ON THE COIFMAN TYPE INEQUALITY FOR THE OSCILLATION OF ONE-SIDED AVERAGES

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Abstract. In this paper we study the Coifman type estimate for an oscillation operator related to the one-sided discrete square function, S^+. We prove that for any A_∞^+ weight w, the $L^p(w)$-norm of this operator, and therefore the $L^p(w)$-norm of S^+, is dominated by a constant times the $L^p(w)$-norm of the one-sided Hardy-Littlewood maximal function iterated two times. For the k-th commutator with a BMO function we show that $k+2$ iterates of the one-sided Hardy-Littlewood maximal function are sufficient.

1. Introduction

In [5], Coifman and Fefferman proved that if T is a Calderón-Zygmund operator, w is an A_∞ weight and M is the Hardy-Littlewood maximal operator, then, for each p, $0 < p < \infty$, there exists C such that

$$\int |Tf|^p w \leq C \int (Mf)^p w\,,$$

whenever the left-hand side is finite. Inequalities of the type

$$\int |Tf|^p w \leq C \int (M_T f)^p w\,,$$

where T is an operator and M_T is a maximal operator which, in general, will depend on T, are known as Coifman type inequalities.

Recently, de la Torre and Torrea [26] and Lorente, Riveros and de la Torre [14] have studied inequalities with weights for the one-sided discrete square function defined as follows: for f locally integrable in $\mathbb{R}$ and $s > 0$, let us consider the averages

$$A_s f(x) = \frac{1}{s}\int_x^{x+s} f(y)dy.$$

The one-sided discrete square function of f is given by

$$S^+ f(x) = \left(\sum_{n\in\mathbb{Z}} |A_{2^n} f(x) - A_{2^{n-1}} f(x)|^2\right)^{1/2}.$$

We write S^+ instead of S to emphasize that this is a one-sided operator, i.e., $S^+f(x) = S^+(f\chi_{(x,\infty)})(x)$.

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In [14] it was shown that if $0 < p < \infty$ and $w \in A_\infty^+$, then

$$\int_{\mathbb{R}} (S^+ f)^p \omega \leq \int_{\mathbb{R}} ((M^+)^3 f)^p \omega, \qquad f \in L_c^\infty\,, \tag{1.1}$$

whenever the left-hand side is finite, where $(M^+)^k$ stands for the k-th iteration of M^+, and

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f|.$$

A natural question left open by this result is the following: can we improve the result using fewer iterates of M^+ in (1.1)? In this note we study a bigger operator for which two iterates are enough. Therefore the inequality (1.1) is improved in two ways: a bigger operator on the left and a smaller operator on the right. The operator that we will study is the oscillation of the averages,

$$\mathcal{O}^+ f(x) = \left(\sum_{n\in\mathbb{Z}} \sup_{s\in J_n} |A_{2^n} f(x) - A_s f(x)|^2 \right)^{1/2},$$

where $J_n = [2^n, 2^{n+1})$. It is clear that $S^+ f(x) \leq \mathcal{O}^+ f(x)$ for all $x \in \mathbb{R}$.

If we look at the definition of $\mathcal{O}^+ f(x)$, we see that the sequence $\{\tau_n(x)\}$, defined by $\tau_n(x) = \sup_{s\in J_n} |A_{2^n} f(x) - A_s f(x)|$, measures the oscillation of the $A_s f(x)$ in the interval J_n. Then we take the ℓ^2 norm of this sequence. Operators of this kind are of interest in ergodic theory, [3], [8] and singular integrals. The behavior of $\mathcal{O}^+$ with respect to the Lebesgue's measure has already been studied. Concretely, in lemma 2.1 in [4] it is proved that $\mathcal{O}^+$ is of weak type (1,1) and strong type (p,p), $1 < p < \infty$, with respect to the Lebesgue's measure. It is natural to ask what happens if we change the measure, i.e., which are the good weights for the operator $\mathcal{O}^+$? It is worth mentioning that recently, in [9], it has been proved that the oscillation of the Hilbert Transform is bounded in $L^p(w)$, for $w \in A_p$, $1 < p < \infty$. In this paper we shall obtain that $\mathcal{O}^+$ is bounded from $L^p(w)$ to $L^p(w)$, for $w \in A_p^+$, $1 < p < \infty$, as a consequence of the Coifman type inequality stated in the following theorem:

Theorem 1.1. *Let $\omega \in A_\infty^+$ and $0 < p < \infty$. Then, there exists $C > 0$ such that*

$$\int_{\mathbb{R}} (S^+ f)^p \omega \leq \int_{\mathbb{R}} (\mathcal{O}^+ f)^p \omega \leq C \int_{\mathbb{R}} ((M^+)^2 f)^p \omega, \qquad f \in L_c^\infty,$$

whenever the left-hand side is finite.

Remark 1.2. As a consequence of the above theorem, if $1 < p < \infty$ and $\omega \in A_p^+$, we obtain that $\mathcal{O}^+$ and S^+ are bounded in $L^p(\omega)$. For S^+ this was first proved in [26].

Remark 1.3. This theorem improves inequality (1.1) for S^+ substituting $(M^+)^3$ by the smaller operator $(M^+)^2$. It is an open question if (1.1) holds with M^+ instead of $(M^+)^2$, which holds for standard one-sided Calderón-Zygmund operators.

The paper is organized as follows: In Section 2 we introduce notation and recall some basic results about one-sided weights and maximal operators associated to Young functions. In section 3 we prove Theorem 1.1 and in section 4 we study the commutators of S^+ and $\mathcal{O}^+$ with a BMO function b. For $p > 1$ we prove Coifman type inequalities that imply the boundedness of the commutators in $L^p(w)$ whenever $w \in A_p^+$. For $p = 1$ we also obtain a weak type inequality for the k-th order commutator.

2. Definitions and basic facts about one-sided operators

Definition 2.1. *The one-sided Hardy-Littlewood maximal operators M^+ and M^- are defined for locally integrable functions f by*

$$M^+f(x) = \sup_{h>0} \frac{1}{h}\int_x^{x+h} |f| \quad \textit{and} \quad M^-f(x) = \sup_{h>0} \frac{1}{h}\int_{x-h}^{x} |f|.$$

The one-sided weights are defined as follows,

$$\sup_{a<b<c} \frac{1}{(c-a)^p}\int_a^b \omega \left(\int_b^c \omega^{1-p'}\right)^{p-1} < \infty, \quad 1<p<\infty, \tag{A_p^+}$$

$$M^-\omega(x) \leq C\omega(x) \quad \text{a.e.} \tag{A_1^+}$$

A_∞^+ is defined as the union of the A_p^+ classes,

$$A_\infty^+ = \cup_{p\geq 1} A_p^+. \tag{A_∞^+}$$

The A_p^- classes are defined reversing the orientation of $\mathbb{R}$. It is interesting to note that $A_p = A_p^+ \cap A_p^-$, $A_p \subsetneq A_p^+$ and $A_p \subsetneq A_p^-$. (See [23], [15], [16], [17] for more definitions and results.)

It was proved in [26], that $\omega \in A_p^+$, $1<p<\infty$, if, and only if, S^+ is bounded from $L^p(\omega)$ to $L^p(\omega)$, and that $\omega \in A_1^+$, if, and only if, S^+ is of weak-type (1,1) with respect to ω.

Definition 2.2. *Let b be a locally integrable function. We say that $b \in BMO$ if*

$$||b||_{BMO} = \sup_I \frac{1}{|I|}\int_I |b-b_I| < \infty,$$

where I denotes any bounded interval and $b_I = \frac{1}{|I|}\int_I b$.

Definition 2.3. *Let f be a locally integrable function. The one-sided sharp maximal function is defined by*

$$M^{+,\#}(f)(x) = \sup_{h>0} \frac{1}{h}\int_x^{x+h} \left(f(y) - \frac{1}{h}\int_{x+h}^{x+2h} f\right)^+ dy.$$

For $\delta > 0$ we define

$$M_\delta^{+,\#}f(x) = \left(M^{+,\#}|f|^\delta(x)\right)^{1/\delta}.$$

It is proved in [18] that

$$\begin{aligned} M^{+,\#}(f)(x)\Delta &\leq \sup_{h>0}\inf_{a\in\mathbb{R}} \frac{1}{h}\int_x^{x+h} (f(y)-a)^+ dy + \frac{1}{h}\int_{x+h}^{x+2h} (a-f(y))^+ dy \\ &\leq C||f||_{BMO}. \end{aligned}$$

Now we give definitions and results about Young functions. A function $B: [0,\infty) \to [0,\infty)$ is a Young function if it is continuous, convex, increasing and satisfies $B(0)=0$ and $B(t)\to\infty$ as $t\to\infty$. A Young function B is said to satisfy Δ_2-condition (or $B \in \Delta_2$) if there exists a constant C such that $B(2t) \leq CB(t)$ for $t \geq 0$. The Luxemburg norm of a function f, given by B is

$$||f||_B = \inf\left\{\lambda > 0 : \int B\left(\frac{|f|}{\lambda}\right) \leq 1\right\},$$

and so the B-average of f over I is

$$||f||_{B,I} = \inf\left\{\lambda > 0 : \frac{1}{|I|}\int_I B\left(\frac{|f|}{\lambda}\right) \le 1\right\}.$$

We will denote by $\overline{B}$ the complementary function associated to B (see [1]). The following version of Hölder's inequality holds,

$$\frac{1}{|I|}\int_I |f\,g| \le 2||f||_{B,I}||g||_{\overline{B},I}.$$

This inequality can be extended to three functions (see [19]). If A, B, C are Young functions such that

$$A^{-1}(t)B^{-1}(t) \le C^{-1}(t),$$

then

$$||fg||_{C,I} \le 2||f||_{A,I}||g||_{B,I}. \tag{2.1}$$

Definition 2.4. *For each locally integrable function f, the one-sided maximal operators associated to the Young function B are defined by*

$$M_B^+ f(x) = \sup_{x<b} \|f\|_{B,(x,b)} \quad \textit{and} \quad M_B^- f(x) = \sup_{a<x} \|f\|_{B,(a,x)}.$$

We will be dealing with the Young functions $B_k(t) = e^{t^{1/k}} - 1$ and $\overline{B_k}(t) = t(1 + \log^+(t))^k$, $k \in \mathbb{N}$. The maximal operator associated to $\overline{B_k}$, $M^+_{\overline{B_k}}$ will be denoted by $M^+_{L(1+\log^+ L)^k}$. It is proved in [22] that $M^+_{L(1+\log^+ L)^k}$ is pointwise equivalent to $(M^+)^{k+1}$.

It is convenient to look at our operators as vector valued. Let us consider the sequence

$$H(x) = \left\{\frac{1}{2^n}\chi_{(-2^n,0)}(x) - \frac{1}{2^{n-1}}\chi_{(-2^{n-1},0)}(x)\right\}_{n\in\mathbb{Z}}$$

and let us define the operator $U : f \to Uf$ by

$$Uf(x) = \int_{\mathbb{R}} H(x-y)f(y)dy.$$

Then it is clear that $S^+f(x) = ||Uf(x)||_{\ell^2}$. If instead of the sequence H we consider for each $s > 0$ the sequence $K(x) = \{K_{n,s}(x)\}_{n\in\mathbb{Z},}$, where

$$K_{n,s}(x) = \left(\frac{1}{2^n}\chi_{(-2^n,0)}(x) - \frac{1}{s}\chi_{(-s,0)}(x)\right)\chi_{J_n}(s)\,,$$

we can define the operator V acting on locally integrable functions f, as $Vf(x) = \int_{\mathbb{R}} K(x-y)f(y)\,dy$.

If for functions $h : \mathbb{R}\times\mathbb{Z} \to \mathbb{R}$ we define the norm

$$||h||_E = \left(\sum_{n\in\mathbb{Z}} \sup_{s\in J_n} |h(s,n)|^2\right)^{1/2},$$

then $\mathcal{O}^+f(z) = ||Vf(z)||_E$.

3. Proof of Theorem 1.1

The key point in the proof of Theorem 1.1 is the following pointwise estimate: for each $0<\delta<1$, there exists C so that for any locally integrable function,

$$M_\delta^{+,\#}(\mathcal{O}^+f)(x)\leq CM^+f(x)+CM^+_{L(1+\log^+L)}f(x). \tag{3.1}$$

Let us prove inequality (3.1). For $0<\delta<1$ we have

$$M_\delta^{+,\#}(\mathcal{O}^+f)(x)\leq C\sup_{h>0}\inf_{c\in\mathbb{R}}\left(\frac{1}{h}\int_x^{x+2h}\left||\mathcal{O}^+f(y)|^\delta-|c|^\delta\right|dy\right)^{1/\delta}. \tag{3.2}$$

Let $x\in\mathbb{R}$ and $h>0$. Let us consider the unique $i\in\mathbb{Z}$ such that $2^i\leq h<2^{i+1}$ and let denote by J the interval $J=[x,x+2^{i+3})$. If we write $f=f_1+f_2$, where $f_1=f\chi_J$, and choose $c=\mathcal{O}^+(f_2)(x)$, we have

$$\begin{aligned}&\left(\frac{1}{h}\int_x^{x+2h}\left||\mathcal{O}^+f(y)|^\delta-|\mathcal{O}^+f_2(x)|^\delta\right|dy\right)^{1/\delta}\\ &\leq C\left(\frac{1}{2^i}\int_x^{x+2^{i+2}}\left|\mathcal{O}^+f(y)-\mathcal{O}^+f_2(x)\right|^\delta dy\right)^{1/\delta}\\ &\leq C\left(\frac{1}{2^i}\int_x^{x+2^{i+2}}\left|\mathcal{O}^+f_1(y)\right|^\delta dy\right)^{1/\delta}+C\left(\frac{1}{2^i}\int_x^{x+2^{i+2}}\left|\mathcal{O}^+f_2(y)-\mathcal{O}^+f_2(x)\right|^\delta dy\right)^{1/\delta}\\ &=I+II.\end{aligned} \tag{3.3}$$

Using Lemma 2.1 (1) in [4], that is, the fact that $\mathcal{O}^+$ is weak type (1,1) with respect to the Lebesgue's measure, and Kolmogorov's inequality, we get

$$I\leq C\frac{1}{2^i}\int_x^{x+2^{i+3}}|f(y)|dy\leq CM^+f(x). \tag{3.4}$$

In order to estimate II, we first use Jensen's inequality and obtain

$$\begin{aligned}II&=C\left(\frac{1}{2^i}\int_x^{x+2^{i+2}}\left|||Vf_2(y)||_E-||Vf_2(x)||_E\right|^\delta dy\right)^{1/\delta}\\ &\leq C\frac{1}{2^i}\int_x^{x+2^{i+2}}||Vf_2(y)-Vf_2(x)||_E\,dy.\end{aligned} \tag{3.5}$$

Let us estimate $||Vf_2(y) - Vf_2(x)||_E$.

$$
\begin{aligned}
&||Vf_2(y) - Vf_2(x)||_E \\
&= \left\| \left\{ \left(\frac{1}{2^n} \int_y^{y+2^n} f_2 - \frac{1}{s} \int_y^{y+s} f_2 \right) \chi_{J_n}(s) \right\}_{n\in\mathbb{Z}, s\in\mathbb{R}} \right. \\
&\qquad \left. - \left\{ \left(\frac{1}{2^n} \int_x^{x+2^n} f_2 - \frac{1}{s} \int_x^{x+s} f_2 \right) \chi_{J_n}(s) \right\}_{n\in\mathbb{Z}, s\in\mathbb{R}} \right\|_E \\
&\leq \left\| \left\{ \left(\frac{1}{2^n} \int_y^{y+2^n} f_2 - \frac{1}{2^n} \int_x^{x+2^n} f_2 \right) \chi_{J_n}(s) \right\}_{n\in\mathbb{Z}, s\in\mathbb{R}} \right\|_E \\
&\qquad + \left\| \left\{ \left(\frac{1}{s} \int_x^{x+s} f_2 - \frac{1}{s} \int_y^{y+s} f_2 \right) \chi_{J_n}(s) \right\}_{n\in\mathbb{Z}, s\in\mathbb{R}} \right\|_E \\
&= III + IV. \qquad (3.6)
\end{aligned}
$$

Observe that since $y \in (x, x+2^{i+2})$ and f_2 has support in $(x+2^{i+3}, \infty)$, it follows that, if $n \leq i+2$, then $x+2^n \leq x+2^{i+2}$ and $y+2^n \leq x+2^{i+2}+2^n \leq x+2^{i+2}+2^{i+2} = x+2^{i+3}$. As a consequence the only non-zero terms in III are those with $n > i+2$. Therefore

$$
III \leq \left(\sum_{n=i+3}^{\infty} \left| \frac{1}{2^n} \int_{x+2^n}^{y+2^n} f \right|^2 \right)^{1/2}. \qquad (3.7)
$$

Let us consider the Young function $B_1(t) = \mathrm{e}^t - 1$. Then $\overline{B_1}(t) = t(1+\log^+ t)$ and $B_1^{-1}(t) = \log^+(1+t)$. Using the generalized Hölder's inequality, we obtain that, for $n \geq i+3$,

$$
\begin{aligned}
\left| \frac{1}{2^n} \int_{x+2^n}^{y+2^n} f \right| &\leq \frac{1}{2^n} \int_{x+2^n}^{x+2^{n+1}} |f| \chi_{[x+2^n, y+2^n)} \\
&\leq C M^+_{\overline{B_1}} f(x) \left|\left|\chi_{[x+2^n,y+2^n)}\right|\right|_{B_1,[x+2^n,x+2^{n+1})} \\
&= C M^+_{\overline{B_1}} f(x) \frac{1}{B_1^{-1}\left(\frac{2^n}{y-x}\right)} \leq C M^+_{\overline{B_1}} f(x) \frac{1}{B_1^{-1}\left(2^{n-i-2}\right)}, \qquad (3.8)
\end{aligned}
$$

where in the last inequality we have used that $y - x \leq 2^{i+2}$ and that B_1^{-1} is nondecreasing.

Putting together (3.7) and (3.8) we obtain

$$
\begin{aligned}
III &\leq C M^+_{\overline{B_1}} f(x) \left(\sum_{n=i+3}^{\infty} \frac{1}{\left(B_1^{-1}\left(2^{n-i-2}\right)\right)^2} \right)^{1/2} \\
&\leq C M^+_{\overline{B_1}} f(x) \left(\sum_{n=i+3}^{\infty} \frac{1}{(n-i-2)^2} \right)^{1/2} = C M^+_{\overline{B_1}} f(x). \qquad (3.9)
\end{aligned}
$$

Let us estimate IV. For $n \in \mathbb{Z}$, set

$$
\beta_n = \sup_{s \in J_n} \left| \frac{1}{s} \int_x^{x+s} f_2 - \frac{1}{s} \int_y^{y+s} f_2 \right|.
$$

Then, if $\beta_n \neq 0$, we have that there exists $s_n \in J_n$ such that

$$\left|\frac{1}{s_n}\int_x^{x+s_n} f_2 - \frac{1}{s_n}\int_y^{y+s_n} f_2\right| > \frac{1}{2}\beta_n.$$

If $n \leq i+1$ then $y+s_n \leq y+2^{n+1} \leq x+2^{i+2}+2^{n+1} \leq x+2^{i+3}$. Therefore in IV we may assume $n \geq i+2$.

Using again generalized Hölder's inequality, we get that, for $n \geq i+2$,

$$\begin{aligned}\beta_n &\leq C\frac{1}{s_n}\int_{x+s_n}^{y+s_n} |f_2| \leq C\frac{1}{s_n}\int_x^{x+2^{n+2}} |f|\chi_{[x+s_n,y+s_n)} \\ &\leq C\frac{2^{n+2}}{s_n} M^+_{B_1}f(x)\left|\left|\chi_{[x+s_n,y+s_n)}\right|\right|_{B_1,(x,x+2^{n+2})} \\ &= CM^+_{B_1}f(x)\frac{1}{B_1^{-1}\left(\frac{2^{n+2}}{y-x}\right)} \leq CM^+_{B_1}f(x)\frac{1}{B_1^{-1}(2^{n-i})}. \end{aligned} \tag{3.10}$$

Then,

$$IV \leq CM^+_{B_1}f(x)\left(\sum_{n=i+2}^{\infty}\frac{1}{\left(B_1^{-1}(2^{n-i})\right)^2}\right)^{1/2} = CM^+_{B_1}f(x). \tag{3.11}$$

Collecting inequalities (3.2)–(3.6), (3.9) and (3.11), we obtain (3.1). On the other hand, we have that $M^+_{B_1}f = M^+_{L(1+\log^+ L)}f$ is pointwise equivalent to $(M^+)^2 f$ (see [22]). As a consequence, (3.1) gives

$$M^{+,\#}_\delta(\mathcal{O}^+ f)(x) \leq C(M^+)^2 f(x), \quad \text{a.e. } x \in \mathbb{R}.$$

To finish the proof of Theorem 1.1 we only have to observe that since $w \in A^+_\infty$, there exists $r > 1$, such that $w \in A^+_r$. Then, for δ small enough, we get that $r < p/\delta$ and thus, $w \in A^+_{p/\delta}$. Therefore, by theorem 4 in [18], we get

$$\begin{aligned}\int_{\mathbb{R}} \left|\mathcal{O}^+ f\right|^p \omega &\leq \int_{\mathbb{R}} \left(M^+_\delta\left(\mathcal{O}^+ f\right)\right)^p \omega = \int_{\mathbb{R}}\left(M^+\left(\mathcal{O}^+ f\right)^\delta\right)^{p/\delta}\omega \\ &\leq C\int_{\mathbb{R}}\left(M^{+,\#}_\delta\left(\mathcal{O}^+ f\right)\right)^p \omega \leq C\int_{\mathbb{R}}\left((M^+)^2 f\right)^p \omega, \end{aligned} \tag{3.12}$$

whenever the left hand side is finite. □

4. Commutators

The commutators of singular integrals with BMO functions have been extensively studied (see [2],[6],[24],[25],[20],[21],[10],[11],[12],[13]). Since S^+ can be considered as a singular integral whose kernel satisfies a weaker condition (see [14]), it is interesting to know if the results about commutators of singular integrals can be extended to S^+. In [14] we have proved that the classical results about boundedness with weights can be extended to S^+ and, furthermore, can be improved allowing a wider class of weights, since S^+ is a one-sided operator. The results in [20] and [21] have been improved in [11] for one-sided singular integrals. Observe that for standard Calderón-Zygmund singular integrals (satisfying the usual Lipschitz condition) one obtains Mf instead of $M^2 f$ in Theorem 1.1. Therefore we can not expect to obtain the same results for the commutator of S^+ as we obtained in [11] for one-sided singular integrals. However, we can give estimates of the same kind, increasing in one the iterations of M^+. Concretely, for the k-th order commutator of S^+ and $\mathcal{O}^+$ we have:

Theorem 4.1. *Let $b \in BMO$ and $k = 0, 1, 2, \dots$. Let us define the k-th order commutator of S^+ and $\mathcal{O}^+$ by*

$$S_b^{+,k} f(x) = \left\| \int_{\mathbb{R}} (b(x) - b(y))^k H(x-y) f(y) dy \right\|_{\ell^2},$$

and

$$\mathcal{O}_b^{+,k} f(x) = \left\| \int_{\mathbb{R}} (b(x) - b(y))^k K(x-y) f(y) dy \right\|_{E}.$$

(Observe that for $k = 0$ we obtain S^+ and $\mathcal{O}^+$.) Then, for $0 < p < \infty$ and $w \in A_\infty^+$, there exists $C > 0$ such that,

$$\int_{\mathbb{R}} (S_b^{+,k} f)^p w \le \int_{\mathbb{R}} (\mathcal{O}_b^{+,k} f)^p w \le C \int_{\mathbb{R}} ((M^+)^{k+2} f)^p w, \qquad f \in L_c^\infty,$$

whenever the left-hand side is finite.

Remark 4.2. In [10], the $L^{\mathcal{A},k}$-Hörmander condition was introduced. If we just use that H, the vector valued kernel of S^+, satisfies the $L^{\mathcal{A},k}$-Hörmander condition for $\mathcal{A}(t) = e^{\frac{1}{1+k+\epsilon}}$, then theorem 3.3 in [10] gives $(M^+)^{k+3}$ instead of $(M^+)^{k+2}$ in the previous inequality for $S_b^{+,k}$.

Remark 4.3. In particular, we have that for $1 < p < \infty$ and $w \in A_p^+$, $\mathcal{O}_b^{+,k}$ and $S_b^{+,k}$ are bounded in $L^p(\omega)$, which was proved for $S_b^{+,k}$ using a different approach in [12].

In [26] it was proved that S^+ is of weak type (1,1) with respect to w, iff $w \in A_1^+$. The commutator is more singular than the operator. A fact that is not apparent in the $L^p(w)$ norm but it makes a difference near $L^1(w)$.

Theorem 4.4. *Let $b \in BMO$, $w \in A_\infty^+$ and $k = 0, 1, 2....$ Then, there exists $C > 0$ such that*

$$\begin{aligned} w(\{x \in \mathbb{R} : S_b^{+,k} f(x) > \lambda\}) &\le w(\{x \in \mathbb{R} : \mathcal{O}_b^{+,k} f(x) > \lambda\}) \\ &\le C \int_{\mathbb{R}} \frac{|f(x)|}{\lambda} \log^+ \left(1 + \frac{|f(x)|}{\lambda}\right)^{k+1} M^- w(x)\, dx, \quad f \in L_c^\infty, \end{aligned}$$

whenever the left-hand side is finite.

Remark 4.5. If $w \in A_1^+$, we can put w instead of M^-w in the right hand side.

The following lemma will allow us to use induction in the proof of Theorem 4.1.

Lemma 4.6. *Let $0 < \delta < \gamma < 1$, $b \in BMO$ and $k \in \mathbb{N} \cup \{0\}$. Then there exists $C > 0$ such that for any locally integrable f,*

$$\begin{aligned} M_\delta^{+,\#} \left(\mathcal{O}_b^{+,k} f \right)(x) &\le C \sum_{j=0}^{k-1} M_\gamma^+ (\mathcal{O}_b^{+,j} f)(x) + C M^+_{L(1+\log^+ L)^{1+k}} f(x) \\ &\le C \sum_{j=0}^{k-1} M_\gamma^+ (\mathcal{O}_b^{+,j} f)(x) + C (M^+)^{k+2} f(x) \ a.e. \end{aligned}$$

Proof. The case $k = 0$ follows from inequality (3.1) in the proof of Theorem 1.1. Let us prove the case $k \geq 1$. Let λ be an arbitrary constant. Then, $b(x) - b(y) = (b(x) - \lambda) - (b(y) - \lambda)$ and

$$\begin{aligned}
\mathcal{O}_b^{+,k} f(x) &= \left\| \int_{\mathbb{R}} (b(x) - b(y))^k K(x-y) f(y) dy \right\|_E \\
&= \left\| \sum_{j=0}^{k} C_{j,k} (b(x) - \lambda)^j \int_{\mathbb{R}} (b(y) - \lambda)^{k-j} K(x-y) f(y) dy \right\|_E \\
&\leq \left\| \int_{\mathbb{R}} (b(y) - \lambda)^k K(x-y) f(y) dy \right\|_E \\
+ \left\| \sum_{j=1}^{k} C_{j,k} \right. & \left. (b(x) - \lambda)^j \int_{\mathbb{R}} (b(y) - \lambda)^{k-j} K(x-y) f(y) dy \right\|_E \\
&= \mathcal{O}^+((b-\lambda)^k f)(x) \\
+ \left\| \sum_{j=1}^{k} \sum_{s=0}^{k-j} \right. & \left. C_{j,k,s} (b(x) - \lambda)^{s+j} \int_{\mathbb{R}} (b(x) - b(y))^{k-j-s} K(x-y) f(y) dy \right\|_E \\
&\leq \mathcal{O}^+((b-\lambda)^k f)(x) + \sum_{m=0}^{k-1} C_{k,m} |b(x) - \lambda|^{k-m} \mathcal{O}_b^{+,m} f(x) \,, \qquad (4.1)
\end{aligned}$$

where $C_{j,k}$ (respectively $C_{j,k,s}$) are absolute constants depending only on j and k (respectively j, k and s). Let $x \in \mathbb{R}$ and $h > 0$. Let $i \in \mathbb{Z}$ be such that $2^i \leq h < 2^{i+1}$ and set $J = [x, x + 2^{i+3})$. Then, write $f = f_1 + f_2$, where $f_1 = f\chi_J$ and set $\lambda = b_J$. Then, for any $a \in \mathbb{R}$ we have

$$\begin{aligned}
&\left(\frac{1}{h} \int_x^{x+2h} \left| (\mathcal{O}_b^{+,k} f(y))^\delta - |a|^\delta \right| dy \right)^{\frac{1}{\delta}} \leq \left(\frac{1}{h} \int_x^{x+2h} \left| (\mathcal{O}_b^{+,k} f(y)) - a \right|^\delta dy \right)^{\frac{1}{\delta}} \\
&\leq C \left[\sum_{m=0}^{k-1} \left(\frac{1}{h} \int_x^{x+2h} |b(y) - b_J|^{(k-m)\delta} (\mathcal{O}_b^{+,m} f(y))^\delta dy \right)^{\frac{1}{\delta}} \right. \\
&\qquad + \left(\frac{1}{h} \int_x^{x+2h} |\mathcal{O}^+((b - b_J)^k f_1)(y)|^\delta dy \right)^{\frac{1}{\delta}} \\
&\qquad\qquad \left. + \left(\frac{1}{h} \int_x^{x+2h} |\mathcal{O}^+((b - b_J)^k f_2)(y) - a|^\delta dy \right)^{\frac{1}{\delta}} \right] \\
&= (I) + (II) + (III). \qquad (4.2)
\end{aligned}$$

Let us estimate (I). Since $0 < \delta < \gamma < 1$, we can choose q such that $1 < q < \frac{\gamma}{\delta}$. Then, using Hölder's and John-Nirenberg's inequalities, we get

$$
\begin{aligned}
(I) &\leq C\sum_{m=0}^{k-1}\left(\frac{1}{h}\int_x^{x+2h}|b(y)-b_J|^{(k-m)\delta q'}dy\right)^{\frac{1}{\delta q'}}\times\\
&\times\left(\frac{1}{h}\int_x^{x+2h}|\mathcal{O}_b^{+,m}f(y)|^{\delta q}dy\right)^{\frac{1}{\delta q}}\\
&\leq C\sum_{m=0}^{k-1}\left[\left(\frac{1}{h}\int_x^{x+8h}|b(y)-b_J|^{(k-m)\delta q'}dy\right)^{\frac{1}{\delta q'(k-m)}}\right]^{k-m}\times\\
&\times\left(\frac{1}{h}\int_x^{x+2h}|\mathcal{O}_b^{+,m}f(y)|^{\delta q}dy\right)^{\frac{1}{\delta q}}\\
&\leq C\sum_{m=0}^{k-1}M_{\delta q}^{+}(\mathcal{O}_b^{+,m}f)(x)\leq C\sum_{m=0}^{k-1}M_{\gamma}^{+}(\mathcal{O}_b^{+,m}f)(x). \qquad (4.3)
\end{aligned}
$$

Kolmogorov's inequality plus the fact that $\mathcal{O}^+$ is of weak type $(1,1)$ with respect to the Lebesgue measure imply

$$(II)\leq C\frac{1}{h}\int_x^{x+2^{i+3}}|b(y)-b_J|^k|f(y)|dy.$$

Using now the generalized Hölder's inequality with $B_{k+1}(t)=e^{t^{1/(k+1)}}-1$ and $\overline{B_{k+1}}(t)=t(1+\log^+t)^{k+1}$ we get,

$$(II)\leq C|||b-b_J|^k||_{B_{k+1},J}||f||_{\overline{B_{k+1}},J}.$$

It follows from John-Nirenberg's inequality that

$$
\begin{aligned}
(II) &\leq C||b-b_J||_{B_1,J}^{k+1}||f||_{\overline{B_{k+1}},J}\leq C||b||_{BMO}^{k+1}M_{\overline{B_{k+1}}}^{+}f(x)\\
&\leq C(M^+)^{k+2}f(x). \qquad (4.4)
\end{aligned}
$$

For (III) we take $a=\mathcal{O}^+((b-b_J)^kf_2)(x)$. Then, by Jensen's inequality,

$$
\begin{aligned}
(III) &\leq C\frac{1}{2^i}\int_x^{x+2^{i+3}}|\mathcal{O}^+((b-b_J)^kf_2)(y)-\mathcal{O}^+((b-b_J)^kf_2)(x)|\,dy\\
&\leq C\frac{1}{2^i}\int_x^{x+2^{i+3}}||V((b-b_J)^kf_2)(y)-V((b-b_J)^kf_2)(x)||_E\,dy. \qquad (4.5)
\end{aligned}
$$

For $j\geq 3$, let $I_j=[x+2^j,x+2^{j+1})$ and $\tilde{I}_j=[x,x+2^{j+1})$. As in inequality (3.6) we have

$$
\begin{aligned}
&||V((b-b_J)^kf_2)(y)-V((b-b_J)^kf_2)(x)||_E\\
&\leq\left|\left|\left\{\left(\frac{1}{2^n}\int_y^{y+2^n}(b-b_J)^kf_2-\frac{1}{2^n}\int_x^{x+2^n}(b-b_J)^kf_2\right)\chi_{J_n}(s)\right\}_{n\in\mathbb{Z},s\in\mathbb{R}}\right|\right|_E\\
&\quad+\left|\left|\left\{\left(\frac{1}{s}\int_x^{x+s}(b-b_J)^kf_2-\frac{1}{s}\int_y^{y+s}(b-b_J)^kf_2\right)\chi_{J_n}(s)\right\}_{n\in\mathbb{Z},s\in\mathbb{R}}\right|\right|_E\\
&=(III_n)+(III_s). \qquad (4.6)
\end{aligned}
$$

For (III_n), we proceed as in the estimate of (III) in Theorem 1.1. Since $y \in (x, x + 2^{i+2})$ and f_2 has support in $(x + 2^{i+3}, \infty)$, it follows that, if $n \leq i+2$, then $x + 2^n \leq x + 2^{i+2}$ and $y + 2^n \leq x + 2^{i+2} + 2^n \leq x + 2^{i+2} + 2^{i+2} = x + 2^{i+3}$. As a consequence, we only have to take into account $n > i+2$. Therefore

$$\begin{aligned}
(III_n) &= \left(\sum_{n=i+3}^{\infty} \left| \frac{1}{2^n} \int_{x+2^n}^{y+2^n} f(b-b_J)^k \right|^2 \right)^{1/2} \\
&\leq C \left(\sum_{n=i+3}^{\infty} \left| \frac{1}{2^n} \int_{x+2^n}^{y+2^n} f(b-b_{I_n})^k \right|^2 \right)^{1/2} \\
&\qquad + C \left(\sum_{n=i+3}^{\infty} \left| \frac{1}{2^n} \int_{x+2^n}^{y+2^n} f(b_{I_n}-b_J)^k \right|^2 \right)^{1/2} \\
&= C \left(\sum_{n=i+3}^{\infty} |(IV_n)|^2 \right)^{1/2} + C \left(\sum_{n=i+3}^{\infty} |(V_n)|^2 \right)^{1/2}. \qquad (4.7)
\end{aligned}$$

Using the generalized Hölder's inequality (2.1) with $A = B_1$, $B = \overline{B_{k+1}}$ and $C = \overline{B_k}$, followed by John-Nirenberg's inequality we get

$$\begin{aligned}
(IV_n) &\leq C \frac{\sqrt{2}}{2^n} \int_{I_n} |b(t) - b_{I_n}|^k |f(t)| \chi_{(x+2^n, y+2^n)}(t)\, dt \\
&\leq C ||(b-b_{I_n})^k||_{B_k, \tilde{I_n}} ||f \chi_{(x+2^n, y+2^n)}||_{\overline{B_k}, \tilde{I_n}} \\
&\leq C ||b||_{BMO}^k ||f||_{\overline{B_{k+1}}, \tilde{I_n}} ||\chi_{(x+2^n, y+2^n)}||_{B_1, \tilde{I_n}} \\
&\leq C M^+_{\overline{B_{k+1}}} f(x) \frac{1}{B_1^{-1}(2^{n-i-2})}. \qquad (4.8)
\end{aligned}$$

For (V_n) again the generalized Hölder's inequality is used to obtain

$$\begin{aligned}
(V_n) &\leq C(n-i-1)^k ||f||_{\overline{B_{k+1}}, \tilde{I_n}} ||\chi_{(x+2^n, y+2^n)}||_{B_{k+1}, \tilde{I_n}} \\
&\leq C(n-i-1)^k M^+_{\overline{B_{k+1}}} f(x) \frac{1}{B_{k+1}^{-1}(2^{n-i-2})}. \qquad (4.9)
\end{aligned}$$

Putting together inequalities (4.8) and (4.9) we get

$$\begin{aligned}
(III_n) &\leq C M^+_{\overline{B_{k+1}}} f(x) \left(\sum_{n \geq i+3} \frac{1}{(B_1^{-1}(2^{n-i-2}))^2} \right)^{1/2} \\
&\qquad + C M^+_{\overline{B_{k+1}}} f(x) \left(\sum_{n \geq i+3} (n-i-1)^{2k} \frac{1}{(B_{k+1}^{-1}(2^{n-i-2}))^2} \right)^{1/2} \\
&\leq C M^+_{\overline{B_{k+1}}} f(x) \leq C (M^+)^{k+2} f(x). \qquad (4.10)
\end{aligned}$$

Let us estimate (III_s). As in Theorem 1.1, for $n \in \mathbb{Z}$, set

$$\beta_n = \sup_{s \in J_n} \left| \frac{1}{s} \int_x^{x+s} (b-b_J)^k f_2 - \frac{1}{s} \int_y^{y+s} (b-b_J)^k f_2 \right|.$$

Then, if $\beta_n \neq 0$ there exists $s_n \in J_n$, such that

$$\left| \frac{1}{s_n} \int_x^{x+s_n} (b-b_J)^k f_2 - \frac{1}{s_n} \int_y^{y+s_n} (b-b_J)^k f_2 \right| > \frac{1}{2}\beta_n.$$

If $n \leq i+1$ then $y+s_n \leq y+2^{n+1} \leq x+2^{i+2}+2^{n+1} \leq x+2^{i+3}$. Therefore we only have to consider $n \geq i+2$ in the estimate of III_s. Then

$$\begin{aligned}
\beta_n &\leq C\frac{1}{s_n}\int_{x+s_n}^{y+s_n} |(b-b_J)^k f_2| \\
&\leq C\frac{2^{n+2}}{s_n}\frac{1}{2^{n+2}}\int_x^{x+2^{n+2}} |(b(t)-b_J)^k f(t)\chi_{[x+s_n,y+s_n)}(t)|dt \\
&\leq C\frac{1}{2^{n+2}}\int_{\tilde{I}_{n+1}} |(b(t)-b_{I_{n+1}})^k f(t)\chi_{[x+s_n,y+s_n)}(t)|dt \\
&\quad + C\frac{1}{2^{n+2}}\int_{\tilde{I}_{n+1}} |(b_{I_{n+1}}-b_J)^k f(t)\chi_{[x+s_n,y+s_n)}(t)|dt.
\end{aligned}$$

By the generalized Hölder's inequality (2.1) with the Young functions used in (4.8) and (4.9), we get

$$\begin{aligned}
\beta_n &\leq C||(b-b_{I_{n+1}})^k||_{B_k,\tilde{I}_{n+1}}||f||_{\overline{B_{k+1}},\tilde{I}_{n+1}}||\chi_{(x+s_n,y+s_n)}||_{B_1,\tilde{I}_{n+1}} \\
&\quad + C(n-i)^k||f||_{\overline{B_{k+1}},\tilde{I}_{n+1}}||\chi_{(x+s_n,y+s_n)}||_{B_{k+1},\tilde{I}_{n+1}} \\
&\leq CM^+_{\overline{B_{k+1}}}f(x)\left(\frac{1}{B_1^{-1}(\frac{2^{n+2}}{y-x})} + (n-i)^k\frac{1}{B_{k+1}^{-1}(\frac{2^{n+2}}{y-x})}\right) \\
&\leq CM^+_{\overline{B_{k+1}}}f(x)\left(\frac{1}{B_1^{-1}(2^{n-i})} + \frac{(n-i)^k}{B_{k+1}^{-1}(2^{n-i})}\right).
\end{aligned}$$

Then,

$$\begin{aligned}
(III_s) &\leq CM^+_{\overline{B_{k+1}}}f(x)\left[\left(\sum_{n=i+2}^{\infty}\left(\frac{1}{B_1^{-1}(2^{n-i})}\right)^2\right)^{1/2} + \left(\sum_{n=i+2}^{\infty}\left(\frac{(n-i)^k}{B_{k+1}^{-1}(2^{n-i})}\right)^2\right)^{1/2}\right] \\
&\leq CM^+_{\overline{B_{k+1}}}f(x) \leq C(M^+)^{k+2}f(x). \qquad (4.11)
\end{aligned}$$

Collecting now inequalities (4.2)–(4.6), (4.10) and (4.11) we finish the proof of Lemma 4.6. □

Proof of Theorem 4.1. Let us observe that from the definition of $||\cdot||_E$, it follows that $S_b^{+,k}f \leq \mathcal{O}_b^{+,k}f$, therefore the first inequality in Theorem 4.1 holds trivially. For the second one, we will proceed by induction on k. The case $k=0$ is Theorem 1.1. Let now $k \in \mathbb{N}$ and suppose that Theorem 4.1 holds for $j=1,...,k-1$. In order to prove the case $j=k$ we proceed as in (3.12): since $w \in A_\infty^+$, there exists $r>1$, such that $w \in A_r^+$. Then, for δ and γ small enough, $0<\delta<\gamma<1$, we get that $r<p/\gamma<p/\delta$

and thus, $w \in A^+_{p/\gamma} \subset A^+_{p/\delta}$. Then by theorem 4 in [18] and Lemma 4.6 we have

$$\begin{aligned} ||\mathcal{O}_b^{+,k} f||_{L^p(w)} &\le ||M_\delta^+(\mathcal{O}_b^{+,k} f)||_{L^p(w)} \le C||M_\delta^{+,\#}(\mathcal{O}_b^{+,k} f)||_{L^p(w)} \\ &\le C\sum_{j=0}^{k-1} ||M_\gamma^+(\mathcal{O}_b^{+,j} f)||_{L^p(w)} + C||(M^+)^{k+2} f||_{L^p(w)}. \end{aligned} \tag{4.12}$$

Since $w \in A^+_{p/\gamma}$ we obtain

$$||M_\gamma^+(\mathcal{O}_b^{+,j} f)||_{L^p(w)} = \int_{\mathbb{R}} \left(M^+(\mathcal{O}_b^{+,j} f)^\gamma\right)^{p/\gamma} w \le C||\mathcal{O}_b^{+,j} f||_{L^p(w)}.$$

Then, by recurrence, we can continue the chain of inequalities in (4.12) by

$$\le C\sum_{j=0}^{k-1} ||(M^+)^{j+2} f||_{L^p(w)} + C||(M^+)^{k+2} f||_{L^p(w)} \le C||(M^+)^{k+2} f||_{L^p(w)}.$$

□

Proof of Theorem 4.4. To prove this theorem we shall use the following results:

(i) For any weight w, we have that

$$w(\{x \in \mathbb{R} : (M^+)^{k+2} f(x) > \lambda\}) \le C \int_{\mathbb{R}} \frac{|f|}{\lambda} \log^+ \left(1 + \frac{|f|}{\lambda}\right)^{k+1} M^- w.$$

(ii) Let $1 < p_0 < \infty$ and $\mathcal{F}$ be a family of couples of non-negative functions such that, for $w \in A^+_\infty$,

$$\int_{\mathbb{R}} f(x)^{p_0} w(x)dx \le C \int_{\mathbb{R}} g(x)^{p_0} w(x)dx, \tag{4.13}$$

for all $(f, g) \in \mathcal{F}$. Let $\phi \in \Delta_2$ and such that there exist some exponents $0 < r_0 < s_0 < \infty$, such that $\phi(t^{r_0})^{s_0}$ is quasi convex. Then, for all $w \in A^+_\infty$,

$$\sup_{\lambda>0} \phi(\lambda) w(\{x \in \mathbb{R} : f(x) > \lambda\}) \le C \sup_{\lambda>0} \phi(\lambda) w(\{x \in \mathbb{R} : g(x) > \lambda\})$$

for all $(f, g) \in \mathcal{F}$, such that the left hand side is finite.

Result (i) is a direct consequence of theorem 3 in [22], since the pair $(w, M^- w) \in A^+_1$.

The proof of (ii) follows exactly as in theorem 3.1 in [7], then we omit it.

The Coifman type estimate in Theorem 4.1 gives inequality (4.13) for the family of functions $(\mathcal{O}_b^{+,k} f, (M^+)^{k+2} f)$ $(k \ge 0)$. Also observe that $\phi(t) = \frac{1}{\overline{B_{k+1}}(1/t)}$, where $\overline{B_{k+1}}(t) = t(1 + \log^+ t)^{k+1}$, belongs to Δ_2 and $\phi(t^r)$, is quasi convex for $r > 1$ large enough.

In order to prove Theorem 4.4, it suffices to consider $\lambda = 1$ (the general case follows by applying the result to the function f/λ). We may also asume $||b||_{\mathrm{BMO}} = 1$. Then by (ii) and (i),

$$\begin{aligned}
w(\{x\in\mathbb{R}:\mathcal{O}_b^{+,k}f(x)>1\}) &\leq \sup_{t>0}\phi(t)w(\{x\in\mathbb{R}:\mathcal{O}_b^{+,k}f(x)>t\})\\
&\leq C\sup_{t>0}\phi(t)w(\{x\in\mathbb{R}:(M^+)^{k+2}f(x)>t\})\\
&\leq C\sup_{t>0}\phi(t)\int_{\mathbb{R}}\overline{B_{k+1}}\left(\frac{|f|}{t}\right)M^-w\\
&\leq C\sup_{t>0}\phi(t)\overline{B_{k+1}}\left(\frac{1}{t}\right)\int_{\mathbb{R}}\overline{B_{k+1}}\left(|f|\right)M^-w\\
&\leq C\int_{\mathbb{R}}\overline{B_{k+1}}\left(|f|\right)M^-w.
\end{aligned}$$

□

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