WEIGHTED INEQUALITIES FOR COMMUTATORS OF ONE-SIDED SINGULAR INTEGRALS

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Abstract. We prove weighted inequalities for commutators of one-sided singular integrals (given by a Calderón-Zygmund kernel with support in $(-\infty, 0)$) with BMO functions. We give the one-sided version of the results in [C. Pérez, Sharp estimates for commutators of singular integrals via iterations of the Hardy-Littlewood maximal function, The Journal of Fourier Analysis and Applications, vol. 3 (6), 1997, pages 743-756] and [C. Pérez, Endpoint estimates for commutators of singular integral operators, Journal of Functional Analysis, vol 128 (1), 1995, pages 163-185]. We improve these results for one-sided singular integrals by putting in the right hand side of the inequalities a smaller operator and a wider class of weights.

1. Introduction

In this paper we obtain non standard weighted inequalities for commutators of singular integral operators given by a Calderón-Zygmund kernel K with support in $(-\infty, 0)$. This estimates will reflect a higher degree of singularity compared with the standard Calderón-Zygmund singular integral operators.

Let T denote a Calderón-Zygmund singular integral operator and M denote the Hardy-Littlewood maximal operator. Coifman proved in [C] that T and M satisfy

$$\int_{\mathbb{R}^n} |Tf|^p w \leq C \int_{\mathbb{R}^n} |Mf|^p w, \tag{1.1}$$

for $0 < p < \infty$, $w \in A_\infty(\mathbb{R}^n)$ and f such that the left hand side is finite. This is a very important estimate in weighted theory since it implies the boundedness of T from $L^p(w)$ into $L^p(w)$, for $p > 1$, when $w \in A_p$.

Combining (1.1) with certain sharp two weighted inequalities for M one can derive a two weighted estimate for T with no asumption on the weight w: If T is a Calderón-Zygmund singular integral operator, Pérez, [P1], proves that for, $1 < p < \infty$,

$$\int_{\mathbb{R}^n} |Tf|^p w \leq C \int_{\mathbb{R}^n} |f|^p M^{[p]+1} w, \tag{1.2}$$

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where M^k is the k-times iterated of the Hardy-Littlewood maximal operator. The case $1 < p \leq 2$ was first obtained in [W], but for singular integral operators with much stronger conditions on the kernel, namely they must be of convolution type with C^∞ kernel.

It is possible to generalize inequalities (1.1) and (1.2) for a large family of singular integral operators, i.e., the higher order commutators introduced by Coifman, Rochberg and Weiss in [CRcW]. Let K be a Calderón-Zygmund kernel. For appropriate b and f we define

$$T_b^k f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^k K(x-y) f(y) dy,$$

$k = 0, 1, 2 \dots$ (in the principal value sense). For $k = 1$ the operator is usually denoted by $[M_b, T] = M_b \circ T - T \circ M_b$, where M_b is the operator $M_b f = bf$, and b is called the symbol of the operator. These generalizations were given by Pérez in [P2]:

Theorem A[P2]**.** *Let $0 < p < \infty$, $w \in A_\infty$ and $b \in BMO$. Then there exists a constant C such that*

$$\int_{\mathbb{R}^n} |T_b^k f|^p w \leq C ||b||_{BMO}^{kp} \int_{\mathbb{R}^n} \left(M^{k+1} f\right)^p w,$$

for all f such that the left hand side is finite.

Theorem B[P2]**.** *Let $1 < p < \infty$ and $b \in BMO$. Then for each weight w there exists a constant C such that*

$$\int_{\mathbb{R}^n} |T_b^k f|^p w \leq C ||b||_{BMO}^{kp} \int_{\mathbb{R}^n} |f|^p M^{[(k+1)p]+1} w.$$

Recently, Aimar, Forzani and Martín-Reyes [AFM] have studied singular integral operators associated to a Calderón-Zygmund kernel with support in $(-\infty, 0)$ or $(0, \infty)$. They prove that the maximal operators which control these singular integrals are the one-sided Hardy-Littlewood maximal operators M^+ and M^- defined for locally integrable functions f by

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f| \quad \text{and} \quad M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f|,$$

and the good weights for these operators are the one-sided weights introduced by Sawyer [S]. Their result improves (1.1) for singular integrals with kernel supported in $(-\infty, 0)$ in two ways, by putting in the right hand side a smaller operator and by allowing a wider class of weights for which the inequality holds. More precisely they prove that if T is a singular integral operator given by a kernel with support in $(-\infty, 0)$ then there exists C such that

$$\int_{\mathbb{R}} |Tf|^p w \leq C \int_{\mathbb{R}} |M^+ f|^p w, \tag{1.3}$$

for $0 < p < \infty$ and $w \in A_\infty^+(\mathbb{R})$ (see [MPT] for the definition of $A_\infty^+(\mathbb{R})$).

The aim of this paper is to study the results of C. Pérez for this kind of singular integrals and to extend them in the double sense as in [AFM]. Our results are the following:

Theorem 1. *Let $0 < p < \infty$, $k = 0, 1, \dots,$ $w \in A_\infty^+$ and $b \in BMO$. Let K be a Calderón-Zygmund kernel with support in $(-\infty, 0)$ and let $T_b^{+,k}$ defined (in the principal value sense) by*

$$T_b^{+,k} f(x) = \int_x^\infty (b(x) - b(y))^k K(x-y) f(y) dy.$$

Then there exists C such that

$$\int_{\mathbb{R}} |T_b^{+,k} f|^p w \leq C ||b||_{BMO}^{kp} \int_{\mathbb{R}} \left((M^+)^{k+1} f\right)^p w,$$

for all bounded functions f with compact support.

Corollary 1. *Under the same hypotheses as in Theorem 1, if $1 < p < \infty$ and $w \in A_p^+$ then there exists C such that*

$$\int_{\mathbb{R}} |T_b^{+,k} f|^p w \leq C ||b||_{BMO}^{kp} \int_{\mathbb{R}} |f|^p \, w,$$

for all bounded functions f with compact support.

We also give a weak type result that generalizes the result in [P3] for this kind of singular integrals:

Theorem 2. *Let $w \in A_\infty^+$, $b \in BMO$ and $T_b^{+,k}$ be as in Theorem 1. Then there exists C such that*

$$w(\{x : |T_b^{+,k} f(x)| > \lambda\}) \leq C \phi_k(||b||_{BMO}^k) \int_{\mathbb{R}} \frac{|f(x)|}{\lambda} \left(1 + \log^+(|f(x)|/\lambda)\right)^k M^- w(x) dx,$$

for all bounded functions f with compact support, where $\phi_k(t) = t(1 + \log^+ t)^k$.

Corollary 2. *Under the same hypotheses as in Theorem 2, if $w \in A_1^+$, then there exists C such that*

$$w(\{x : |T_b^{+,k} f(x)| > \lambda\}) \leq C \phi_k(||b||_{BMO}^k) \int_{\mathbb{R}} \frac{|f(x)|}{\lambda} \left(1 + \log^+(|f(x)|/\lambda)\right)^k w(x) dx,$$

for all bounded functions f with compact support.

Theorem 3. *Let $1 < p < \infty$, $b \in BMO$ and $T_b^{+,k}$ be as in Theorem 1. Then, for each weight w, there exists C such that*

$$\int_{\mathbb{R}} |T_b^{+,k} f|^p w \leq C ||b||_{BMO}^{kp} \int_{\mathbb{R}} |f|^p (M^-)^{[(k+1)p]+1} w, \tag{1.4}$$

for all bounded functions f with compact support.

The case $k = 0$, i.e., the generalization of the result in [P1] for these singular integrals, can be found in [RRoT].

Clearly, every theorem has a corresponding one, reversing the orientation of $\mathbb{R}$.

2. Definitions and preliminaries

We introduce some definitions and tools that we need to prove the main results.

Definition 2.1. We shall say that a function K in $L^1_{\text{loc}}(\mathbb{R}\setminus\{0\})$ is a Calderón–Zygmund kernel if the following properties are satisfied:

(a) there exists a finite constant B_1 such that

$$\left|\int_{\epsilon<|x|<N} K(x)\,dx\right| \;\le\; B_1,$$

for all ϵ and all N with $0<\epsilon<N$, and furthermore, $\lim_{\epsilon\to 0^+}\int_{\epsilon<|x|<1} K(x)\,dx$ exists,

(b) there exists a finite constant B_2 such that

$$|K(x)| \le \frac{B_2}{|x|},$$

for all $x \neq 0$,

(c) there exists a finite constant B_3 such that

$$|K(x-y)-K(x)| \le B_3|y||x|^{-2},$$

for all x and y with $|x|>2|y|$.

A one-sided singular integral T^+ is a singular integral associated to a Calderón–Zygmund kernel with support in $(-\infty,0)$; therefore, in that case,

$$T^+f(x) = \lim_{\epsilon\to 0^+}\int_{x+\epsilon}^{\infty} K(x-y)f(y)\,dy.$$

Examples of such kernels are given in [AFM].

F. J. Martín-Reyes and A. de la Torre introduced the one-sided sharp functions in [MT].

Definition 2.2. Let f be a locally integrable function. The one-sided sharp maximal function is defined by

$$M^{+,\#}f(x) = \sup_{h>0}\frac{1}{h}\int_x^{x+h}\left(f(y)-\frac{1}{h}\int_{x+h}^{x+2h} f\right)^+ dy.$$

It is proved in [MT] that

$$M^{+,\#}f(x) \le \sup_{h>0}\inf_{a\in\mathbb{R}}\frac{1}{h}\int_x^{x+h}(f(y)-a)^+\,dy + \frac{1}{h}\int_{x+h}^{x+2h}(a-f(y))^+\,dy \le ||f||_{BMO}.$$

See [MT] for other results and definitions.

We shall also need the following maximal operators:

$$M_\epsilon^+ f(x) = (M^+|f|^\epsilon(x))^{1/\epsilon} \quad \text{and} \quad M_\delta^{+,\#} f(x) = \left(M^{+,\#}|f|^\delta(x)\right)^{1/\delta}.$$

Now we give definitions and results about Young functions. A function $B : [0,\infty) \to [0,\infty)$ is a Young function if it is continuous, convex and increasing satisfying $B(0) = 0$ and $B(t) \to \infty$ as $t \to \infty$. The Luxemburg norm of a function f, given by B is

$$||f||_B = \inf\left\{\lambda > 0 : \int B\left(\frac{|f|}{\lambda}\right) \le 1\right\},$$

and so the B-average of f over I is

$$||f||_{B,I} = \inf\left\{\lambda > 0 : \frac{1}{|I|}\int_I B\left(\frac{|f|}{\lambda}\right) \le 1\right\}.$$

We will denote by $\overline{B}$ the complementary function associated to B (see [BS]). Then the generalized Hölder's inequality

$$\frac{1}{|I|}\int_I |f\,g| \le ||f||_{B,I}||g||_{\overline{B},I},$$

holds. There is a further generalization that turns to be out useful for our purposes (see [O]). If A, B, C are Young functions such that

$$A^{-1}(t)B^{-1}(t) \le C^{-1}(t),$$

then

$$||fg||_{C,I} \le 2||f||_{A,I}||g||_{B,I}.$$

Definition 2.3. For each locally integrable function f, the one-sided maximal operators associated to the Young function B are defined by

$$M_B^+ f(x) = \sup_{x<b} \|f\|_{B,(x,b)} \quad \text{and} \quad M_B^- f(x) = \sup_{a<x} \|f\|_{B,(a,x)}.$$

Definition 2.4. Let B be a Young function. We say that B satisfies the B_p condition, or that $B \in B_p$, $p > 1$, if there exists $c > 0$ such that

$$\int_c^\infty \frac{B(t)}{t^p}\frac{dt}{t} \approx \int_c^\infty \left(\frac{t^{p'}}{\overline{B}(t)}\right)^{p-1}\frac{dt}{t} < \infty.$$

The B_p condition appears for the first time in [P4]. The interest of definition 2.4 is that it implies the boundedness of M_B^+ from $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$ for $1 < p < \infty$. In fact one has

Theorem C[RRoT]. *Let $1 < p < \infty$, w be a weight and B be a Young function. Then the following statements are equivalent:*

a) $B \in B_p$.

b) *There exists C such that* $\int (M_B^+ f)^p w \le C \int |f|^p M^- w$.

We will be working most of the time with $B(t) = t(1 + \log^+ t)^k$, $k \ge 0$ and for this B, it is proved in [RRoT] that

$$M_B^+ f \approx (M^+)^{k+1} f. \tag{2.1}$$

3. Proofs of the results

To prove Theorem 1 we need the following lemma:

Lemma 1. *Let* $0 < \delta < 1$. *Then*

(a) *There exists* $C = C_\delta > 0$ *such that*

$$M_\delta^{+,\#}\left(T^+ f\right)(x) \le CM^+ f(x).$$

(b) *For each* $b \in BMO$, $\delta < \epsilon < 1$ *and* $k = 1, 2, \dots,$ *there exists* $C = C_{\delta,\epsilon} > 0$ *such that*

$$M_\delta^{+,\#}\left(T_b^{+,k} f\right)(x) \le C\sum_{j=0}^{k-1} ||b||_{BMO}^{k-j} M_\epsilon^+(T_b^{+,j} f)(x) + C||b||_{BMO}^k (M^+)^{k+1} f(x).$$

Proof. We start by proving (b). Let λ be an arbitrary constant. Then $b(x) - b(y) = (b(x) - \lambda) - (b(y) - \lambda)$ and

(3.1)
$$\begin{aligned}
T_b^{+,k} f(x) &= \int_{\mathbb{R}} (b(x) - b(y))^k K(x-y) f(y)dy \\
&= \sum_{j=0}^{k} C_{j,k}(b(x) - \lambda)^j \int_{\mathbb{R}} (b(y) - \lambda)^{k-j} K(x-y) f(y)dy \\
&= T^+((b-\lambda)^k f)(x) + \sum_{j=1}^{k} C_{j,k}(b(x) - \lambda)^j \int_{\mathbb{R}} (b(y) - \lambda)^{k-j} K(x-y) f(y)dy \\
&= T^+((b-\lambda)^k f)(x) \\
&\quad + \sum_{j=1}^{k}\sum_{s=0}^{k-j} C_{j,k,s}(b(x) - \lambda)^{s+j} \int_{\mathbb{R}} (b(x) - b(y))^{k-j-s} K(x-y) f(y)dy \\
&= T^+((b-\lambda)^k f)(x) + \sum_{m=0}^{k-1} C_{k,m}(b(x) - \lambda)^{k-m} T_b^{+,m} f(x),
\end{aligned}$$

where $m = k - j - s$. Let us fix x and $h > 0$ and let $I = [x, x+8h]$. Then we write $f = f_1 + f_2$ where $f_1 = f\chi_I$. Taking into account (3.1), for all $a \in \mathbb{R}$, we have the following:

(3.2)

$$\left(\frac{1}{h}\int_x^{x+h}\left||T_b^{+,k}f(y)|^\delta-|a|^\delta\right|dy\right)^{\frac{1}{\delta}}+\left(\frac{1}{h}\int_{x+h}^{x+2h}\left||T_b^{+,k}f(y)|^\delta-|a|^\delta\right|dy\right)^{\frac{1}{\delta}}$$

$$\leq\left(\frac{1}{h}\int_x^{x+h}|T_b^{+,k}f(y)-a|^\delta dy\right)^{\frac{1}{\delta}}+\left(\frac{1}{h}\int_{x+h}^{x+2h}|T_b^{+,k}f(y)-a|^\delta dy\right)^{\frac{1}{\delta}}$$

$$\leq C\left[\sum_{m=0}^{k-1}\left(\frac{1}{h}\int_x^{x+2h}|b(y)-\lambda|^{(k-m)\delta}|T_b^{+,m}f(y)|^\delta dy\right)^{\frac{1}{\delta}}\right.$$

$$\left.+\left(\frac{1}{h}\int_x^{x+2h}|T^+((b-\lambda)^kf)(y)-a|^\delta dy\right)^{\frac{1}{\delta}}\right]$$

$$\leq C\left[\sum_{m=0}^{k-1}\left(\frac{1}{h}\int_x^{x+2h}|b(y)-\lambda|^{(k-m)\delta}|T_b^{+,m}f(y)|^\delta dy\right)^{\frac{1}{\delta}}\right.$$

$$\left.+\left(\frac{1}{h}\int_x^{x+2h}|T^+((b-\lambda)^kf_1)(y)|^\delta dy\right)^{\frac{1}{\delta}}+\left(\frac{1}{h}\int_x^{x+2h}|T^+((b-\lambda)^kf_2)(y)-a|^\delta dy\right)^{\frac{1}{\delta}}\right]$$

$$=(I)+(II)+(III).$$

Let $\lambda=b_I=\frac{1}{8h}\int_x^{x+8h}b(y)dy$. Since $0<\delta<\epsilon<1$, we can choose q such that $1<q<\frac{\epsilon}{\delta}$. Then, using Hölder's inequality for q and q', we get

(3.3)

$$(I)\leq C\sum_{m=0}^{k-1}\left(\frac{1}{h}\int_x^{x+2h}|b(y)-b_I|^{(k-m)\delta q'}dy\right)^{\frac{1}{\delta q'}}\left(\frac{1}{h}\int_x^{x+2h}|T_b^{+,m}f(y)|^{\delta q}dy\right)^{\frac{1}{\delta q}}$$

$$\leq C\sum_{m=0}^{k-1}\left[\left(\frac{1}{h}\int_x^{x+8h}|b(y)-b_I|^{(k-m)\delta q'}dy\right)^{\frac{1}{\delta q'(k-m)}}\right]^{k-m}\left(\frac{1}{h}\int_x^{x+2h}|T_b^{+,m}f(y)|^{\delta q}dy\right)^{\frac{1}{\delta q}}$$

$$\leq C\sum_{m=0}^{k-1}||b||_{BMO}^{k-m}M_{\delta q}^+(T_b^{+,m}f)(x)$$

$$\leq C\sum_{m=0}^{k-1}||b||_{BMO}^{k-m}M_\epsilon^+(T_b^{+,m}f)(x).$$

Using that T^+ is of weak type (1,1), Kolmogorov's inequality gives that

$$(II)\leq C\frac{1}{h}\int_x^{x+2h}|b-b_I|^k|f|\chi_I(y)dy.$$

And by the generalized Hölder's inequality for $B(t) = t(1+\log^+ t)^k$ and $\overline{B}(t) \approx e^{t^{1/k}}$ we get,

$$(II) \leq C||b - b_I||_{\overline{B},I}||f\chi_I||_{B,I}.$$

Now if $D(t) = e^t$, using the John-Nirenberg's inequality, we have

$$(II) \leq C||b - b_I||^k_{D,I}||f\chi_I||_{B,I} \leq C||b||^k_{BMO}M^+_B f(x) \leq C||b||^k_{BMO}(M^+)^{k+1}f(x). \tag{3.4}$$

For (III) we take $a = T^+((b - b_I)^k f_2)(x + 2h)$. Then, by Jensen's inequality,

$$(III) \leq C\frac{1}{h}\int_x^{x+2h} |T^+((b - b_I)^k f_2)(y) - T^+((b - b_I)^k f_2)(x + 2h)|dy. \tag{3.5}$$

For $j \geq 3$, let $I_j = [x + 2^j h, x + 2^{j+1}h]$ and $\tilde{I_j} = [x, x + 2^{j+1}h]$. Using property (c) of the kernel K, for every $y \in [x, x + 2h]$, we have

$$|T^+((b - b_I)^k f_2)(y) - T^+((b - b_I)^k f_2)(x + 2h)| \leq \tag{3.6}$$

$$\begin{aligned}\int_{x+8h}^{\infty} \frac{x + 2h - y}{(t - (x + 2h))^2}|b(t) - b_I|^k|f(t)|dt &\leq Ch\sum_{j=3}^{\infty}\int_{x+2^j h}^{x+2^{j+1}h} \frac{|b(t) - b_I|^k}{(t - (x + 2h))^2}|f(t)|dt \\ &\leq Ch\sum_{j=3}^{\infty}\frac{2^{j+1}}{(2^j - 2)^2 h}\frac{1}{2^{j+1}h}\int_{\tilde{I_j}} |b(t) - b_I|^k|f(t)|dt.\end{aligned}$$

Observe that by the generalized Hölder's inequality and using again the John-Nirenberg's inequality, we obtain

$$\begin{aligned}&\frac{1}{2^{j+1}h}\int_{\tilde{I_j}} |b(t) - b_I|^k|f(t)|dt \\ &\leq \frac{C}{2^{j+1}h}|b_{\tilde{I_j}} - b_I|^k\int_{\tilde{I_j}} |f(t)|dt + \frac{C}{2^{j+1}h}\int_{\tilde{I_j}} |b(t) - b_{\tilde{I_j}}|^k|f(t)|dt \\ &\leq C(2j)^k||b||^k_{BMO}M^+f(x) + C||b - b_{\tilde{I_j}}||_{\overline{B},\tilde{I_j}}||f\chi_{\tilde{I_j}}||_{B,\tilde{I_j}} \\ &\leq C(2j)^k||b||^k_{BMO}M^+f(x) + C||b||^k_{BMO}(M^+)^{k+1}f(x).\end{aligned} \tag{3.7}$$

So inequalities (3.5), (3.6) and (3.7) give

$$\begin{aligned}(III) &\leq C\sum_{j=3}^{\infty}\frac{2^{j+1}}{(2^j - 2)^2}(2j)^k||b||^k_{BMO}M^+f(x) + C\sum_{j=3}^{\infty}\frac{2^{j+1}}{(2^j - 2)^2}||b||^k_{BMO}(M^+)^{k+1}f(x) \\ &\leq C||b||^k_{BMO}(M^+)^{k+1}f(x).\end{aligned} \tag{3.8}$$

Putting together inequalities (3.2), (3.3), (3.4) and (3.8), we obtain that

$$M_\delta^{+,\#}\left(T_b^{+,k}f\right)(x) \le C||b||_{BMO}^k (M^+)^{k+1}f(x) + C\sum_{m=0}^{k-1}||b||_{BMO}^{k-m}M_\epsilon^+(T_b^{+,m}f)(x). \quad \square$$

The proof of part (a) follows the same pattern as the proof of (b) but it is easier and therefore we omit it.

We will now prove Theorem 1.

Proof of Theorem 1. Observe that the case $k=0$ is the inequality for singular integrals with support in $(-\infty,0)$ (see [AFM]). We will proceed by induction on k. So assume that the theorem is true for all $j \le k$ and let us see how it follows the case $k+1$. Since $w \in A_\infty^+$, there exists $r>1$ such that $w \in A_r^+$. Observe that for all $\delta>0$ small enough, we have that $r < \frac{p}{\delta}$ and thus, $w \in A_{\frac{p}{\delta}}^+$. To apply Theorem 4 in [MT] we need $||M_\delta^+(T_b^{+,k+1}f)||_{L^p(w)}$ to be finite. Suppose this for the moment. Then, by Lemma 1, for all ϵ with $\delta<\epsilon<1$, we have

$$\begin{aligned}
||T_b^{+,k+1}f||_{L^p(w)} &\le ||M_\delta^+(T_b^{+,k+1}f)||_{L^p(w)} \\
&\le C||M_\delta^{+,\#}(T_b^{+,k+1}f)||_{L^p(w)} \\
&\le C\sum_{j=0}^{k}||b||_{BMO}^{k+1-j}||M_\epsilon^+(T_b^{+,j}f)||_{L^p(w)} + C||b||_{BMO}^{k+1}||(M^+)^{k+2}f||_{L^p(w)}.
\end{aligned}$$

We choose $\epsilon>0$ such that $r<\frac{p}{\epsilon}$. Then $w \in A_{\frac{p}{\epsilon}}^+$ and we obtain

$$||M_\epsilon^+(T_b^{+,j}f)||_{L^p(w)}^p = \int_{\mathbb{R}}(M^+(|T_b^{+,j}f|^\epsilon))^{\frac{p}{\epsilon}}w \le C\int_{\mathbb{R}}(|T_b^{+,j}f|^\epsilon)^{\frac{p}{\epsilon}}w = C||T_b^{+,j}f||_{L^p(w)}^p.$$

Then, by recurrence

$$\begin{aligned}
||T_b^{+,k+1}f||_{L^p(w)} &\le C\sum_{j=0}^{k}||b||_{BMO}^{k+1-j}||T_b^{+,j}f||_{L^p(w)} \\
&\qquad + C||b||_{BMO}^{k+1}||(M^+)^{k+2}f||_{L^p(w)} \\
&\le C\sum_{j=0}^{k}||b||_{BMO}^{k+1-j}||b||_{BMO}^{j}||(M^+)^{j+1}f||_{L^p(w)} \\
&\qquad + C||b||_{BMO}^{k+1}||(M^+)^{k+2}f||_{L^p(w)} \\
&\le C||b||_{BMO}^{k+1}||(M^+)^{k+2}f||_{L^p(w)}.
\end{aligned}$$

If w is bounded, then

$$\begin{aligned}
||M_\delta^+(T_b^{+,k+1}f)||_{L^p(w)} &\le C||M_\delta^+(T_b^{+,k+1}f)||_{L^p(dx)} \\
&\le C||T_b^{+,k+1}f||_{L^p(dx)} \le C||b||_{BMO}^{k+1}||f||_{L^p(dx)} < \infty.
\end{aligned}$$

Then the theorem is proved if w is bounded. For the general case, we consider $w_N = \min\{w, N\}$. It is not hard to prove that $w_N \in A_\infty^+$ (A_p^+ is a lattice) with constant independent of N. Therefore we have

$$\int_{\mathbb{R}} |T_b^{+,k} f|^p w_N \le C||b||_{BMO}^{kp} \int_{\mathbb{R}} \left((M^+)^{k+1} f\right)^p w_N.$$

Now, we obtain the desired result applying the monotone convergence theorem. $\square$

To prove Theorem 2 we need the following two lemmas.

Lemma 2. *Let $f \in L^1_{loc}(\mathbb{R})$ and $\lambda > 0$. Then for every weight w there exists $C > 0$ such that*

$$w(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > \lambda\}) \le C \int_{\mathbb{R}} \frac{|f(y)|}{\lambda} \left(1 + \log^+ \frac{|f(y)|}{\lambda}\right)^k M^- w(y) dy.$$

Proof. This lemma is a consequence of (2.1) and Theorem 2.5 in [RRoT] with $B(t) = t(1 + \log^+ t)^k$, since $(w, M^- w) \in A_1^+$. $\square$

Lemma 3. *Let $\phi_k(t) = t(1 + \log^+ t)^k$, $k = 0, 1, \ldots$, $b \in BMO$ and $w \in A_\infty^+$. Then there exists $C > 0$ such that*

$$\sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : |T_b^{+,k} f(x)| > t\})$$
$$\le C\phi_k(||b||_{BMO}^k) \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > t\}).$$

for all bounded functions f with compact support.

Proof. We first suppose that $||b||_{BMO} = 1$. We shall prove the following,

$$\sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : |T_b^{+,k} f(x)| > t\})$$
$$\le C \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > t\}).$$

Now, set $b_m = b$ if $-m \le b \le m$, $b_m = m$ if $b \ge m$ and $b_m = -m$ if $b \le -m$. Also, set $w_N = \inf\{w, N\}$. As we have said before, $w_N \in A_\infty^+$ with constant independent of N. On the other hand $||b_m||_{BMO} \le C'||b||_{BMO} = C'$ with C' independent of m. In order to simplify notation, rename $b = b_m$ and $w = w_N$. Observe that for all $\delta > 0$ we have

$$w(\{x \in \mathbb{R} : |T_b^{+,k} f(x)| > t\}) \le w(\{x \in \mathbb{R} : M_\delta^+(T_b^{+,k} f)(x) > t\}).$$

Let us consider the functional

$$L_{b,w,\phi_k,\delta}(f) = L_\delta(f) = \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : M_\delta^+(T_b^{+,k} f)(x) > t\}).$$

We claim that for some $\gamma > 0$ and every $0 < \epsilon < 1$ we have

$$L_\delta(f) \le \epsilon^\gamma C L_\delta(f) + C \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > t\}). \tag{3.9}$$

If $L_\delta(f) < \infty$ then the result (for b_m and w_N) follows from (3.9), choosing ϵ small enough.

In what follows we prove that $L_\delta(f) < \infty$. In [MT] it was proved that if $w \in A_\infty^+$ and $M^+ f \in L^{p_0}(w)$ for some p_0, then

$$w(\{x \in \mathbb{R} : M^+ f(x) > t, M^{+,\#} f(x) \le t\epsilon\}) \le C\epsilon^\gamma w(\{x \in \mathbb{R} : M^+ f(x) > \frac{t}{2}\}), \tag{3.10}$$

for some $\gamma > 0$. Observe that we have $M_\delta^+(T_b^{+,k} f) \in L^{p_0}(w)$ for some p_0, since f is bounded with compact support, $w \le N$ and $|b| \le m$. Then

(3.11)

$$\begin{aligned} &w(\{x \in \mathbb{R} : M_\delta^+(T_b^{+,k} f)(x) > t\}) \\ &\quad = w(\{x \in \mathbb{R} : M^+(|T_b^{+,k} f|^\delta)(x) > t^\delta, M^{+,\#}(|T_b^{+,k} f|^\delta)(x) \le t^\delta \epsilon\}) \\ &\qquad\qquad + w(\{x \in \mathbb{R} : M^+(|T_b^{+,k} f|^\delta)(x) > t^\delta, M^{+,\#}(|T_b^{+,k} f|^\delta)(x) > t^\delta \epsilon\}) \\ &\quad \le C\epsilon^\gamma w(\{x \in \mathbb{R} : M_\delta^+(T_b^{+,k} f)(x) \ge t/2^{\frac{1}{\delta}}\}) + w(\{x \in \mathbb{R} : M_\delta^{+,\#}(T_b^{+,k} f)(x) > t\epsilon^{1/\delta}\}) \\ &\quad = I + II. \end{aligned}$$

Using Lemma 1 for $\epsilon = \delta r$, and $1 < r < \frac{1}{\delta}$, we have

$$\begin{aligned} II \le\ & w(\{x \in \mathbb{R} : \sum_{j=0}^{k-1} (C')^{k-j} M_{\delta r}^+(T_b^{+,j} f)(x) > \frac{t\epsilon^{\frac{1}{\delta}}}{2C}\}) \\ & + w(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > \frac{t\epsilon^{\frac{1}{\delta}}}{2C(C')^k}\}). \end{aligned} \tag{3.12}$$

Bearing in mind (3.11) and (3.12) we obtain

(3.13)

$$\begin{aligned} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : M_\delta^+(T_b^{+,k} f)(x) > t\}) \le\ & \frac{C\epsilon^\gamma}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : M_\delta^+(T_b^{+,k}) f(x) > \frac{t}{2^{\frac{1}{\delta}}}\}) \\ & + \sum_{j=0}^{k-1} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : M_{\delta r}^+(T_b^{+,j} f)(x) > \frac{t\epsilon^{\frac{1}{\delta}}}{2Ck(C')^{k-j}}\}) \\ & + \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > \frac{t\epsilon^{\frac{1}{\delta}}}{2C(C')^k}\}) \\ =\ & I' + II' + III'. \end{aligned}$$

Observe that there exists C such that $\phi_k(2t) \le C\phi_k(t)$ for all $t > 0$ (*i.e.* ϕ_k is doubling). Let $l \in \mathbb{N}$ be such that $2^{\frac{1}{\delta}} < 2^l$. Using that ϕ_k is non-decreasing, we get

$$\phi_k\left(\frac{2^{\frac{1}{\delta}}}{t}\right) \le \phi_k\left(\frac{2^l}{t}\right) \le C\phi_k\left(\frac{1}{t}\right).$$

Then

$$I' \le \frac{C\epsilon^{\gamma}}{\phi_k(\frac{2^{\frac{1}{\delta}}}{t})} w(\{x \in \mathbb{R} : M_{\delta}^{+}(T_b^{+,k} f)(x) > \frac{t}{2^{\frac{1}{\delta}}}\}) \le C\epsilon^{\gamma} L_{\delta}(f).$$

Now let $a_j = \frac{2Ck(C')^{k-j}}{\epsilon^{\frac{1}{\delta}}}$ and $h \in \mathbb{Z}$ be such that $a_j \le 2^h$, for all j. Therefore

$$\phi_k\left(\frac{a_j}{t}\right) \le \phi_k\left(\frac{2^h}{t}\right) \le C\phi_k\left(\frac{1}{t}\right).$$

As a consequence,

$$\begin{aligned} II' &\le C\sum_{j=0}^{k-1} \frac{1}{\phi_k(\frac{a_j}{t})} w(\{x \in \mathbb{R} : M_{\delta r}^{+}(T_b^{+,j} f)(x) > \frac{t}{a_j}\}) \\ &\le C\sum_{j=0}^{k-1} \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : M_{\delta r}^{+}(T_b^{+,j} f)(x) > t\}). \end{aligned} \tag{3.14}$$

Now for each $j = 0, 1 \ldots, k-1$, let us estimate $\sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : M_{\delta r}^{+}(T_b^{+,j} f)(x) > t\})$.

Using that ϕ_k is doubling and non-decreasing, it follows from (3.10) and Lemma 1 (a) that, for all $0 < \epsilon < 1$,

$$\begin{aligned} \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x : M_{\epsilon}^{+}(T^{+} f)(x) > t\}) &\le \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x : M_{\epsilon}^{+,\#}(T^{+} f)(x) > t\}) \\ &\le C \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x : M^{+} f(x) > t\}) \\ &\le C \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x : (M^{+})^{k+1} f(x) > t\}). \end{aligned}$$

Fix $J < k-1$ and suppose that, for every $0 \le j \le J$ and for all $0 < \epsilon < 1$, there exists C such that

$$\sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : M_{\epsilon}^{+}(T_b^{+,j} f)(x) > t\}) \le C \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : (M^{+})^{k+1} f(x) > t\}). \tag{3.15}$$

We will prove, that (3.15) holds for $j = J+1$. Using again that ϕ_k is doubling, non-decreasing, (3.10) and Lemma 1 (b) we obtain

$$\begin{aligned}
\sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x : M_\epsilon^+(T_b^{+,J+1}f)(x) > t\}) &\leq C \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x : M_\epsilon^{+,\#}(T_b^{+,J+1}f)(x) > t\}) \\
&\leq C \left[\sum_{i=0}^{J} \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x : M_{\epsilon'}^+(T_b^{+,i}f)(x) > t\}) + w(\{x : (M^+)^{J+1}f(x) > t\}) \right] \\
&\leq C \sum_{i=0}^{J} \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x : (M^+)^{k+1}f(x) > t\}) \\
&\qquad + C \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x : (M^+)^{J+1}f(x) > t\}) \\
&\leq C \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x : (M^+)^{k+1}f(x) > t\}),
\end{aligned}$$

where $\epsilon < \epsilon' < 1$. As a consequence, for $\epsilon = \delta r$, (3.15) together with (3.14) gives

$$II' \leq C \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : (M^+)^{k+1}f(x) > t\}).$$

Finally let $a = \frac{\epsilon^{\frac{1}{\delta}}}{2C(C')^k}$. Then

$$\begin{aligned}
III' &\leq \frac{C}{\phi_k(\frac{1}{at})} w(\{x \in \mathbb{R} : (M^+)^{k+1}f(x) > at\}) \\
&\leq C \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : (M^+)^{k+1}f(x) > t\}).
\end{aligned}$$

Putting all these estimates together we get (3.9).

Therefore if we prove that $L_{b,w,\phi_k,\delta}f < \infty$, using (3.9) we obtain

$$L_{b,w,\phi_k,\delta}(f) \leq C \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : (M^+)^{k+1}f(x) > t\}).$$

Assume now that supp $f \subset (-R, R)$, for some $R > 0$. Then for $x \leq -2R$ we have

$$\begin{aligned}
|T_b^{+,k}f(x)| &\leq C \int_{-R}^{R} \frac{|b(x) - b(y)|^k}{|x-y|} |f(y)| dy \\
&\leq \frac{2Cm^k}{|x|} \int_{x}^{R} |f(y)| dy \\
&\leq Cm^k M^+ f(x).
\end{aligned} \tag{3.16}$$

Using that $0 < \delta < 1$, the fact that M^+ is of weak type $(1,1)$ with respect to the pair $(w, M^-w) \in A_1^+$, the analogous to M^{k+1} given in Lemma 2 and (3.16), we get

$$\begin{aligned}
\frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : M_\delta^+(T_b^{+,k} f)(x) > t\}) &\leq \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : M_\delta^+(\chi_{(-2R,2R)} T_b^{+,k} f)(x) > t/2\}) \\
&\quad + \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : M_\delta^+(\chi_{(-\infty,-2R)} T_b^{+,k} f)(x) > t/2\}) \\
&\leq \frac{1}{\phi_k(\frac{1}{t})} \frac{C}{t} \int_{-2R}^{2R} |T_b^{+,k} f(x)| M^- w(x) dx + \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > C_m t\}) \\
&\leq C4NR \left(\frac{1}{4R} \int_{-2R}^{2R} |T_b^{+,k} f(x)|^2 dx \right)^{\frac{1}{2}} + \frac{C}{\phi_k(\frac{1}{t})} \int_{\mathbb{R}} \phi_k\left(\frac{|f(x)|}{C_m t} \right) M^- w(x) dx \\
&\leq C4NR \left(\frac{1}{4R} \int_{-R}^{R} |f(x)|^2 dx \right)^{\frac{1}{2}} + CN \int_{-R}^{R} \phi_k(|f(x)|) dx.
\end{aligned}$$

Since f is bounded and with compact support the last expresion is finite.

Then, we have obtained the following:

$$\begin{aligned}
&\sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w_N(\{x \in \mathbb{R} : |T_{b_m}^{+,k} f(x)| > t\}) \\
&\qquad\qquad \leq C \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w_N(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > t\}).
\end{aligned}$$

Observe that $\{b_m^j f\}$ converges to $b^j f$ in $L^1(dx)$, since f is bounded with compact support and $b \in BMO$ implies that b is locally in $L^p(dx)$ for all $p \geq 1$. Then, taking into account that T^+ is of weak type (1,1) with respect to the Lebesgue measure, we obtain that $\{T^+(b_m^j f)\}$ converges to $T^+(b^j f)$ in measure. This implies that, for a subsequence, we have almost everywhere convergence. On the other hand, $\{b_m^j T^+ f\}$ converges to $b^j T^+ f$ almost everywhere. As a consequence, a subsequence of $\left\{|T_{b_m}^{+,k} f|\right\}$ converges to $|T_b^{+,k} f|$ almost everywhere. We shall continue denoting this subsequence by $\left\{|T_{b_m}^{+,k} f|\right\}$. Then, by Fatou's lemma,

$$\begin{aligned}
&\sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w_N(\{x \in \mathbb{R} : |T_b^{+,k} f(x)| > t\}) \\
&\qquad = \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} \int_{\mathbb{R}} \lim_{m\to\infty} w_N(x) \chi_{\{x \in \mathbb{R}: |T_{b_m}^{+,k} f(x)| > t\}} dx \\
&\qquad \leq \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} \liminf_{m\to\infty} w_N(\{x \in \mathbb{R} : |T_{b_m}^{+,k} f(x)| > t\}) \\
&\qquad \leq C \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w_N(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > t\}).
\end{aligned}$$

Letting N go to infinity we obtain the desired result.

Now, for general $b \in BMO$ ($||b||_{BMO} > 0$), we consider $h = \frac{b}{||b||_{BMO}}$. Then, since $T_h^{+,k} f = \frac{1}{||b||_{BMO}^k} T_b^{+,k} f$ and taking into account that ϕ_k is submultiplicative, we have

$$\begin{aligned}
&\sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : |T_b^{+,k} f(x)| > t\}) \\
&\qquad = \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : |T_h^{+,k} f(x)| > \frac{t}{||b||_{BMO}^k}\}) \\
&\qquad \leq \phi_k(||b||_{BMO}^k) \sup_{t>0} \frac{1}{\phi_k\left(\frac{||b||_{BMO}^k}{t}\right)} w(\{x \in \mathbb{R} : T_h^{+,k} f(x) > \frac{t}{||b||_{BMO}^k}\}) \\
&\qquad \leq C\phi_k(||b||_{BMO}^k) \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : M^{k+1} f(x) > t\}). \quad \square
\end{aligned}$$

Proof of Theorem 2. It suffices to consider the case $\lambda = 1$. (For $\lambda > 0$ the result follows by considering $\frac{f}{\lambda}$). By Lemma 3, the fact that ϕ_k is submultiplicative and by Lemma 2, we get,

$$\begin{aligned}
w(\{x \in \mathbb{R} : |T_b^{+,k} f(x)| > 1\}) &\leq \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : |T_b^{+,k} f(x)| > t\}) \\
&\leq C\phi_k(||b||_{BMO}^k) \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} w(\{x \in \mathbb{R} : (M^+)^{k+1} f(x) > t\}) \\
&\leq C\phi_k(||b||_{BMO}^k) \sup_{t>0} \frac{1}{\phi_k(\frac{1}{t})} \phi_k(\frac{1}{t}) \int_{\mathbb{R}} \phi_k(|f(x)|) M^- w(x) dx \\
&= C\phi_k(||b||_{BMO}^k) \int_{\mathbb{R}} |f(x)|(1 + \log^+ |f(x)|)^k M^- w(x) dx. \quad \square
\end{aligned}$$

Proof of Theorem 3. By duality, (1.4) is equivalent to

$$\int_{\mathbb{R}} |T_b^{-,k} f|^{p'} ((M^-)^{[(k+1)p]+1} w)^{1-p'} \leq C \int_{\mathbb{R}} |f|^{p'} w^{1-p'}.$$

Observe that $((M^-)^{[(k+1)p]+1} w)^{1-p'} \in A_\infty^-$, and by Theorem 1, we get

$$\int_{\mathbb{R}} |T_b^{-,k} f|^{p'} ((M^-)^{[(k+1)p]+1} w)^{1-p'} \leq C \int_{\mathbb{R}} ((M^-)^{k+1} f)^{p'} ((M^-)^{[(k+1)p]+1} w)^{1-p'}.$$

Therefore it suffices to prove that

$$\int_{\mathbb{R}} ((M^-)^{k+1} f)^{p'} ((M^-)^{[(k+1)p]+1} w)^{1-p'} \leq C \int_{\mathbb{R}} |f|^{p'} w^{1-p'}. \tag{3.17}$$

Now observe that proving (3.17) is equivalent to

$$\int_{\mathbb{R}} ((M^-)^{k+1} (f w^{\frac{1}{p}}))^{p'} ((M^-)^{[(k+1)p]+1} w)^{1-p'} \leq C \int_{\mathbb{R}} |f|^{p'}. \tag{3.18}$$

If $\phi_k(t) = t(1+\log^+ t)^k$, then (3.18) is equivalent to

$$\int_{\mathbb{R}} ((M^-_{\phi_k})(fw^{\frac{1}{p}}))^{p'}((M^-)^{[(k+1)p]+1}w)^{1-p'} \leq C\int_{\mathbb{R}} |f|^{p'}. \tag{3.19}$$

For large t, $\phi_k^{-1}(t) \approx \dfrac{t}{\log(t)^k}$. Then, for $\epsilon > 0$,

$$\phi_k^{-1}(t) \approx \frac{t^{\frac{1}{p}}}{\log(t)^{k+\frac{p-1+\epsilon}{p}}} \times t^{\frac{1}{p'}} \log(t)^{\frac{p-1+\epsilon}{p}} = A^{-1}(t) \times B^{-1}(t),$$

where $A(t) \approx t^p \log(t)^{(k+1)p-1+\epsilon}$ and $B(t) \approx \dfrac{t^{p'}}{\log(t)^{1+(p'-1)\epsilon}}$. Then, by the generalized Hölder's inequality, we have

$$(M^-_{\phi_k})(fw^{\frac{1}{p}}) \leq CM^-_B(f)M^-_A(w^{\frac{1}{p}}) \leq CM^-_B(f)(M^-_D(w))^{\frac{1}{p}},$$

where $D(t) = t(\log t)^{(k+1)p-1+\epsilon}$. We choose ϵ such that $(k+1)p-1+\epsilon = [(k+1)p]$. Then

$$\begin{aligned}
&\int_{\mathbb{R}} ((M^-_{\phi_k})(fw^{\frac{1}{p}}))^{p'}((M^-)^{[(k+1)p]+1}w)^{1-p'} \\
&\qquad \leq C\int_{\mathbb{R}} (M^-_B(f))^{p'}((M^-_D(w))^{\frac{p'}{p}}((M^-)^{[(k+1)p]+1}w)^{1-p'} \\
&\qquad \leq C\int_{\mathbb{R}} (M^-_B(f))^{p'}((M^-_D(w))^{p'-1}((M^-_D(w))^{1-p'} \\
&\qquad \leq C\int_{\mathbb{R}} |f|^{p'},
\end{aligned}$$

where the last inequality follows from Theorem C, since $B \in B_{p'}$. □

References

[AFM] H. Aimar, L. Forzani, F.J. Martín-Reyes, *On weighted inequalities for one-sided singular integrals*, Proc. Amer. Math. Soc. **125** (1997), 2057-2064.

[BS] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, New York (1988).

[C] R. Coifman, *Distribution function inequalities for singular integrals*, Proc. Acad. Sci. USA **69** (1972), 2838-2839.

[CRcW] R. Coifman, R. Rochberg, G.Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103 (2)** (1976), 611-635.

[MPT] F.J. Martín-Reyes, L. Pick and A. de la Torre, *A^+_∞ condition,*, Canadian Journal of Mathematics **45** (1993), 1231-1244.

[MT] F.J. Martín-Reyes , A. de la Torre, *One Sided BMO Spaces*, Jour. of London Math. Soc. **(2) 49** (1994), 529-542.

[O] R. O'Neil, *Fractional integration in Orlicz spaces*, Trans. Amer. Math. Soc. **115** (1963), 300-328.

[P1] C. Pérez, *Weighted norm inequalities for singular integral operators*, J.London Math. Soc. **49** (1994), 296-308.

[P2] ______, *Sharp estimates for commutators of singular integrals via iterations of the Hardy-Littlewood maximal function*, The Journal of Fourier Analysis and Applications **3 (6)** (1997), 743-756.

[P3] ______, *Endpoint estimates for commutators of singular integral operators*, Journal of Functional Analysis **128 (1)** (1995), 163-185.

[P4] ______, *On sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted L^p-spaces with different weights*, Proc. London Math. Soc. **71 (3)** (1995), 135-157.

[RRoT] M.S. Riveros, L. de Rosa, A. de la Torre, *Sufficient conditions for one-sided operators*, J. Fourier Anal. Appl **6 (6)** (2000), 607-621.

[S] E. T. Sawyer, *Weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Trans. Amer. Math. Soc. **297** (1986), 53-61.

[W] J. M. Wilson, *Weighted norm inequalities for the continuous square functions*, Trans. Amer. Math. Soc. **314** (1989), 661-692.

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