NORM INEQUALITIES RELATING ONE-SIDED SINGULAR INTEGRALS AND THE ONE-SIDED MAXIMAL FUNCTION

M. S. Riveros and A. de la Torre

Abstract. In this paper we prove that if a weight w satisfies the C_q^+ condition, then the $L^p(w)$ norm of a one-sided singular integral is bounded by the $L^p(w)$ norm of the one-sided Hardy-Littlewood maximal function, for $1 < p < q < \infty$.

1. Introduction

One-sided singular integrals were defined by Aimar, Forzani and Martín-Reyes in [AFM] as singular integrals T^+f whose kernel has support on $(-\infty, 0)$. In the same paper they proved that a weight w satisfies $\int |T^+f|^p w \le C \int |f|^p w$, for all $f \in L^p(w)$ if, the weight satisfies the one-sided A_p^+ condition, introduced by Sawyer [S1], that characterizes the boundedness of the one-sided Hardy-Littlewood maximal operator $M^+f(^x) = \sup_{h>0} h^{-1} \int_x^{x+h} |f|$.

A crucial step in the proof, is the fact that if $w \in A_\infty^+$, then

$$\int |T^+f|^r w \le C \int [M^+f]^r w, \tag{1.1}$$

for any $1 < r$. We recall de definitions of the A_p^+ classes. $w \in A_p^+$, $1 < p$ if there exists a constant C such that for all $a < b < c$

$$\int_a^b w \left(\int_b^c w^{1-p'} \right)^{p-1} \le C(c-a)^p \tag{A_p^+}$$

where $p + p' = pp'$. A weight w is in A_∞^+ if there exist positive constants C and ϵ such that for any $a < b < c$ and any measurable set $E \subset (a, b)$,

$$\frac{\int_E w}{\int_a^c w} \le C \left(\frac{|E|}{c-b} \right)^\epsilon \tag{A_∞^+}$$

These definitions and many properties of A_p^+ and A_∞^+ can be found in [MPT]. A natural question arises. Can we find conditions weaker than A_∞^+ that are sufficient for (1.1). In [S2] Sawyer considered the following conditon, introduced first by Muckenhoupt in [Mu].

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There exists two positive constants C and ϵ such that for every interval $I \in \mathbb{R}$ and every measurable subset $E \subset I$ we have

$$(C_p) \qquad \int_E w \le C \left(\frac{|E|}{|I|}\right)^\epsilon \int [M\chi_I]^p w < \infty,$$

where M is the Hardy-Littlewood maximal operator. Sawyer proved that for a standard singular integral Tf, C_q is suficient for

$$(1.2) \qquad \int |Tf|^p w \le C \int [Mf]^p w$$

provided $q > p$. He does not require $\int [M\chi_I]^p w < \infty$. Observe that if $\int [M\chi_I]^q w = \infty$ for some I, then $\int [M\chi_J]^q w = \infty$ for every interval J. Then for every $f \ge 0$ and $p \le q$ we have that $\int [Mf]^p w = \infty$. In this paper we introduce a one-sided version of this condition C_p^+, and prove that if $q > p$, then

$$\int |T^+ f|^p w \le C \int [M^+ f]^p w.$$

The definition of C_p^+ is as follows.

Definition. A weight w satisfies C_p^+ if there exist $\epsilon > 0$ and $C > 0$, so that for any $a < b < c$, with $c - b < b - a$, and any measurable set $E \subseteq (a, b)$, the following holds:

$$(C_p^+) \qquad \int_E w \le C \left(\frac{|E|}{(c-b)}\right)^\epsilon \int_{\mathbb{R}} [M^+ \chi_{(a,c)}]^p w < \infty.$$

Observe that if $w \in A_\infty^+$ then $w \in \cap_{p>1} C_p^+$. We give examples of weights that satisfy C_p^+ condition for all $p > 1$ but they do not satisfy A_∞^+ condition.

The class of one-sided singular integrals is a subclass of the standard singular integrals and our theorem says that for this subclass we can obtain a more precise result. On one hand, we obtain a smaller right hand side, with $M^+ f$ instead of Mf. On the other hand, the condition C_p^+ is different from C_p. These facts make the proof more complicated than in the standard case although it follows the same lines as the paper by Sawyer.

Now we recall the definition of one-sided singular integrals studied in [AFM]. We say that a function k in $L^1_{\mathrm{loc}}(\mathbb{R} - \{0\})$ is a Calderón–Zygmund kernel if the following properties are satisfied:

(a) There exists a finite constant B_1 such that

$$\left| \int_{\epsilon < |x| < N} k(x)\, dx \right| \le B_1$$

for all ϵ and all N with $0 < \epsilon < N$. Furthermore $\lim_{\epsilon \to 0^+} \int_{\epsilon < |x| < N} k(x)\, dx$ exists.

(b) There exists a finite constant B_2 such that

$$|k(x)| \le \frac{B_2}{|x|}$$

for all $x \neq 0$.

(c) There exists a finite constant B_3 such that

$$|k(x-y)-k(x)| \leq B_3|y||x|^{-2}$$

for all x and y with $|x| > 2|y| > 0$.

A one-sided singular integral is

$$T^+f(x) = \lim_{\epsilon \to 0} \int_{x+\epsilon}^{\infty} k(x-y)f(y)\,dy,$$

where k is a Calderón–Zygmund kernel, with support in $\mathbb{R}^-$. We also define

$$T^{*+}f(x) = \sup_{\epsilon>0} \left| \int_{x+\epsilon}^{\infty} k(x-y)f(y)\,dy, \right|.$$

Examples of such kernels are given in [AFM].

We end this section with some notation. A weight w is a non-negative, locally integrable function. If E is a measurable set, $w(E)$ denotes the integral of w over E. Throughout the paper the letter C represents a positive constant that may change from time to time.

2. Statement and proof of the result

Theorem 1. *Let T^+f be a one-sided singular integral, $1 < p < q < \infty$ and assume that w satisfies C_q^+ , then*

$$\int_{\mathbb{R}} |T^+f|^p w \leq C \int_{\mathbb{R}} [M^+f]^p w$$

for all f such that the right hand side is finite.

Remark. If $w(x) = e^x$ then $w \in A_1^+ \subset A_\infty^+ \subset C_p^+$, $p > 1$. But $\int [M\chi_I]^p w = \infty$, and therefore $w \notin C_p$, $p > 1$.

The proof is based on a series of lemmas that we now state and prove.

Lemma 1. *Let us assume that w satisfies C_q^+, $1 < q < \infty$, then for any $\delta > 0$ there exists $C(\delta)$ such that for any disjoint family of intervals $\{J_j\}$ contained in $I = (a,b)$ we have:*

$$\text{(i)} \qquad \int_I \sum_j [M^+\chi_{J_j}]^q w \leq C(\delta)w(I) + \delta \int_{\mathbb{R}} [M^+\chi_I]^q w$$

and

$$\text{(ii)} \qquad \int_{\mathbb{R}} \sum_j [M^+\chi_{J_j}]^q w \leq C \int_{\mathbb{R}} [M^+\chi_I]^q w.$$

Proof. First, we claim that (i) implies (ii). Indeed,

$$\begin{aligned}\int_{\mathbb{R}}\left(\sum_j [M^+\chi_{J_j}]^q\right) w &= \int_I \left(\sum_j [M^+\chi_{J_j}]^q\right) w + \int_{(-\infty,a)}\left(\sum_j [M^+\chi_{J_j}]^q\right) w\\ &\le C(\delta)w(I) + \delta\int_{\mathbb{R}}[M^+\chi_I]^q w + \int_{(-\infty,a)} \frac{\sum_j |J_j|^q}{(b-x)^q} w\\ &\le C(\delta)w(I) + \delta\int_{\mathbb{R}}[M^+\chi_I]^q w + \int_{(-\infty,a)} \frac{|I|^q}{(b-x)^q} w\\ &\le C(\delta)w(I) + (\delta+1)\int_{\mathbb{R}}[M^+\chi_I]^q w\\ &\le 2C(\delta)\int_{\mathbb{R}}[M^+\chi_I]^q w + (\delta+1)\int_{\mathbb{R}}[M^+\chi_I]^q w.\end{aligned}$$

To prove (i) we use the fact that there exists $\alpha > 0$ such that for every $\lambda > 0$ we have

$$|E_\lambda| = |\{x : \sum_j [M^+\chi_{J_j}]^q(x) > \lambda\}| \le Ce^{-\alpha\lambda}|I| \tag{2.1}$$

(for details see [FeSt]). We define a sequence of points as follows: $x_0 = a$ and for $i \in \mathbb{N}$, $x_i - x_{i-1} = b - x_i$ and consider the sets $E^i_\lambda = E_\lambda \cap (x_i, x_{i+1})$. For $x \in (x_i, x_{i+1})$ we may assume that J_j in $\sum_j |M^+\chi_{J_j}|^q(x)$ are all contained in (x_i, b). It follows from (2.1) that

$$|E^i_\lambda| \le Ce^{-\alpha\lambda}(b - x_i) = Ce^{-\alpha\lambda}(x_{i+2} - x_{i+1}).$$

If we now use condition C^+_q for the set E^i_λ and the points x_i, x_{i+1}, x_{i+2} we get

$$w(E^i_\lambda) \le Ce^{-\alpha\lambda\epsilon}\int [M^+\chi_{(x_i,x_{i+2})}]^q w.$$

It is easy to see that $\sum_{i>1} M^+\chi_{(x_i,x_{i+2})} \le CM^+\chi_I$ and adding up we get

$$w(E_\lambda \cap I) \le Ce^{-\alpha\lambda\epsilon}\int [M^+\chi_I]^q w.$$

Therefore,

$$\begin{aligned}\int_I \sum_j [M^+\chi_{J_j}]^q w &= \int_0^{\lambda_0}\int_{E_\lambda\cap I} w\, d\lambda + \int_{\lambda_0}^{\infty}\int_{E_\lambda\cap I} w\, d\lambda\\ &\le \lambda_0 w(I) + \int_{\lambda_0}^{\infty} w(E_\lambda\cap I)\, d\lambda\\ &\le \lambda_0 w(I) + C\int_{\lambda_0}^{\infty} e^{-\alpha\lambda\epsilon}\, d\lambda \int [M^+\chi_I]^q w\\ &\le C(\delta)w(I) + \delta\int [M^+\chi_I]^q w,\end{aligned}$$

if we choose λ_0 big enough. $\square$

For the next lemma we need to define a new operator, $M^+_{p,q}$. Let f be a nonnegative measurable function. Let us consider

$$\Omega_k = \{x : f(x) > 2^k\} = \cup_i I_i^k,$$

where I_i^k are the connected components of Ω_k. Then

$$[M^+_{p,q} f(x)]^p = \sum_{k,i} 2^{pk}[M^+\chi_{I_i^k}(x)]^q.$$

Lemma 2. *Let $1 < p < q < \infty$, $w \in C_q^+$, and f non-negative, bounded and of compact support. Then*

$$\int [M^+_{p,q}(M^+ f)]w \le C \int [M^+ f]^p w.$$

Proof. Let $\Omega_k = \{x : M^+ f(x) > 2^k\} = \cup_j I_j^k$, where I_j^k are the connected components of Ω_k. Let $N \ge 1$, note that $\Omega_k \subseteq \Omega_{k-N}$ for all k. Given a connected component of Ω_{k-N}, I_i^{k-N} we estimate $|\Omega_k \cap I_i^{k-N}|$. First, we put $f = g + h$ with $g = f\chi_{I_i^{k-N}}$. Observe that if $x \in I_i^{k-N} = (a, b)$, then $M^+h(x) \le M^+f(b) \le 2^{k-N}$. So if $x \in \Omega_k \cap I_i^{k-N}$, then

$$M^+g(x) \ge M^+f - M^+h \ge 2^k - 2^{k-N} \ge \frac{1}{2}2^k.$$

Now using the fact that the operator M^+ is of weak type $(1, 1)$ with respect to Lebesgue meausure we get

$$\begin{aligned} |\Omega_k \cap I_i^{k-N}| &\le |\{x : M^+g(x) \ge \frac{1}{2}2^k\}| \le C2^{-k} \int g \\ &= C2^{-k} \int_{I_i^{k-N}} f \le C2^{-k}|I_i^{k-N}|\, M^+f(a) \le C2^{-N}|I_i^{k-N}|. \end{aligned} \tag{2.2}$$

Let $S(k) = 2^{kp} \sum_j \int [M^+\chi_{I_j^k}]^q w$ and $S(k, N, i) = 2^{kp} \sum_{j: I_j^k \subseteq I_i^{k-N}} \int [M^+\chi_{I_j^k}]^q w$.

Then

$$\begin{aligned} S(k, N, i) &= 2^{kp} \sum_{j: I_j^k \subseteq I_i^{k-N}} \int_{I_i^{k-N}} [M^+\chi_{I_j^k}]^q w + 2^{kp} \sum_{j: I_j^k \subseteq I_i^{k-N}} \int_{(I_i^{k-N})^C} [M^+\chi_{I_j^k}]^q w \\ &= I + II. \end{aligned}$$

By Lemma 1

$$I \le C(\delta)2^{kp} w(I_i^{k-N}) + \delta 2^{kp} \int [M^+\chi_{I_i^{k-N}}]^q w,$$

where $\delta > 0$ is choosen later. Now, by (2.2)

$$II \leq C2^{kp} \int_{-\infty}^{a} \frac{\sum |I_j^k|^q}{(b-x)^q} w \leq C2^{kp} \int_{-\infty}^{a} \frac{(C2^{-N}|I_i^{k-N}|)^q}{(b-x)^q} w$$
$$\leq C2^{N(p-q)} 2^{p(k-N)} \int [M^+ \chi_{I_i^{k-N}}]^q w.$$

So we get

$$S(k) = \sum_i S(k,N,i) \leq C(\delta) 2^{kp} \sum_i w(I_i^{k-N}) + [\delta 2^{Np} + C2^{N(p-q)}] S(k-N).$$

As $p < q$, we can choose δ small and N big enough such that

$$S(k) \leq C(\delta) 2^{kp} w(\Omega_{k-N}) + \frac{1}{2} S(k-N).$$

Now

$$S_M = \sum_{k \leq M} S(k) \leq \frac{1}{2} S_M + C \int [M^+ f]^p w,$$

for all M. If we prove that under the assumptions on f, we have $S_M < \infty$, we are finished. Let us suppose that supp$f \subset I = (a,b)$. There exists L such that $2^L < 1/(b-a) \int_a^b f \leq 2^{L+1}$.

If $k \geq L+1$, then $\Omega_k \subset I^- \cup I$, where $I^- = (2a-b, a)$. Indeed, if $x < 2a - b$, then

$$M^+ f(x) = \sup_{h > a-x > b-a} \frac{1}{h} \int_x^{x+h} \leq \frac{1}{b-a} \int_a^b f \leq 2^{L+1}.$$

If I_j^k are the connected componets of Ω_k, using Lemma 1 and since $q > p$, we have

$$\sum_{k=L+1}^{M} \sum_j 2^{kp} \int [M^+ \chi_{I_j^k}]^q w \leq \sum_{k=L+1}^{M} 2^{kp} \int [M^+ \chi_{I^- \cup I}]^q w \leq C \int [M^+ \chi_I]^p w < \infty.$$

If $k \leq L$ we can show again that $\Omega_k \subset 2^{L-k+2}(I^-) \cup I$, where $2^n(I^-) = (c_n, a)$, with $(a - c_n) = 2^n(b-a)$. Then by Lemma 1 we have

$$\sum_{k \leq L} \sum_j 2^{kp} \int [M^+ \chi_{I_j^k}]^q w \leq C \sum_{k \leq L} 2^{kp} \int [M^+ \chi_{2^{L-k+2}(I^-) \cup I}]^q w.$$

Now its easy to see, using $p < q$, that

$$\sum_{k \leq L} 2^{kp} [M^+ \chi_{2^{L-k+2}(I^-) \cup I}(x)]^q \leq C 2^{Lp} [M^+ \chi_I(x)]^p < \infty. \quad \square$$

Lemma 3. *Let $1 < p < q < \infty$, $w \in C_q^+$ and let f be a non-negative bounded function with compact support. Then*

$$\int [M_{p,q}^+(T^{*+}f)]^p w \le C[\int [T^{*+}f]^p w + \int [M^+ f]^p w].$$

Proof. Let $\Omega_k = \{x : T^{*+}f(x) > 2^k\} = \cup_j I_j^k$, where I_j^k are the connected components of Ω_k. Observe that in the proof of the "good lambda inequality" in [AFM, Lemma 2.7], what they really show is
(2.3)
$$|\{x \in I_i^{k-N} : T^{*+}f(x) > 2^k\}| \le C2^{-N}|I_i^{k-N}| \text{ if } I_i^{k-N} \nsubseteq \{x : M^+f(x) > 2^{k-N}\}.$$

Let $O_k = \{x : M^+f(x) > 2^k\} = \cup J_j^k$, where J_j^k are the connected components of O_k. For each I_i^{k-N} we have two cases

(1) $I_i^{k-N} \subseteq O_{k-N}$,
(2) $I_i^{k-N} \nsubseteq O_{k-N}$.

Case (1) There exists l_i such that $I_i^{k-N} \subseteq J_{l_i}^{k-N}$.
Case (2). (2.3) implies

$$\sum_{j:I_j^k \subseteq I_i^{k-N}} |I_j^k| = |\{x \in I_i^{k-N} : T^{*+}f(x) > 2^k\}| \le C2^{-N}|I_i^{k-N}|. \tag{2.4}$$

Let $S(k) = 2^{kp}\sum_j \int [M^+\chi_{I_j^k}]^q w$ and $S(k,N,i) = 2^{kp}\sum_{j:I_j^k \subseteq I_i^{k-N}} \int [M^+\chi_{I_j^k}]^q w.$

Then

$$\begin{aligned} S(k,N,i) &= 2^{kp}\sum_{j:I_j^k \subseteq I_i^{k-N}} \int_{I_i^{k-N}} [M^+\chi_{I_j^k}]^q w + 2^{kp}\sum_{j:I_j^k \subseteq I_i^{k-N}} \int_{(I_i^{k-N})^C} [M^+\chi_{I_j^k}]^q w \\ &= I + II. \end{aligned}$$

By Lemma 1 we have that

$$I \le C(\delta)2^{kp}w(I_i^{k-N}) + \delta 2^{kp}\int [M^+\chi_{I_i^{k-N}}]^q w,$$

where $\delta > 0$. We denote $(a_i^{k-N}, b_i^{k-N}) = I_i^{k-N}$, then by (2.4) we obtain

$$\begin{aligned} II &\le C2^{kp}\int_{-\infty}^{a_i^{k-N}} \frac{\sum_{j:I_j^k \subseteq I_i^{k-N}} |I_j^k|^q}{(b_i^{k-N} - x)^q} w \le C2^{kp}\int_{-\infty}^{a_i^{k-N}} \frac{(C2^{-N}|I_i^{k-N}|)^q}{(b_i^{k-N} - x)^q} w \\ &\le C2^{kp-Nq}\int [M^+\chi_{I_i^{k-N}}]^q w. \end{aligned}$$

Adding I and II we get

$$S(k,N,i) \le C(\delta)2^{kp}w(I_i^{k-N}) + (\delta + C2^{-Nq})2^{kp}\int [M^+\chi_{I_i^{k-N}}]^q w.$$

Then

$$S(k)=\sum_{i:I_i^{k-N}\text{ is in case (1)}} S(k,N,i)+\sum_{i:I_i^{k-N}\text{ is in case (2)}} S(k,N,i)=III+IV.$$

For III we observe that I_j^k is contained in exatly one J_l^{k-N} and by Lemma 1 we have

$$\begin{aligned}III&=\sum_{i:I_i^{k-N}\subseteq J_{l_i}^{k-N}} S(k,N,i)=\sum_{i:I_i^{k-N}\subseteq J_{l_i}^{k-N}}\sum_{j:I_j^k\subseteq I_i^{k-N}} 2^{kp}\int[M^+\chi_{I_j^k}]^q w\\&\le\sum_l\sum_{j:I_j^k\subseteq J_{l_i}^{k-N}} 2^{kp}\int[M^+\chi_{I_j^k}]^q w\\&\le C\sum_l 2^{kp}\int[M^+\chi_{J_l^{k-N}}]^q w.\end{aligned}$$

To estimate IV we observe that

$$\begin{aligned}IV&\le C(\delta)2^{kp}\sum_i w(I_i^{k-N})+(\delta+C2^{-Nq})2^{kp}\sum_i\int[M^+\chi_{I_i^{k-N}}]^q w\\&\le C2^{kp}w(\Omega_{k-N})+\frac12 S(k-N),\end{aligned}$$

choosing δ small and N big enough. Combining III and IV we get

$$S(k)\le\frac12 S(k-N)+C2^{kp}w(\Omega_{k-N})+C2^{kp}\sum_l\int[M^+\chi_{J_l^{k-N}}]^q w.$$

Using Lemma 2

$$\begin{aligned}S_M=\sum_{k\le M}S_k&\le\frac12 S_M+C\int[T^{*+}f]^p w+C\int[M^+_{p,q}(M^+f)]^p w\\&\le\frac12 S_M+C\left(\int[T^{*+}f]^p w+\int[M^+f]^p w\right),\end{aligned}$$

and since $S_M<\infty$ (see Lemma 2), we get

$$\int[M^+_{p,q}(T^{*+}f)]^p w\le C\left(\int[T^{*+}f]^p w+\int[M^+f]^p w\right).\quad\square$$

Proof of Theorem 1. First we observe that $|T^+f|\le T^{*+}f$, so it is enough to prove the theorem for T^{*+}. Let f be a non-negative bounded function with compact support.
Let $\Omega_k=\{x:T^{*+}f(x)>2^k\}=\cup_j J_j^k$ where J_j^k, are the connected components of Ω_k. Let us fix $(a,b)=J_j^k$. We partition (a,b) as follows. Let $x_0=a$, and we choose x_{i+1} such that $x_{i+1}-x_i=b-x_{i+1}$ and we let $I_i^k=(x_i,x_{i+1})$. By "the good lambda inequality" in [AFM Lemma 2.7] we have that

$$|E_i^k|=|\{x\in I_i^k:T^{*+}f(x)>2^{k+1},M^+f(x)\le\gamma 2^k\}|\le C\gamma|I_i^k|\quad \textit{for } 0<\gamma<1.$$

From C_q^+ condition we have

$$w(E_i^k) \leq C\gamma^\epsilon \int [M^+\chi_{I_i^k \cup I_{i+1}^k}]^q w.$$

Summing over all i and using Lemma 1we infer that

$$\begin{aligned} w(\{x \in J_j^k : T^{*+}f(x) > 2^{k+1}, M^+f(x) \leq \gamma 2^k\}) &\leq C\gamma^\epsilon \sum_i \int [M^+\chi_{I_{i,j}^k \cup I_{i+1,j}^k}]^q w \\ &\leq C\gamma^\epsilon \int [M^+\chi_{J_j^k}]^q w. \end{aligned}$$

Now, summing over all j we have that

$$w(\{x \in \Omega_k : T^{*+}f(x) > 2^{k+1}, M^+f(x) \leq \gamma 2^k\}) \leq C\gamma^\epsilon \sum_j \int [M^+\chi_{J_j^k}]^q w.$$

Then by Lemma 3,

$$\begin{aligned} \int (T^{*+}f)^p w &= \sum_k \int_{\Omega_k - \Omega_{k+1}} (T^{*+}f)^p w \leq 2^p \sum_k 2^{kp} w(\Omega_k) \\ &= C \sum_k 2^{kp} [w(\{x \in \Omega_k : T^{*+}f > 2^{k+1}, M^+f \leq \gamma 2^k\}) \\ &\qquad + w(\{x \in \Omega_k : T^{*+}f > 2^{k+1}, M^+f > \gamma 2^k\})] \\ &\leq \sum_{j,k} (C\gamma^\epsilon 2^{kp} \int [M^+\chi_{J_j^k}]^q w) + C \sum_k 2^{kp} w(\{x \in \Omega_k : M^+f(x) > \gamma 2^k\}) \\ &\leq C\gamma^\epsilon [\int [T^{*+}f]^p w + \int [M^+f]^p w] + C \int [M^+f]^p w. \end{aligned}$$

Finally we prove that under the assumptions on f, we have that $\int [T^{*+}f]^p w < \infty$, and choosing γ small enough we finish the proof. To see that $\int [T^{*+}f]^p w < \infty$, let $\text{supp} f \subset I = (a, b)$ and $I^- = (2a - b, a)$. If $x < 2a - b$, then $T^{*+}f(x) \leq CM^+f(x)$, so

$$\int_{-\infty}^{2a-b} [T^{*+}f]^p w \leq \int_{-\infty}^{2a-b} [M^+f]^p w < \infty.$$

Since $T^{*+}f$ is a singular integral and f is bounded, it is known that $\int_{I^- \cup I} e^{\alpha T^{*+}f} < \infty$ for some $\alpha > 0$. Thus

$$|E_\lambda| = |\{x \in I^- \cup I : T^{*+}f(x) > \lambda\}| \leq Ce^{-\lambda\alpha} |I^- \cup I|$$

for all $\lambda > 0$. Applying the C_q^+ condition to the set E_λ and the points $2a-b, b, 2b-a$, we get

$$w(E_\lambda) \leq Ce^{-\lambda\alpha\epsilon} \int [M^+\chi_{I^- \cup I \cup I^+}]^q w,$$

where $I^+ = (b, 2b - a)$. Integrating with respect to λ, using that $p < q$, and proceeding as in the final step of the proof of Lemma 1, we have

$$\int_{I^- \cup I} [T^{*+}f]^p w \leq C \int [M^+\chi_{I^- \cup I \cup I^+}]^p w < \infty.$$

□

As observed in the introduction $A_\infty^+ \subseteq \cap_{p>1} C_p^+$. We now show that the inclusion is proper.

Proposition 1. *Let $w \in A_\infty$, then $w\chi_{(-\infty,0)} \in \cap_{p>1} C_p^+$.*

Proof. First we observe that $w\chi_{(-\infty,0)} \notin A_\infty^+$. Let us consider $a < b < c$ such that $c - b < b - a$ and E a mesurable set such that $E \subset (a,b)$. We have several cases
(i) $a < b < c < 0$. In this case there is nothing to prove because $A_\infty \Longrightarrow A_\infty^+ \Longrightarrow \cap_{p>1} C_p^+$.
(ii) $a < b < 0 < c$. There exist $\epsilon > 0$ and $C > 0$ such that

$$\begin{aligned} w\chi_{(-\infty,0)}(E) = w(E) &\le C\left(\frac{|E|}{b-a}\right)^\epsilon w(a,b) \le C\left(\frac{|E|}{c-b}\right)^\epsilon \int_a^b [M^+\chi_{(a,b)}]^p w \\ &\le C\left(\frac{|E|}{c-b}\right)^\epsilon \int_{-\infty}^0 [M^+\chi_{(a,b)}]^p w \le C\left(\frac{|E|}{c-b}\right)^\epsilon \int_{-\infty}^0 [M^+\chi_{(a,c)}]^p w. \end{aligned}$$

(iii) $a < 0 < b < c$, and $b \le -2a$. Suppose that $E \subseteq (a,0)$. Note that since $b - a \le -3a$,

$$\begin{aligned} w\chi_{(-\infty,0)}(E) = w(E) &\le C\left(\frac{|E|}{0-a}\right)^\epsilon w(a,0) \le C\left(\frac{|E|}{b-a}\right)^\epsilon \int_a^0 [M^+\chi_{(a,0)}]^p w \\ &\le C\left(\frac{|E|}{c-b}\right)^\epsilon \int_{-\infty}^0 [M^+\chi_{(a,c)}]^p w. \end{aligned}$$

If $E \nsubseteq (a,0)$, then

$$\begin{aligned} w\chi_{(-\infty,0)}(E) = w(E\cap(-\infty,0)) &\le C\left(\frac{|E\cap(-\infty,0)|}{c-b}\right)^\epsilon \int_{-\infty}^0 [M^+\chi_{(a,c)}]^p w \\ &\le C\left(\frac{|E|}{c-b}\right)^\epsilon \int_{-\infty}^0 [M^+\chi_{(a,c)}]^p w. \end{aligned}$$

(iv) $a < 0 < b < c$ and $b > -2a$.

$$w\chi_{(-\infty,0)}(E) \le w(E) \le C\left(\frac{|E|}{b-a}\right)^\epsilon w(a,b) \le C\left(\frac{|E|}{c-b}\right)^\epsilon w(a,b).$$

If we prove that $w(a,b) \le C\int_{-\infty}^0 [M^+\chi_{(a,c)}]^p w$, we have finished the proof. Using that w satisfies the doubling condition and that $b > -2a$ if and only if $a + b > b/2$ we have

$$\begin{aligned} \int_{-\infty}^0 [M^+\chi_{(a,c)}]^p w &\ge \int_{-\infty}^0 [M^+\chi_{(a,b)}]^p w \ge \int_{-b}^a [M^+\chi_{(a,b)}]^p w = \int_{-b}^a \left(\frac{b-a}{b-x}\right)^p w \\ &\ge \int_{-b}^a \left(\frac{1}{2}\right)^p w \ge \frac{C}{2^p} w(-b,a) \ge Cw(a,b). \end{aligned}$$

□

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FaMAF. Universidad Nacional de Córdoba (5000) Córdoba, Argentina
E-mail address: sriveros@mate.uncor.edu

Análisis Matemático, Facultad de Ciencias. Universidad de Málaga (29071) Málaga, Spain
E-mail address: torre@anamat.cie.uma.es