The Journal of Fourier Analysis and Applications
Volume 11, Issue 5, 2005

Weighted Estimates for Singular Integral Operators Satisfying Hörmander's Conditions of Young Type

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Communicated by Michael W. Frazier

ABSTRACT. The following open question was implicit in the literature: Are there singular integrals whose kernels satisfy the L^r-Hörmander condition for any $r > 1$ but not the L^∞-Hörmander condition? We prove that the one-sided discrete square function, studied in ergodic theory, is an example of a vector-valued singular integral whose kernel satisfies the L^r-Hörmander condition for any $r > 1$ but not the L^∞-Hörmander condition. For a Young function A we introduce the notion of L^A-Hörmander. We prove that if an operator satisfies this condition, then one can dominate the $L^p(w)$ norm of the operator by the $L^p(w)$ norm of a maximal function associated to the complementary function of A, for any weight w in the A_∞ class and $0 < p < \infty$. We use this result to prove that, for the one-sided discrete square function, one can dominate the $L^p(w)$ norm of the operator by the $L^p(w)$ norm of an iterate of the one-sided Hardy-Littlewood Maximal Operator, for any w in the A_∞^+ class.

1. Introduction

Let T be a singular integral operator of the type

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y) f(y)\, dy \,,$$

where the kernel K has bounded Fourier transform, and let Mf be the Hardy-Littlewood maximal function. A classical result of Coifman [4] states that if the kernel satisfies the

Math Subject Classifications. Primary 42B20; secondary 42B25.
Keywords and Phrases. Singular integrals, weights, maximal operators.
Acknowledgements and Notes. First author was supported by Ministerio de Ciencia y Tecnología BFM 2001-1638 and Junta de Andalucía; second author was partially supported by CONICET, Agencia Córdoba Ciencias y SECYT-UNC; third author was supported by Ministerio de Ciencia y Tecnología BFM 2001-1638 and Junta de Andalucía.

ISSN 1069-5869 DOI: 10.1007/s00041-005-4039-4

following Lipschitz condition: There are numbers $\alpha > 0$ and $C > 0$ such that

$$|K(x-y) - K(-y)| \le C\frac{|x|^\alpha}{|y|^{\alpha+n}}, \quad \text{whenever} \quad |y| > 2|x| \tag{1.1}$$

then, for any $0 < p < \infty$ and any $w \in A_\infty$, there exists a constant C such that

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\,dx \le C \int_{\mathbb{R}^n} (Mf(x))^p w(x)\,dx\,, \tag{1.2}$$

for every f such that the left-hand side is finite. Recently, Martell, Pérez, and Trujillo [7] have proved that (1.2) fails if instead of condition (1.1) we assume that K satisfies the weaker Hörmander condition

$$\sup_{x\in\mathbb{R}^n} \int_{|y|>2|x|} |K(x-y) - K(-y)|\,dy < \infty\,. \tag{1.3}$$

Actually they prove that (1.2) fails even if the kernel K satisfies certain intermediate conditions between (1.1) and (1.3). These conditions are the L^r-Hörmander conditions defined as follows:

Definition 1. Let $1 \le r \le \infty$, we say that the kernel K satisfies the L^r-Hörmander condition, if there are numbers $c_r > 1$ and $C_r > 0$ such that for any $x \in \mathbb{R}^n$ and $R > c_r|x|$

$$\sum_{m=1}^{\infty} \left(2^m R\right)^n \left(\frac{1}{\left(2^m R\right)^n} \int_{2^m R<|y|\le 2^{m+1}R} |K(x-y) - K(-y)|^r\,dy\right)^{\frac{1}{r}} \le C_r\,, \tag{1.4}$$

if $r < \infty$, and

$$\sum_{m=1}^{\infty} \left(2^m R\right)^n \sup_{2^m R<|y|\le 2^{m+1}R} |K(x-y) - K(-y)| \le C_\infty\,, \tag{1.5}$$

in the case $r = \infty$.

We will denote by H_r the class of kernels satisfying the L^r-Hörmander condition.

Observe that these classes are nested, namely

$$H_\infty \subset H_r \subset H_s \subset H_1, \quad 1 < s < r$$

and that H_1 is the class of kernels satisfying the Hörmander condition (1.3). For these classes some weighted estimates are known. See [13] and [2].

Theorem. *Let $1 < r \le \infty$. Assume that the operator T is bounded in some L^p, $1 < p < \infty$, and the kernel K belongs to H_r, then for any $0 < p < \infty$ and $w \in A_\infty$ there is a constant C such that*

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\,dx \le C \int_{\mathbb{R}^n} (M_{r'}f(x))^p\, w(x)\,dx\,, \tag{1.6}$$

whenever the left-hand side is finite.

We recall that for any $1 \le t$, the maximal operator M_t is defined as $M_tf(x) = (M|f|^t(x))^{\frac{1}{t}} \ge Mf(x)$. In [7] it is proved that this theorem is sharp in the following sense:

Theorem. *Let $1 \leq r < \infty$ and $1 \leq t < r'$. There exists a singular integral operator T, bounded in some L^p, $1 < p < \infty$, and whose kernel is in H_r, for which the following inequality does **not** hold:*

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} (M_t f(x))^p\, w(x)\, dx\ , \tag{1.7}$$

for any function f for which the left-hand side is finite, where $0 < p < \infty$, $w \in A_\infty$.

A natural question, left open by this result, is the following:

What happens between H_∞ and the intersection of the H_r, $1 \leq r < \infty$?

More precisely: Are there kernels which belong to H_r for every finite r but do not belong to H_∞?

For such kernels, if there are any, the best known result is that the following inequality holds

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} (M_t f(x))^p\, w(x)\, dx\ , \tag{1.8}$$

for any $1 < t$. Since those kernels do not belong to H_∞ we can not assert that

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} (Mf(x))^p\, w(x)\, dx\ . \tag{1.9}$$

This, however, does not exclude that these operators could satisfy an inequality of the type

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} (M_A f(x))^p\, w(x)\, dx \tag{1.10}$$

where M_A is some maximal operator such that $Mf(x) \leq M_A f(x) \leq M_t f(x)$, for any function f and any $1 < t$.

In this note we give a positive answer to these questions.

In order to state our results we need to recall some definitions. A function $B : [0, \infty) \to [0, \infty)$ is a Young function if it is continuous, convex, increasing and satisfies $B(0) = 0$ and $B(t) \to \infty$ as $t \to \infty$. The Luxemburg norm of a function f, induced by B, is

$$||f||_B = \inf \left\{ \lambda > 0 : \int B\left(\frac{|f|}{\lambda}\right) \leq 1 \right\}\ ,$$

and the B-average of f over a cube, (or a ball) Q is

$$||f||_{B,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q B\left(\frac{|f|}{\lambda}\right) \leq 1 \right\}\ .$$

We will denote by $\overline{B}$ the complementary function associated to B (see [3]). Then the generalized Hölder's inequality

$$\frac{1}{|Q|} \int_Q |f g| \leq ||f||_{B,Q} ||g||_{\overline{B},Q}\ , \tag{1.11}$$

holds.

The behavior of $B(t)$ for $t \leq t_0$ does not affect the value of $||f||_{B,Q}$. Therefore, if $A(t) \approx B(t)$ for $t \geq t_0$, then $||f||_{A,Q} \approx ||f||_{B,Q}$. This means that we will not be concerned about the value of the Young functions for t small.

Definition 2. For each locally integrable function f, the maximal operator associated to the Young function B is defined by

$$M_B f(x) = \sup_{x \in Q} \|f\|_{B,Q} ,$$

where the sup is taken over all the cubes, or balls, that contain x.

We will be using the following Young functions: $B(t) = t^r$, $B(t) = e^{t^{1/k}} - 1$, $B(t) = t(1 + \log^+(t))^k$. The maximal operators associated to these functions are M_r, $M_{\exp L^{1/k}}$ and $M_{L(1+\log^+ L)^k}$. If $k \geq 0$, $k \in \mathbb{Z}$, then $M_{L(1+\log^+ L)^k}$ is pointwise equivalent to M^{k+1}, where M^k is the k-times iterated of M (see [11]). It is also known that

$$Mf(x) \leq CM_{L(1+\log^+ L)^k} f(x) \leq CM_r f(x) ,$$

for all $k > 0$ and $r > 1$.

Definition 3. Let A be a Young function. We say that the kernel K satisfies the L^A-Hörmander condition, if there are numbers $c_A > 1$ and $C_A > 0$ such that for any x and $R > c_A|x|$,

$$\sum_{m=1}^{\infty} \left(2^m R\right)^n \left\| (K(x - \cdot) - K(-\cdot))\, \chi_{\{2^m R<|y|\leq 2^{m+1}R\}}(\cdot) \right\|_{A,B(0,2^{m+1}R)} \leq C_A .$$

We will denote by H_A the class of all kernels satisfying this condition.
The main results on this article are:

Theorem A. *Assume that T is a singular integral operator, bounded in some L^p, $1 < p < \infty$, whose kernel K belongs to H_A. Then, for any $0 < p < \infty$ and $w \in A_\infty$, there exists C such that*

$$\int_{\mathbb{R}^n} |Tf|^p w \leq C \int_{\mathbb{R}^n} (M_{\overline{A}} f)^p w ,$$

for any $f \in C^\infty$ with compact support.

Similar results can be proved for vector valued operators or one-sided operators.

Theorem B. *There is a vector valued, one-sided operator S bounded in all L^p, $1 < p < \infty$, whose kernel K belongs to H_r for every finite $r \geq 1$ but does not belong to H_∞. It does satisfy the L^A-Hörmander condition with $A(t) = \exp(t^{\frac{1}{1+\epsilon}}) - 1$, $(\epsilon > 0)$.*

As a corollary we obtain that for this operator the inequality

$$\int_{\mathbb{R}} |Sf(x)|^p w(x)\, dx \leq C \int_{\mathbb{R}} (M_t f(x))^p\, w(x)\, dx, \quad \text{any} \quad t > 1,\ 0 < p < \infty,\ w \in A_\infty \tag{1.12}$$

may be improved to

$$\int_{\mathbb{R}} |Sf(x)|^p w(x)\, dx \leq C \int_{\mathbb{R}} \left(\left(M^+\right)^3 f(x)\right)^p w(x)\, dx , \tag{1.13}$$

where $(M^+)^3$ is the one-sided Hardy-Littlewood maximal operator iterated three times and w is a weight in the A_∞^+ class.

Remark 1. We do not know if our operator satisfies (1.2). It is an open question if (1.2) holds for an operator whose kernel is in $\cap H_r \setminus H_\infty$.

Remark 2. As a by-product of the analysis developed for the study of the example of Theorem B we give an easy example of an operator whose kernel is not in H_∞ but satisfies (1.2).

The organization of the article is as follows. In Section 2 we give the proof of Theorem A and state, without proof, the corresponding version for the vector valued case. Since our example for Theorem B is a vector valued operator with kernel supported on $(-\infty, 0)$, we dedicate Section 3 to the proof of the one-sided version of Theorem A. Finally, in Section 4 we give an example of an operator whose kernel belongs to $\cap H_r \setminus H_\infty$.

2. Proof of Theorem A

The sharp maximal function is defined as

$$M^\# f(x) = \sup_{x\in Q} \inf_{a\in\mathbb{R}} \frac{1}{|Q|}\int_Q |f(y)-a|\,dy\ . \tag{2.1}$$

Although this operator is dominated pointwise by a multiple of the Hardy-Littlewood maximal function, there is a theorem that states some kind of reverse inequality. See [5].

Theorem. *For any $0 < p < \infty$ and $w \in A_\infty$ there exists C such that*

$$\int_{\mathbb{R}^n} (Mf(x))^p w(x)\,dx \le C\int_{\mathbb{R}^n} \left(M^\# f(x)\right)^p w(x)\,dx\ , \tag{2.2}$$

whenever the left-hand side is finite.

Since it is easy to see that $\int \left(M|Tf|^\delta(x)\right)^{\frac{p}{\delta}} w(x)\,dx$ is finite whenever f is a C^∞-function with compact support, $0 < \delta < 1$, and $w \in A_\infty$, it follows from the preceding theorem and from the inequality

$$|Tf(x)| \le \left(M|Tf|^\delta(x)\right)^{\frac{1}{\delta}}\ ,$$

that, in order to prove Theorem A, it is enough to prove

Theorem 1. *Let T be a singular integral operator, bounded in some L^p, $1 < p < \infty$, whose kernel K satisfies the L^A-Hörmander condition. Then, for any $0 < \delta < 1$, there is a constant C_δ such that for any f and x,*

$$\left(M^\#|Tf|^\delta(x)\right)^{\frac{1}{\delta}} \le C_\delta M_{\overline{A}} f(x)\ . \tag{2.3}$$

Proof. It follows from (1.11) that for any Young function A, $H_A \subset H_1$ and therefore T is of weak type $(1, 1)$. It also follows that $Mf(x) \le CM_A f(x)$ for any f and x.

Let x_0 be fixed and let Q be any cube containing x_0. We will denote by $d(Q)$ its diameter. Let $\tilde{Q}$ be a cube concentric with Q with side equal to $5c_A$ times the side of Q. If

$y \notin \tilde{Q}$ then $|y - x_0| > 2c_A d(Q)$. We split f in the form $f = f_1 + f_2$ where $f_1 = f\chi_{\tilde{Q}}$. It will be enough to prove

$$\left(\frac{1}{|Q|}\int_Q \left||Tf(x)|^\delta - |Tf_2(x_0)|^\delta\right| dx\right)^{\frac{1}{\delta}} \leq CM_{\overline{A}}f(x_0) . \tag{2.4}$$

In order to prove this inequality it is enough to prove:

$$\frac{1}{|Q|}\int_Q |Tf_1(x)|^\delta\, dx \leq C(Mf(x_0))^\delta , \tag{2.5}$$

and

$$\frac{1}{|Q|}\int_Q \left||Tf_2(x)|^\delta - |Tf_2(x_0)|^\delta\right| dx \leq C(M_{\overline{A}}f(x_0))^\delta . \tag{2.6}$$

For (2.5) we use that our operator T is of weak type (1, 1) and Kolmogorov's inequality.

$$\begin{aligned}\frac{1}{|Q|}\int_Q |Tf_1(x)|^\delta\, dx &\leq C_\delta\left(\frac{1}{|Q|}\int_{\mathbb{R}^n}|f_1(x)|\, dx\right)^\delta \\ &= C_\delta\left(\frac{1}{|Q|}\int_{\tilde{Q}}|f(x)|\, dx\right)^\delta \leq C_{n,\delta}(Mf(x_0))^\delta .\end{aligned}$$

To prove (2.6) we need to use the fact that our kernel satisfies H_A. From

$$\left||Tf_2(x)|^\delta - |Tf_2(x_0)|^\delta\right| \leq |Tf_2(x) - Tf_2(x_0)|^\delta ,$$

it follows that is enough to estimate $|Tf_2(x) - Tf_2(x_0)|^\delta$.

If $x \in Q$ and $R = c_A d(Q) > c_A|x - x_0|$, we have

$$\begin{aligned}|Tf_2(x) - Tf_2(x_0)| &= \left|\int_{y\notin\tilde{Q}}(K(x-y) - K(x_0-y))\, f(y)\, dy\right| \\ &\leq \int_{|y-x_0|>2R}|K(x-y) - K(x_0-y)||f(y)|\, dy \\ &= \sum_{m=1}^{\infty}\int_{2^m R<|y-x_0|\leq 2^{m+1}R}|K(x-y) - K(x_0-y)||f(y)|\, dy .\end{aligned}$$

If we use Hölder's inequality (1.11), we may dominate the last term by

$$\begin{aligned}&\sum_{m=1}^{\infty}\left(2^m R\right)^n \left\|(K(x-\cdot) - K(x_0-\cdot))\chi_{\{2^m R<|y-x_0|\leq 2^{m+1}R\}}(\cdot)\right\|_{A,B(x_0,2^{m+1}R)} M_{\overline{A}}f(x_0) \\ &\quad \leq CM_{\overline{A}}f(x_0) .\end{aligned}$$

Hence,

$$|Tf_2(x) - Tf_2(x_0)|^\delta \leq C(M_{\overline{A}}f(x_0))^\delta ,$$

and (2.6) follows. □

The theorem can be extended to vector valued operators $Tf(x) = \text{p.v.}\int K(x - y)f(y)\, dy$, where now K takes values in a Banach space X.

Definition 4. We say that the kernel K, satisfies the L^A-Hörmander condition if there are numbers $c_A > 1$ and $C_A > 0$ such that for any x and $R > c_A|x|$,

$$\sum_{m=1}^{\infty} \left(2^m R\right)^n \left\| \|(K(x-\cdot) - K(-\cdot))\|_X \, \chi_{\{2^m R<|y|\le 2^{m+1}R\}}(\cdot) \right\|_{A,B(0,2^{m+1}R)} \le C_A \, .$$

The theorem, whose proof we leave to the reader, is

Theorem 2. *Let K be a vector valued kernel, that satisfies the L^A-Hörmander condition and let Tf be the associated singular integral. If T is a bounded operator in some L^p, $1 \le p < \infty$, then, for all $0 < p < \infty$ and $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} \|Tf\|_X^p w \le C \int_{\mathbb{R}^n} (M_{\overline{A}} f)^p w \, ,$$

whenever the left-hand side is finite.

3. The One-Sided Case

In dimension one, there are examples of singular integrals, both real valued, [1], and vector valued, [15], whose kernels are supported in $(-\infty, 0)$. These one-sided singular integrals are particular cases of singular integrals, and thus Theorem A holds for them. But it seems natural to ask if one can do better using the fact that the kernel is supported on $(-\infty, 0)$. More precisely:

Can we improve the inequality

$$\int_{\mathbb{R}} |Tf|^p w \le C \int_{\mathbb{R}} (M_{\overline{A}} f)^p w \, ,$$

allowing, perhaps an operator smaller than $M_{\overline{A}} f$, or a wider class of weights?

The answer is yes on both accounts. We can substitute $M_{\overline{A}} f$ by the corresponding one-sided operator and allow w to be any weight in the class A_∞^+ which is bigger than A_∞. (Any increasing function is in A_∞^+).

The one-sided weights are relevant to the study of the one-sided Hardy-Littlewood maximal operators:

Definition 5. The one-sided Hardy-Littlewood maximal operators M^+ and M^- are defined for locally integrable functions f by

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f| \quad \text{and} \qquad M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f| \, .$$

The A_p^+ classes were introduced by E. Sawyer [14] in the study of the weights for these operators.

He proved the following.

Theorem. *If $p > 1$ the inequality $\int_{\mathbb{R}} M^+ f(x)^p w(x)\,dx \le C \int_{\mathbb{R}} |f(x)|^p w(x)\,dx$ holds for all $f \in L^p(w)$ if, and only if, w satisfies the following condition:*

$\left(A_p^+\right)$: *There exists C such that for any three points $a < b < c$,*

$$\left(\int_a^b w\right)^{\frac{1}{p}} \left(\int_b^c w^{1-p'}\right)^{\frac{1}{p'}} \le C(c-a) \qquad \left(p + p' = pp'\right) . \tag{3.1}$$

The case $p = 1$ was not considered in Sawyer's article but it was proved in [8] that the weak type estimate for this operator holds, i.e.,

$$\int_{\{M^+ f(x) > \lambda\}} w \leq \frac{C}{\lambda} \int |f(x)| w(x)\, dx$$

if and only if:

(A_1^+) : *There exists* C *such that for almost every* x: $\quad M^- w(x) \leq C w(x)$.

The class A_∞^+ is defined as the union of all the A_p^+ classes,

$$A_\infty^+ = \cup_{p \geq 1} A_p^+ \, .$$

The classes A_p^- are defined in a similar way. It is interesting to note that $A_p = A_p^+ \cap A_p^-$, $A_p \subsetneq A_p^+$ and $A_p \subsetneq A_p^-$. (See [14, 8, 9] for more definitions and results.)

Definition 6. Let f be a locally integrable function. The one-sided sharp maximal function is defined by

$$M^{+,\#} f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} \left(f(y) - \frac{1}{h} \int_{x+h}^{x+2h} f \right)^+ dy \, .$$

It is proved in [10] that

$$\begin{aligned} M^{+,\#} f(x) &\leq \sup_{h>0} \inf_{a \in \mathbb{R}} \frac{1}{h} \int_x^{x+h} (f(y) - a)^+ \, dy + \frac{1}{h} \int_{x+h}^{x+2h} (a - f(y))^+ \, dy \\ &\leq C ||f||_{BMO} \, . \end{aligned} \tag{3.2}$$

Here f^+ denotes the positive part of f, i.e., $f^+(x) = \max\{f(x), 0\}$. (See [10] for other results and definitions.)

Definition 7. For each locally integrable function f, the one-sided maximal operators associated to the Young function B are defined by

$$M_B^+ f(x) = \sup_{x<b} \|f\|_{B,(x,b)} \quad \text{and} \quad M_B^- f(x) = \sup_{a<x} \|f\|_{B,(a,x)} \, .$$

We shall also need the following maximal operators:

$$M_r^+ f(x) = \left(M^+ |f|^r (x) \right)^{1/r} \quad \text{and} \quad M_\delta^{+,\#} f(x) = \left(M^{+,\#} |f|^\delta (x) \right)^{1/\delta} \, .$$

We can now state our result.

Theorem 3. *Let* K *be a kernel, supported on* $(-\infty, 0)$*, possibly vector valued, that satisfies the* L^A*-Hörmander condition. Let* Tf *be the associated singular integral. If* T *is a bounded operator in some* L^p, $1 \leq p < \infty$*, then, for any* $0 < p < \infty$ *and* $w \in A_\infty^+$ *there exists* $C > 0$ *such that*

$$\int_{\mathbb{R}} |Tf(x)|^p w(x)\, dx \leq C \int_{\mathbb{R}} \left(M_{\bar{A}}^+ f(x) \right)^p w(x)\, dx \, ,$$

for any $f \in C^\infty$ with compact support.

Proof. We will prove the scalar case, since the vector valued case is analogous. Our proof of Theorem A was based on inequality (2.2). The one-sided version of this theorem is the following [10]:

Theorem. *For any $0 < p < \infty$ and $w \in A^+_\infty$ there exists C such that*

$$\int_{\mathbb{R}} \left|M^+ f(x)\right|^p w(x)\,dx \le C \int_{\mathbb{R}} \left(M^{+,\#} f(x)\right)^p w(x)\,dx\ , \tag{3.3}$$

whenever the left-hand side is finite.

It follows from this theorem that it is enough to prove

$$\left(M^{+,\#}|Tf|^\delta(x)\right)^{\frac{1}{\delta}} \le C_\delta M^+_A f(x)\ .$$

If we use (3.2) we get that

$$\begin{aligned} M^{+,\#} f(x) &\le \sup_{h>0} \inf_{a\in\mathbb{R}} \frac{1}{h}\int_x^{x+h} |f(y)-a|\,dy + \frac{1}{h}\int_{x+h}^{x+2h} |a-f(y)|\,dy \\ &\le C \sup_{h>0} \inf_{a\in\mathbb{R}} \frac{1}{h}\int_x^{x+h} |f(y)-a|\,dy\ . \end{aligned}$$

Therefore, it is enough to prove that, for fixed x_0, there is, for every positive h, a real number a_h, that may depend on x_0 and h, such that

$$\left(\frac{1}{h}\int_{x_0}^{x_0+h} \left||Tf(x)|^\delta - |a_h|^\delta\right|\,dx\right)^{\frac{1}{\delta}} \le C\big(M^+_A f\big)(x_0)\ . \tag{3.4}$$

We define $f_1 = f\chi_{(x_0,x_0+2h)}$, $f_2 = f\chi_{(x_0+2h,\infty)}$ and choose $a_h = Tf_2(x_0)$. We need to prove that,

$$\left(\frac{1}{h}\int_{x_0}^{x_0+h} \left||Tf(x)|^\delta - |Tf_2(x_0)|^\delta\right|\,dx\right)^{\frac{1}{\delta}} \le C\big(M^+_A f\big)(x_0)\ . \tag{3.5}$$

Now we use the one-sided character of our operator to get that for $x \in (x_0, x_0+h)$, $Tf(x) = Tf_1(x) + Tf_2(x)$ and follow the proof of (2.3). For f_1, Kolmogorov's inequality yields

$$\frac{1}{h}\int_{x_0}^{x_0+h} |Tf_1(x)|^\delta\,dx \le C_\delta \left(\frac{1}{h}\int_{x_0}^{x_0+2h} |f(x)|\right)^\delta dx \le C_\delta\big(M^+ f(x_0)\big)^\delta\ .$$

For f_2 we observe that for any $x \in (x_0, x_0+h)$, if $R = c_A h$, we have,

$$\begin{aligned} |Tf_2(x) - Tf_2(x_0)| &= \left|\int_{y>x_0+2h} (K(x-y) - K(x_0-y))\, f(y)\,dy\right| \\ &\le \sum_{m=1}^\infty \int_{2^m h<y-x_0\le 2^{m+1}h} |K(x-y) - K(x_0-y)||f(y)|\,dy\ . \end{aligned}$$

If we use Hölder's inequality (1.11), we may dominate the last term by

$$\sum_{m=1}^{\infty} \left(2^m h\right) \left\| (K(x-\cdot) - K(x_0-\cdot))\chi_{\{2^m h < y - x_0 \le 2^{m+1}h\}}(\cdot) \right\|_{A,(x_0,x_0+2^{m+1}h)} M_A^+ f(x_0)$$
$$\le C M_A^+ f(x_0) . \qquad \square$$

4. Proof of Theorem B

Let us now show an example of a one-sided operator whose kernel is in $\cap H_r \setminus H_\infty$. The example comes from ergodic theory.

Definition 8. Let f be a measurable function defined on $\mathbb{R}$. For each $n \in \mathbb{Z}$ we consider the average $A_n f(x) = \frac{1}{2^n} \int_x^{x+2^n} f$. The Square Function is defined as

$$Sf(x) = \left(\sum_{n=-\infty}^{\infty} |A_n f(x) - A_{n-1} f(x)|^2 \right)^{\frac{1}{2}} .$$

The local version of this operator, namely the operator

$$S_1 f(x) = \left(\sum_{n=-\infty}^{0} |A_n f(x) - A_{n-1} f(x)|^2 \right)^{\frac{1}{2}} ,$$

is of interest in ergodic theory and it has been extensively studied. In particular, it has been proved, [6], that it is of weak type one-one, maps L^p into itself ($p > 1$) and L^∞ into BMO. The operator S is obviously non-linear but it can be interpreted as the norm of a vector valued operator (see [15]).

Definition 9. Given a locally integrable function f we define the sequence valued operator U as follows

$$\begin{aligned} Uf(x) &= \left\{ A_n f(x) - A_{n-1} f(x) \right\}_n \\ &= \left\{ \int_{\mathbb{R}} \frac{1}{2^n} \chi_{(-2^n,0)}(x-y) f(y)\, dy - \int_{\mathbb{R}} \frac{1}{2^{n-1}} \chi_{(-2^{n-1},0)}(x-y) f(y)\, dy \right\}_n \\ &= \left\{ \int_{\mathbb{R}} \left(\frac{1}{2^n} \chi_{(-2^n,0)}(x-y) - \frac{1}{2^{n-1}} \chi_{(-2^{n-1},0)}(x-y) \right) f(y)\, dy \right\}_n \\ &= \int_{\mathbb{R}} K(x-y) f(y)\, dy , \end{aligned}$$

where K is the sequence valued function

$$K(x) = \{K_n(x)\}_n = \left\{ \frac{1}{2^n} \chi_{(-2^n,0)}(x) - \frac{1}{2^{n-1}} \chi_{(-2^{n-1},0)}(x) \right\}_n .$$

Observe that $\|Uf(x)\|_{\ell^2} = Sf(x)$. It is proved in [15] that the kernel satisfies the following condition:

Smoothness Condition. *Assume*

$$x_0 \in \mathbb{R},\ x_0 < x < x_0 + 2^i,\ x_0 + 2^j < y \le x_0 + 2^{j+1} ,$$

where $i < j$ and $i, j \in \mathbb{Z}$. Let K be the vector valued kernel that appears in Definition 9. Then

$$K_n(x-y) - K_n(x_0-y) = \begin{cases} 0, & \text{if} \quad n \notin \{j, j+1\}\,; \\ \frac{1}{2^j}\chi_{(x_0+2^j,\ x+2^j)}(y), & \text{if} \quad n \in \{j, j+1\}\,. \end{cases} \tag{4.1}$$

It follows from this lemma that the kernel does not satisfy H_∞. Indeed, take $x_0 = 0$, $0 < x < 2^i$ and $R = 2^i$, then for any $m \in \mathbb{N}$

$$2^m 2^i \sup_{2^{m+i} < y \le 2^{m+i+1}} \|K(x-y) - K(-y)\|_{\ell^2} = C$$

and H_∞ fails. The following lemma tells us that our kernel satisfies something better that just being in the intersection of all the $H_r,\ r \ge 1$.

Lemma 1. *The kernel K satisfies the L^A-Hörmander condition with $A(t) \approx \exp(t^{\frac{1}{1+\epsilon}})$, $\epsilon > 0$.*

Proof. Let us fix x. Observe that since the support of K is contained in $(-\infty, 0)$, we may assume $x > 0$. We will assume that R is of the form $R = 2^i$ for some integer i and the general case will follow. Let $R > |x|$. Then $R = 2^i > x > 0$. Let $I_m = (0, 2^{m+i+1})$. Then

$$\left\| \|(K(x-\cdot) - K(-\cdot))\|_{\ell^2}\, \chi_{\{2^{m+i} < |y| \le 2^{m+i+1}\}}(\cdot) \right\|_{A, I_m} = \frac{\sqrt{2}}{2^{m+i}} \left\| \chi_{(2^{m+i}, x+2^{m+i})} \right\|_{A, I_m} .$$

An easy computation gives

$$\left\| \chi_{(2^{m+i}, x+2^{m+i})} \right\|_{A, I_m} = \frac{1}{A^{-1}\left(\frac{2^{m+i+1}}{x}\right)} \le \frac{C}{A^{-1}\left(2^{m+1}\right)} .$$

Therefore,

$$\sum_{m=1}^{\infty} \left(2^m R\right) \left\| \|(K(x-\cdot) - K(-\cdot))\|_{\ell^2}\, \chi_{\{2^m R < |y| \le 2^{m+1} R\}}(\cdot) \right\|_{A, B(0, 2^{m+1} R)}$$
$$\le C \sum_{m=1}^{\infty} \frac{1}{(m+1)^{1+\epsilon}} < \infty .$$

□

Remark 3. Since the square function Sf is a one-sided operator we may apply Theorem 3 to get that for any $p > 0$ and any A_∞^+ weight w, there exists a constant C such that

$$\int (Sf(x))^p w(x)\, dx \le C \int \left(\left(M^+\right)^3 f(x)\right)^p w(x)\, dx ,$$

whenever the left-hand side is finite.

Proof. We just observe that $\overline{A}(t) = t(1 + \log^+(t))^{1+\epsilon}$ which for ϵ small is dominated by $B(t) = t(1 + \log^+(t))^2$ and $M_B^+ f$ is pointwise equivalent to $(M^+)^3 f$. □

Since the one-sided Hardy-Littlewood maximal operator is bounded form $L^p(w)$ to itself, and $A_p^+ \subset A_\infty^+$, we obtain a different proof of the boundedness of S from $L^p(w)$ to itself, whenever $w \in A_p^+$ [15].

Theorem 4. *There is a vector valued operator T whose kernel K is in $\cap H_r \setminus H_\infty$ but nevertheless the operator satisfies* (1.2).

Proof. Just consider the operator T defined as $Tf(x) = \|Uf(x)\|_{\ell^\infty}$. The argument given for the square function proves that the kernel K with the ℓ^∞ norm does not satisfy H_∞. But the operator corresponding to this norm is dominated by $2M^+f(x)$ and (1.2) holds trivially (even if the weight w does not satisfy A_∞).

We finish by proving that for any Young function A, there exists a kernel K belonging to H_A. (This example is in the spirit of [7] and was suggested to us by C. Pérez.) □

Theorem 5. *Let A be any Young function. For $\beta > 0$ we consider the function $k_A(t) = A^{-1}\big(\frac{1}{t}(\log\frac{e}{t})^{-(1+\beta)}\big)\chi_{(0,1)}(t)$. The kernel K_A defined by $K_A(t) = k_A(t-4)$ belongs to H_A.*

Proof. It is an argument similar to the one in [7]. We will prove first that $k_A \in L^1 \cap L^A$. To see that $k_A \in L^A$ we just need to find $c > 0$, such that

$$\int_{\mathbb{R}} A\left(\frac{k_A(t)}{c}\right) dt < \infty .$$

An easy computation gives

$$\int_{\mathbb{R}} A\left(k_A(t)\right) dt = \int_0^1 \frac{1}{t}\left(\log\left(\frac{e}{t}\right)\right)^{-(1+\beta)} = \frac{1}{\beta} < \infty ,$$

while Jensen's inequality yields

$$\int_0^1 k_A(t) \le A^{-1}\left(\int_0^1 \frac{1}{t}\left(\log\frac{e}{t}\right)^{-(1+\beta)} dt\right) = A^{-1}\left(\frac{1}{\beta}\right) .$$

We define the operator $Tf(x) = K_A * f(x)$. Since K_A is just a translation of k_A, it belongs to L^1 and then $\|Tf\|_q \le C\|f\|_q$ for any $1 \le q$. We need to prove that K_A satisfies

$$\sum_{m=1}^{\infty} \left(2^m R\right) \left\| \left(K_A(x-\cdot) - K(-\cdot)\right) \chi_{\{2^m R<|y|\le 2^{m+1}R\}}(\cdot) \right\|_{A,B(0,2^{m+1}R)} \le C_A .$$

whenever $R > c_A|x|$. We just sketch the proof. We take $c_A = 1$ and $|x| < R$. For $m \ge 1$ and $2^m R < |y| \le 2^{m+1}R$, one has $2^{m-1}R < |y-x| \le 2^{m+2}R$ and, trivially, $2^{m-1}R < |y| \le 2^{m+2}R$. Now

$$\begin{aligned} &\left\| \left(K_A(x-\cdot) - K_A(-\cdot)\right) \chi_{\{2^m R<|y|\le 2^{m+1}R\}}(\cdot) \right\|_{A,B(0,2^{m+1}R)} \\ &\quad \le C \left\| K_A\, \chi_{\{2^{m-1} R<|y|\le 2^{m+2}R\}}(\cdot) \right\|_{A,B(0,2^{m+1}R)} . \end{aligned}$$

The kernel k_A has support on $(0, 1)$. Therefore if $R > 5$ there is nothing to prove. If $R < 5$ and m_0 is the unique natural number so that $2^{m_0}R \le 5 < 2^{m_0+1}R$. Then, for any $m \ge m_0 + 2$ and $2^{m-1}R < |y+4| < 2^{m+2}R$, it follows that $|y| > 1$ and $k_A(y) = 0$. We need only to estimate

$$S = \sum_{m=1}^{m_0+1} 2^m R \left\| K_A\, \chi_{\{2^{m-1} R<|y|\le 2^{m+2}R\}}(\cdot) \right\|_{A,B(0,2^{m+1}R)} .$$

But, for each m, we have

$$\left\| K_A\, \chi_{\{2^{m-1}R<|y|\le 2^{m+2}R\}}(\cdot) \right\|_{A,B(0,2^{m+1}R)} \le 1 + \frac{1}{2^{m+1}R}\int_{2^{m-1}R<|y+4|\le 2^{m+2}R} A(K_A(y))\,dy.$$

Since the domains of integration are almost disjoint we can add and get

$$S \le C2^{m_0}R + C\int A(K_A(y))\,dy \le C\left(5+\frac{1}{\beta}\right) .$$

□

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Received June 17, 2004

Revision received June 17, 2005

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