lambda)} = 0 \qquad \text{for all} \qquad s \in \mathbb{Z}. \tag{4.3.2}$$

Similarly one proves that $x_j^- V(\boldsymbol{\omega}_{i,a})_{w'(\lambda)} = 0$. It follows from (4.3.2) and Lemma 4.3.1 that $(T_w(\boldsymbol{\lambda}))_j(u)$ is a polynomial of degree $w(\lambda)(h_j)$ and that $(x_j^-)^{w(\lambda)(h_j)}$ maps $V_{T_w(\lambda)}$ isomorphically to $V_{T_{w'}(\lambda)}$. This proves that $\dim(V_{T_w(\lambda)}) = \dim(V_{T_{w'}(\lambda)})$ which in turn implies that $T_j(\mathrm{wt}_\ell(V_{w(\lambda)})) = \mathrm{wt}_\ell(V_{w'(\lambda)})$, since $\dim(V_{w(\lambda)}) = \dim(V_{w'(\lambda)})$. This completes the inductive step.

Now assume $\lambda \neq \omega_i$. Then, there exists $j \in I$ and $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V)$ such that $\mathrm{wt}(\boldsymbol{\mu}) = \lambda + \alpha_j$ and

$$\sum_{s \in \mathbb{Z}} x_{j,s}^- V_{\boldsymbol{\mu}} \cap V_{\boldsymbol{\lambda}} \neq \{0\}.$$

In particular, $V_{\lambda+\alpha_j} \neq 0$. Let us show that $(\lambda + \alpha_j)(h_j) = 2$ which is clear if $\lambda = 0$. Otherwise, if $\lambda = \omega_r$ for some r, it suffices to show that $r \neq j$. This follows from Lemma 1.1.24(ii) since $\omega_r + \alpha_j \in \mathrm{wt}(\omega_i)$ implies $\omega_i - (\omega_r + \alpha_j) \in Q^+$. It now follows from Lemma 1.1.24(iii) that

$$x_{j,s}^+ V_{\lambda+\alpha_j} = 0 \qquad \text{for all} \qquad s \in \mathbb{Z}.$$

In particular, since $(\lambda + \alpha_j)(h_j) = 2$, it follows that $\boldsymbol{\mu}_j(u) = (1 - bu)(1 - cu)$ for some $b, c \in \mathbb{F}^\times$. It then follows from Lemma 4.3.1 that either $\boldsymbol{\lambda} = \boldsymbol{\mu}\boldsymbol{\alpha}_{j,b}^{-1}$ or $\boldsymbol{\lambda} = \boldsymbol{\mu}\boldsymbol{\alpha}_{j,c}^{-1}$. Without loos of generality, we assume that it is the former. Moreover, if $c \neq bq^{-2}$, then $\rho_j(\boldsymbol{\mu}\boldsymbol{\alpha}_{j,c}^{-1})$ is an ℓ-weight of the irreducible $U_q(\tilde{\mathfrak{g}}_j)$-module with Drinfeld polynomial $\rho_j(\boldsymbol{\mu})$ and, hence, $\boldsymbol{\mu}\boldsymbol{\alpha}_{j,c}^{-1} \in \mathrm{wt}_\ell(V)$. Similarly, if $b = c$, the dimension of the ℓ-weight space associated to $\rho_j(\boldsymbol{\mu}\boldsymbol{\alpha}_{j,c}^{-1})$ in the irreducible $U_q(\tilde{\mathfrak{g}}_j)$-module with Drinfeld polynomial $\rho_j(\boldsymbol{\mu})$ is 2 and the last statement follows. $\square$

Remark 4.3.3. Notice that it follows from Theorem 4.3.2 and Corollary 3.1.3 that if $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}_{i,a}))$, then $\boldsymbol{\mu} \leq \boldsymbol{\omega}_{i,a}$. Therefore, Theorem 4.3.2 gives an alternate proof of Theorem 3.3.13 in case $\mathfrak{g}$ is not of exceptional type. $\diamondsuit$

Example 4.3.4. Theorem 4.3.2 actually provides an algorithm for computing $\mathrm{qch}(V_q(\boldsymbol{\omega}_{i,a}))$. Let us give the simplest example with ω_i not minuscule. Thus, let $\mathfrak{g}$ be of type D_n and $i = 2$ (so $\omega_i = \theta$). Notice that $\mathrm{wt}(\omega_2) = \mathscr{W}(\omega_2) \cup \{0\} = R \cup \{0\}$. Therefore, we need to obtain the ℓ-weights with classical weight 0. If $\boldsymbol{\lambda}$ is such an element, by Theorem 4.3.2, it must be of the form $\boldsymbol{\mu}\boldsymbol{\alpha}_{i,b}^{-1}$ for some $j \in I, b \in \mathbb{F}^\times$, and $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}_{2,a}))$ such that $\mathrm{wt}(\boldsymbol{\mu}) = \alpha_j$. Moreover, $\boldsymbol{\mu} = T_w(\boldsymbol{\omega}_{2,a})$ for some $w \in \mathscr{W}_{\omega_2}$. Set

$$w_j = (r_{j-1}\cdots r_1)(r_{j+1}\cdots r_{n-2}r_{n-1}r_n)(r_{n-2}\cdots r_2) \qquad \text{for} \qquad 1 \leq j \leq n-2,$$
$$w_{n-1} = (r_{n-2}\cdots r_1)r_n(r_{n-2}\cdots r_2), \qquad \text{and} \qquad w_n = (r_{n-2}\cdots r_1)r_{n-1}(r_{n-2}\cdots r_2).$$

One then checks that $w_j \in \mathscr{W}_{\omega_2}$ and $w_j(\omega_2) = \alpha_j$. Setting $\boldsymbol{\mu}_j = T_{w_j}(\boldsymbol{\omega}_{2,a})$, it follows that, if $\boldsymbol{\lambda} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}_{2,a})_0)$, then $\boldsymbol{\lambda} = \boldsymbol{\mu}_j\boldsymbol{\alpha}_{j,b}^{-1}$ for some $j \in I$ and where b is a root of the polynomial $\rho_j(\boldsymbol{\mu}_j)$. One easily computes that

$$T_{w_j}(\boldsymbol{\omega}_{2,a}) = \begin{cases} \boldsymbol{\omega}_{j-1,aq^{j+1}}^{-1}\boldsymbol{\omega}_{j,aq^j}\boldsymbol{\omega}_{j,aq^{2n-4-j}}\boldsymbol{\omega}_{j+1,aq^{2n-3-j}}^{-1}, & \text{if } j \leq n-3,\\ \boldsymbol{\omega}_{n-3,aq^{n-1}}^{-1}\boldsymbol{\omega}_{n-2,aq^{n-2}}^{2}\boldsymbol{\omega}_{n-1,aq^{n-1}}\boldsymbol{\omega}_{n,aq^{n-1}}^{-1}, & \text{if } j = n-2,\\ \boldsymbol{\omega}_{n-2,aq^n}^{-1}\boldsymbol{\omega}_{j,aq^{n-1}}\boldsymbol{\omega}_{j,aq^{n-3}}, & \text{if } j = n-1, n. \end{cases}$$

Set

$$\boldsymbol{\lambda}_j = \begin{cases} \boldsymbol{\omega}_{j-1,aq^{j+1}}^{-1}\boldsymbol{\omega}_{j-1,aq^{2n-j-3}}\boldsymbol{\omega}_{j,aq^j}\boldsymbol{\omega}_{j,aq^{2n-j-2}}^{-1} = \boldsymbol{\mu}_j\boldsymbol{\alpha}_{j,aq^{2n-4-j}}^{-1}, & \text{if } 1 \leq j \leq n-2,\\ \boldsymbol{\omega}_{j,aq^{n-3}}\boldsymbol{\omega}_{j,aq^{n+1}}^{-1} = \boldsymbol{\mu}_j\boldsymbol{\alpha}_{j,aq^{n-1}}^{-1}, & \text{if } j = n-1, n. \end{cases}$$

By Theorem 4.3.2, $\boldsymbol{\lambda}_j \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}_{2,a}))$ for all $j = 1, \ldots, n$, and $\mathrm{qch}(V_q(\boldsymbol{\omega}_{2,a}))(\boldsymbol{\lambda}_{n-2}) \geq 2$. Notice also that $\boldsymbol{\mu}_j\boldsymbol{\alpha}_{j,aq^j}^{-1} = \boldsymbol{\lambda}_{j+1}$ if $1 \leq j < n-2$ and $\boldsymbol{\mu}_j\boldsymbol{\alpha}_{j,aq^{n-3}}^{-1} = \boldsymbol{\lambda}_{n-2}$ if $j = n-1, n$. It follows that $\mathrm{wt}_\ell(V_q(\boldsymbol{\omega}_{2,a})_0) = \{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n\}$. We claim that

$$\mathrm{qch}(V_q(\boldsymbol{\omega}_{2,a})) = \sum_{w \in \mathscr{W}_{\omega_2}} T_w(\boldsymbol{\omega}_{2,a}) + \sum_{j \neq n-2} \boldsymbol{\lambda}_j + 2\boldsymbol{\lambda}_{n-2}.$$

In other words, we are left to show that $\mathrm{qch}(V_q(\boldsymbol{\omega}_{2,a}))(\boldsymbol{\lambda}_j) \leq 1$ if $j \neq n-2$ and $\mathrm{qch}(V_q(\boldsymbol{\omega}_{2,a}))(\boldsymbol{\lambda}_{n-2}) \leq 2$. This can be seen as a consequence of the fact that the Frenkel-Mukhin algorithm works for $V_q(\boldsymbol{\omega}_{2,a})$. Indeed, the only ℓ-weight of classic weight 0 coming from the expansion with respect to $\boldsymbol{\mu}_j$ is $\boldsymbol{\lambda}_j$ with multiplicity 1 if $j \neq n-2$ and with multiplicity 2 if $j = n-2$. $\diamondsuit$

Remark 4.3.5. Notice that it follows from the computations above that $\dim(V_q(\boldsymbol{\omega}_{2,a})_0) = n + 1 = \dim(V_q(\omega_2)_0) + 1$ (since $V(\omega_2)$ is the adjoint representation). This shows that $V_q(\boldsymbol{\omega}_{2,a})$ is reducible as a $U_q(\mathfrak{g})$-module and implies that there cannot be an algebra map $U_q(\tilde{\mathfrak{g}}) \to U_q(\mathfrak{g})$. Indeed, if such a map existed we would have an irreducible $U_q(\tilde{\mathfrak{g}})$-module with highest-$\ell$-weight $\boldsymbol{\lambda}$ such that $\mathrm{wt}(\boldsymbol{\lambda}) = \omega_2$ and $\dim(V_q(\boldsymbol{\lambda})) = \dim(V_q(\omega_2))$. But the condition $\mathrm{wt}(\boldsymbol{\lambda}) = \omega_2$ implies $\boldsymbol{\lambda} = \boldsymbol{\omega}_{i,a}$ for some $a \in \mathbb{F}^\times$ contradicting our computations above. For all other simple Lie algebras, except $\mathfrak{sl}_n$ and $\mathfrak{so}_5$, there exists $i \in I$ such that $V_q(\boldsymbol{\omega}_{i,a})$ is reducible as a $U_q(\mathfrak{g})$-module showing that there cannot be analogues of evaluation modules. For $\mathfrak{so}_5$, there are no $\boldsymbol{\lambda} \in \mathscr{P}^+$ such that $\mathrm{wt}(\boldsymbol{\lambda}) = \theta = 2\omega_2$ and $V_q(\boldsymbol{\lambda})$ is irreducible as $U_q(\mathfrak{g})$-module and we reach the same conclusion. $\diamondsuit$

4.4. Bibliographical notes

1. Frenkel-Mukhin algorithm

Our presentation of the Frenkel-Mukhin algorithm and the results of Subsection 4.1 is essentially an extended version of the original one given in [36]. However, we make use of the theory of blocks and elliptic characters, which was not available when [36] was published, to give further insights into the theory of qcharacters. In particular, our versions of Propositions 4.1.4 and 4.1.7 are reinterpretations (and also a refinement) of [36, Lemma 3.4]. Also, our map τ_J is a reinterpretation of its namesake from [36] and our Lemmas 4.1.3 and 4.2.5 correspond to Lemmas 3.3 and 3.6 of [36]. In particular, this gives a reinterpretation of the variables $Z_{j,a}$ used in [36] in terms of the groups of J-elliptic characters $\mathscr{E}_J$. Proposition 4.1.4 is contained in an unpublished joint work with D. Hernandez.

Theorem 4.2.3 was first proved by E. Frenkel and E. Mukhin in [36] in the special case of fundamental representations which, together with Corollary 3.2.7, implies Theorem 3.3.13. The proof presented here is essentially the one given by Hernandez in [47]. Evidently, the connection with the theory of blocks and elliptic characters was not used in the original proofs. Theorem 4.1.19 was originally stated in [36] only for ℓ-minuscule modules. It was remarked in [51] that the proof could be easily modified to obtain the statement presented here. Thus, the proof we gave above is essentially an extended version of the original proof of [36]. By now, a much larger class of modules than that of Kirillov-Reshetikhin modules is known to have qcharacters given by the Frenkel-Mukhin algorithm. For instance, all minimal affinizations in the case that the underlying simple Lie algebra is of type A_n, B_n, or G_2 are known to be ℓ-minuscule. This was probed by Hernandez in [48] where some conditions for other types were also obtained. Very recently, E. Mukhin and C. Young obtained a purely combinatorial description of Drinfeld polynomials for which the Frenkel-Mukhin algorithm works [68]. They also introduced a new class of modules, called the class of snake modules and proved that, if the underlying simple Lie algebra is of type A_n or B_n, then the Drinfeld polynomial of the snake modules satisfy this combinatorial condition. Although in type A_n this does not enlarge the class of modules for which it is known that the Frenkel-Mukhin algorithm works, for type B_n that is the case. In particular, the snake modules include all Kirillov-Reshetikhin modules, minimal affinizations, and modules associated to skew Young diagrams.

The terminology Frenkel-Mukhin quiver and several others used above such as i-stratum have never been used in the literature. We introduced the terms here hoping to make the presentation clearer.

The first example of failure of the Frenkel-Mukhin algorithm was given by W. Nakai and T. Nakanishi in the case that $\mathfrak{g}$ is an algebra of type C_3 [71]. The nature of the failure is very similar to the one of Example 4.2.14: the algorithm fails to generate a dominant ℓ-weight and, as a consequence, it eventually becomes non-admissibly colored. The paper [71] also brings some of the above given examples illustrated via the tableaux description. The work of Nakai and Nakanishi also relates the theory of qcharacters to the study of Jacobi-Trudi determinants [69, 70]. Example 4.2.14 was first given by D. Hernandez and B. Leclerc in [51]. However, the irreducibility of the tensor product $V_q(\omega_{i,a}) \otimes V_q(\omega_{i,a}\omega_{j,aq^3})$ was deduced as an application of the main result of [51] – a relation of the tensor structure of our category of modules with the theory of cluster algebras. Since we did not establish such relation here, we deduced the irreducibility of this tensor product in a more elementary manner. One can similarly consider an algorithm for computing qcharacters starting from the lowest

ℓ-weigh instead of the highest ℓ-weight. It is interesting to notice that Example 4.2.14 is not a counter-example for the lowest ℓ-weight version of the algorithm!

2. Kirillov-Reshetikhin modules

The Kirillov-Reshetikhin modules were not originally defined as presented here. In [65], A. N. Kirillov and N. Reshetikhin predicted the existence of certain modules for the Yangians (rather than quantum affine algebras) whose characters (and of their tensor products) were conjectured to satisfy certain fermionic formulas. Their work was based on mathematical physics considerations related to the Bethe Ansatz. It was later realized that, transported to the context of quantum affine algebras, these modules are nothing but the modules $V_q(\boldsymbol{\omega}_{i,a,m})$. Moreover, the Kirillov-Reshetikhin modules are the minimal affinizations of the $U_q(\mathfrak{g})$-module $V_q(m\omega_i)$. Therefore, the class of minimal affinizations can be regarded as a generalization of the class of Kirillov-Reshetikhin modules. The original work of Kirillov and Reshetikhin was concerned with describing the simple factors of the Kirillov-Reshetikhin modules when regarded as modules for $U_q(\mathfrak{g})$ instead of $U_q(\tilde{\mathfrak{g}})$. The aforementioned conjectural fermionic formulas give a combinatorial way of answering this. The conjecture became known as the Kirillov-Reshetikhin conjecture and many papers were written establishing particular cases of the conjecture as well as connections with other concepts such as the ones of Q-systems and T-systems. The latter moves the problem from one on classic characters to one on qcharacters. A general proof of the Kirillov-Reshetikhin conjecture was then obtained by Hernandez in [47, 49] using the theory of T-systems and other combinatorial aspects of the theory of qcharacters such as the Frenkel-Mukhin algorithm. From a purely theoretical point of view, Theorems 4.2.3 and 4.1.19 give the answer to the conjecture since the multiplicity of the simple factors are detremined by the character which can be read off the qcharacter and the qcharacter of a tensor product is the product of the qcharacters. However, in practical terms, it is not a simple task to do that even after obtaining an explicit formula for the qcharacters of the Kirillov-Reshetikhin modules from the Frenkel-Mukhin algorithm. Because of this, there is a lot of work in the direction of obtaining formulas for the multiplicities of the simple factors of the minimal affinizations when regarded as $U_q(\mathfrak{g})$-modules directly, i.e., independently of the knowledge of the qcharacter: [11, 16, 21, 22, 41, 42, 78, 79] (see also [15] and references therein).

3. Other algorithms

The results of Subsection 4.3 were obtained in a joint work with V. Chari [19]. Formulas as in Example 4.3.4 were obtained for all fundamental representations and all non-exceptional $\mathfrak{g}$ in [20]. The results from [19] were extended to the root of unity setting in [58] and we should soon extend the ones from [20] as well.

There is another algorithm available for computing qcharacters which is known to work for simply laced $\mathfrak{g}$ and any element of $\mathscr{P}^+$. Namely, recall that the qcharacters can be encoded in a ring homomorphism from the Grothendieck ring $\mathscr{G}_q$ of our category of modules to the integral group ring $\mathbb{Z}[\mathscr{P}]$. In [73], under the assumption that $\mathfrak{g}$ is simply laced, Nakajima defined certain polynomials similar to Kazhdan-Lusztig polynomials that arise in the study of category $\mathscr{O}$ by studying cohomology of certain quiver varieties which are now known as Nakajima's quiver varieties (the original Kazhdan-Lusztig polynomials arise from the cohomology of Schubert varieties - Nakajima's varieties are very closely related to Lusztig's quiver varieties). This lead him to define a function $\chi_{q,t} : \mathscr{G}_q \otimes_{\mathbb{Z}} \mathbb{Z}[t,t^{-1}] \to \mathbb{Z}[\mathscr{P}] \otimes_{\mathbb{Z}} \mathbb{Z}[t,t^{-1}]$ called the t-analogue of the qcharacter ring homomorphism. It turns out that at $t = 1$ this function specializes to the qcharacter ring homomorphism. Moreover, the definition of $\chi_{q,t}$ was axiomatized in a purely combinatorial manner leading to an algorithm for computing the qcharacters of the irreducible and local Weyl modules. This algorithm was used in [74] to give explicit formulas for the t-analogues of the qcharacters of the local Weyl modules when $\mathfrak{g}$ is of type A or D. The formulas were presented in terms of tableaux and a connection with the theory of crystals was discovered. The algorithm was also used with the help of a supercomputer to compute the t-analogues of the qcharacters of the fundamental modules when $\mathfrak{g}$ is of type E in [75]. In [44, 45], Hernandez proved a conjecture of Nakajima saying that the existence of the function $\chi_{q,t}$ could be established using only its axiomatic description (without the use of geometry). This allowed him to extend the concepts of t-analogues of qcharacters as well as the Kazhdan-Lusztig like polynomials to general $\mathfrak{g}$. However, due to the lack of a definition of the quiver varieties in general, a proof that the algorithm indeed gives the qcharacters of the irreducible modules when $\mathfrak{g}$ is not simply laced is still missing.

Even if a general formula/algorithm is found, it does not mean that finding other formulas is an uninteresting task. For instance, in the classic theory of characters, there are three very famous formulas: Weyl's formula (which indirectly gives the whole character via the Weyl group action on P), Kostant's formula (which gives the dimension of each weight-space directly via the Weyl group action on P and Kostant's partition function), and Freudental's formula (which is algorithmic, but easier for implementing on computers or for performing small computations than the previous two).

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