J. Math. Anal. Appl. (2008), doi:10.1016/j.jmaa.2008.01.003

GENERALIZED HÖRMANDER'S CONDITIONS, COMMUTATORS AND WEIGHTS

MARÍA LORENTE, JOSÉ MARÍA MARTELL, MARÍA SILVINA RIVEROS, AND ALBERTO DE LA TORRE

Abstract. We present a general framework to deal with commutators of singular integral operators with BMO functions. Hörmander type conditions associated with Young functions are assumed on the kernels. Coifman type estimates, weighted norm inequalities and two-weight estimates are considered. We give applications to homogeneous singular integrals, Fourier multipliers and one-sided operators.

1. Introduction

In 1972, R. Coifman established in [4] that a singular integral operator T with regular kernel (that is, $K \in H^*_\infty$, see the definition below) is controlled by the Hardy-Littlewood maximal function M and for every $0 < p < \infty$ and every Muckenhoupt weight $w \in A_\infty$,

$$\int_{\mathbb{R}^n} |Tf(x)|^p\, w(x)\, dx \le C \int_{\mathbb{R}^n} Mf(x)^p\, w(x)\, dx. \tag{1.1}$$

There have been many attempts of controlling a given singular integral operator by an appropriate maximal function (see [6], [5] and the references therein). In [10] (see also [25] and [29]) singular integral operators with less regular kernels are considered. Implicit in their proofs it is shown that the operators in question are controlled, in the sense of (1.1), by a maximal operator $M_r f(x) = M(|f|^r)(x)^{1/r}$ for some $1 \le r < \infty$. The value of the exponent r is determined by the smoothness of the kernel, namely, the kernel satisfies an $L^{r'}$-Hörmander condition (see the precise definition bellow). Let us point out that in [13] it has been proved that this control is sharp in the sense that one cannot write a pointwise smaller operator M_s with $s < r$. This yields, in particular, that (1.1) do not hold in general with M_r for any $1 \le r < \infty$ for singular integral operators satisfying only the classical Hörmander condition H_1.

An interesting consequence of (1.1) is the following: combining this estimate and some sharp two-weight norm inequalities for the Hardy-Littlewood maximal function

---

Date: July 16, 2007. *Revised*: December 20, 2007.

2000 *Mathematics Subject Classification.* 42B20, 42B25.

Key words and phrases. Calderón-Zygmund operators, homogeneous singular integrals, multipliers, one-sided operators, commutators, BMO, Hörmander's condition of Young type, Muckenhoupt weights, two-weight estimates, vector-valued inequalities.

The first, third and last authors are partially supported by MEC Grant MTM2005-08350-C03-02. The first and the last authors are also supported by Junta de Andalucía Grant FQM354. The second author is partially supported by MEC "Programa Ramón y Cajal, 2005", by MEC Grant MTM2007-60952, and by UAM-CM Grant CCG06-UAM/ESP-0286. The third author is partially supported by CONICET and SECYT-UNC.

(see [21]) one gets the sharp weighted estimate

$$\int_{\mathbb{R}^n} |Tf(x)|^p\, u(x)\, dx \le C \int_{\mathbb{R}^n} |f(x)|^p\, M^{[p]+1}u(x)\, dx, \tag{1.2}$$

for all $1 < p < \infty$ with no assumption on u, where $[p]$ stands for the integer part of p and M^k is the Hardy-Littlewood maximal operator iterated k-times. This was proved in [18] generalizing some partial result (by a different method) in [30].

Estimates like (1.1) also hold for the commutator of a singular integral operator with regular kernel T with a function of bounded mean oscillation, $b \in \mathrm{BMO}$, that is,

$$\sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q|\, dx < \infty,$$

where the sup runs over all cubes $Q \subset \mathbb{R}^n$ with the sides parallel to the coordinate axes and where b_Q stands for the average of b over Q. We define the (first-order) commutator by

$$T_b^1 f(x) = [b, T]f(x) = b(x)\, Tf(x) - T(b\, f)(x).$$

In [19] it was shown that for all $0 < p < \infty$ and $w \in A_\infty$

$$\int_{\mathbb{R}^n} |T_b^1 f(x)|^p\, w(x)\, dx \le C \int_{\mathbb{R}^n} M^2 f(x)^p\, w(x)\, dx. \tag{1.3}$$

It was also proved in [19] that this yields the following two-weight norm inequality: for $1 < p < \infty$ and with no assumption on u,

$$\int_{\mathbb{R}^n} |T_b^1 f(x)|^p\, u(x)\, dx \le C \int_{\mathbb{R}^n} |f(x)|^p\, M^{[2\,p]+1}u(x)\, dx. \tag{1.4}$$

Similar results were proved for the higher order commutators T_b^k defined by induction as $T_b^k = [b, T_b^{k-1}]$ for $k \ge 2$. In this case (1.3) holds with M^{k+1} in place M^2 and in (1.4) the right hand side weight is $M^{[p\,(k+1)]+1}u$.

Let us mention that (1.3) suggests that T_b^1 is more singular than T as T_b^1 is controlled by M^2 that is pointwise bigger than M. Observe that under this point of view T_b^k becomes more singular as k grows.

Estimates like (1.1) appear throughout the literature. In some cases these are implied by a good-λ inequality between T and M. Typically (as it has been explained above) T is a singular integral operator and M is a maximal operator. This turns out to be very useful since one can prove weighted estimates for T by using those satisfied by M, which are in general easier to prove. This has been extensively used in [6], [5] where it is shown that starting with (1.1), with some fixed exponent $0 < p_0 < \infty$, for any pair of operators T and M (indeed, pairs of functions can be written in place of the operators) one can extrapolate and get that the same estimate holds on $L^p(w)$ for all $0 < p < \infty$, $w \in A_\infty$. Further, one can replace the Lebesgue spaces by very general weighted Orlicz spaces and weighted rearrangement invariant quasi-Banach spaces (with some minor hypotheses). This general theory also provides modular extensions of (1.1) —that is, $\phi(|Tf|)$ controlled by $\phi(Mf)$ in $L^1(w)$— with some mild restrictions on the functions ϕ. Moreover, all these estimates hold in a vector-valued sense with no extra work. All this is done with no need to appeal to good-λ inequalities of any kind and roughly speaking implies that T and M behave the same way (provided one is not "close" to L^∞, this is clear in the case of Coifman since the Hardy-Littlewood maximal function is bounded on L^∞ and T may not be).

Taking all this into account, it would be of interest to seek for maximal functions that control different types of singular integral operators in the sense of (1.1). As mentioned above, in [10] (also [25], [29]) weighted norm inequalities were shown for singular integral operators satisfying smoothness conditions in the scale of Lebesgue spaces. In these references the Coifman type estimates proved were not a primary aim and they were used as tools to derive the weighted norm inequalities satisfied by T. Motivated by the one-sided discrete square function considered in [28], in [12] further extensions of the aforementioned results were proved. This vector-valued operator has a kernel that satisfies all the L^r-Hörmander conditions with $1 \le r < \infty$ but the one corresponding to L^∞. Thus, using the techniques in [25] one can prove that this operator can be controlled by M_r for any $r > 1$ and the case $r = 1$ remains open. There are however many maximal operators that lie between M and M_r with $r > 1$: for instance M^2 or more in general M^k; maximal operators associated with Orlicz spaces as $L(\log L)^\alpha$, $L(\log L)^\alpha\,(\log\log L)^\beta$. One may wonder whether one of these maximal functions controls the square function.

Given a Young function $\mathcal{A}$, which gives an Orlicz space $L^{\mathcal{A}}$, in [12] it was introduced the $L^{\mathcal{A}}$-Hörmander condition to show that a singular integral operator whose kernel satisfies such condition is controlled as in (1.1) by $M_{\overline{\mathcal{A}}}$, which is the Hardy-Littlewood maximal function associated with the space $L^{\overline{\mathcal{A}}}$, where $\overline{\mathcal{A}}$ is the conjugate function of $\mathcal{A}$ (see the precise definition below). In [12], this was used to deal with the previous square function showing that it is controlled by the Hardy-Littlewood maximal function associated with the space $L(\log L)^{1+\varepsilon}$, for every $\varepsilon > 0$, which can be controlled by M^3 —let us mention that the result proved there is better since one can use the pointwise smaller operator $(M^+)^3$ where M^+ is the one-sided Hardy-Littlewood maximal function corresponding to the intervals of the form $(x, x + h)$.

The aim of the present paper is to prove Coifman type estimates for commutators of singular integral operators with bounded mean oscillation functions, where different conditions are assumed in the kernel of the operators. We also obtain new weighted norm inequalities for the classical operators, and their corresponding commutators, considered in [10] (see also [29]), namely, for Fourier multipliers and also for homogeneous singular integral operators. We will also show that the techniques developed can be extended to improve the results in the case of one-sided singular integrals and commutators. As a consequence we will also obtain weighted modular end-point estimates, two-weight inequalities and vector-valued estimates for the operators in question.

The plan of the paper is as follows. Next section contains some preliminaries that are needed to state our main definitions and results which are in Section 3. In Theorem 3.3, assuming different Hörmander type conditions on the kernels of the operators in question, we establish Coifman type estimates. As a consequence, vector-valued inequalities and estimates with one and two weights are derived (see Sections 3.2 and 3.3). The technical conditions imposed on the kernel will become clear in the applications presented in Section 4: we obtain weighted norm inequalities for homogeneous singular integrals, Fourier multipliers and also one-sided singular integrals that fit within this theory. The proofs of the general results are in Section 5 and the proofs related to the applications are in Section 6. Finally in Section 7 we will discuss further extensions of the techniques developed on which we consider multilinear commutators as in [22].

2. Preliminaries

Throughout this paper T will denote a singular integral operator of convolution type, that is, T is bounded on $L^2(\mathbb{R}^n)$ and

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y)\, f(y)\, dy$$

with K a measurable function defined away from 0. We are taking convolution operators for simplicity, the results presented in this paper also hold for variable kernels with the appropriate changes. The precise statements and the details are left to the reader.

When $n = 1$ and we further assumed that the kernel K is supported on $(-\infty, 0)$ we say that T is a one-sided singular integral and we write T^+ to emphasize it. The results that we present below for (standard) singular integrals apply to T^+. However, taking advantage of the extra assumption on the kernel, one can be more precise and get better estimates (see Remark 3.4 below).

We are going to consider commutators of these operators with BMO functions. Let us recall that a locally integrable functions b belongs to BMO if

$$\|b\|_{\mathrm{BMO}} = \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q|\, dx < \infty,$$

where the sup runs over all cubes $Q \subset \mathbb{R}^n$ with the sides parallel to the coordinate axes and where b_Q stands for the average of b over Q.

Given T and $b \in \mathrm{BMO}$ we define the k-th order commutator, $k \geq 0$, by

$$T_b^k f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^k\, K(x-y)\, f(y)\, dy.$$

Note that for $k = 0$, we have $T_b^k = T$ and observe that $T_b^k = [b, T_b^{k-1}]$, $k \geq 1$.

We consider weights in the Muckenhoupt classes A_p, $1 \leq p \leq \infty$, which are defined as follows. Let w be a non-negative locally integrable function and $1 \leq p < \infty$. We say that $w \in A_p$ if there exists $C_p < \infty$ such that for every ball $B \subset \mathbb{R}^n$

$$\left(\frac{1}{|B|} \int_B w(x)\, dx\right) \left(\frac{1}{|B|} \int_B w(x)^{1-p'}\, dx\right)^{p-1} \leq C_p.$$

when $1 < p < \infty$, and for $p = 1$,

$$\frac{1}{|B|} \int_B w(y)\, dy \leq C_1\, w(x), \qquad \text{for a.e. } x \in B,$$

which can be equivalently written as $Mw(x) \leq C_1\, w(x)$ for a.e. $x \in \mathbb{R}^n$. Finally we set $A_\infty = \cup_{p \geq 1} A_p$. It is well known that the Muckenhoupt classes characterize the boundedness of the Hardy-Littlewood maximal function on weighted Lebesgue spaces. Namely, $w \in A_p$, $1 < p < \infty$, if and only if M is bounded on $L^p(w)$; and $w \in A_1$ if and only if M maps $L^1(w)$ into $L^{1,\infty}(w)$.

2.1. **One-sided theory.** In $\mathbb{R}$, the one-sided Hardy-Littlewood maximal operators M^+ and M^- are defined for locally integrable functions f by

$$M^+f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(y)|\,dy \qquad \text{and} \qquad M^-f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(y)|\,dy.$$

The classes A_p^+, $1 < p < \infty$, were introduced by E. Sawyer [27] in the study of the weights for these operators proving that M^+ maps $L^p(w)$ into $L^p(w)$, if and only if, $w \in A_p^+$, that is, there exists a constant $C_p < \infty$ such that for all $a < b < c$

$$\frac{1}{(c-a)^p} \left(\int_a^b w(x)\,dx\right) \left(\int_b^c w(x)^{1-p'}\,dx\right)^{p-1} \le C_p.$$

The case $p = 1$ was not considered in Sawyer's paper but it was proved in [14] that M maps $L^1(w)$ into $L^{1,\infty}(w)$ if and only if $w \in A_1^+$, that is, $M^-w(x) \le C_1\, w(x)$. The class A_∞^+ is defined as the union of all the A_p^+ classes, $A_\infty^+ = \cup_{p\ge 1} A_p^+$. The classes A_p^- are defined in a similar way. It is interesting to note that $A_p = A_p^+ \cap A_p^-$, $A_p \subsetneq A_p^+$ and $A_p \subsetneq A_p^-$. (See [27], [14], [15], [16] for more definitions and results.)

2.2. **Young functions and Orlicz spaces.** Let us recall some of the needed background for Orlicz spaces. The reader is referred to [23] and [2] for a complete account of this topic. A function $\mathcal{A} : [0,\infty) \longrightarrow [0,\infty)$ is a Young function if it is continuous, convex, increasing and satisfies $\mathcal{A}(0) = 0$, $\mathcal{A}(\infty) = \infty$. We assume that the Young functions are normalized so that $\mathcal{A}(1) = 1$. The Orlicz space $L^{\mathcal{A}}$ is defined by the Luxemburg norm

$$\|f\|_{\mathcal{A}} = \inf\left\{\lambda > 0 : \int \mathcal{A}\left(\frac{|f(x)|}{\lambda}\right)\,dx \le 1\right\}.$$

We also define an averaged version of $\|\cdot\|_{\mathcal{A}}$ in the following way: given a ball B

$$\|f\|_{\mathcal{A},B} = \inf\left\{\lambda > 0 : \frac{1}{|B|}\int_B \mathcal{A}\left(\frac{|f(x)|}{\lambda}\right)\,dx \le 1\right\}.$$

For instance, when $\mathcal{A}(t) = t^r$ with $r \ge 1$ we have

$$\|f\|_{\mathcal{A},B} = \|f\|_{L^r(B,dx/|B|)} = \left(\frac{1}{|B|}\int_B |f(x)|^r\,dx\right)^{\frac{1}{r}}.$$

Let us observe that if $\mathcal{A}(t) \le C\,\mathcal{B}(t)$ for $t \ge t_0$ then

$$\frac{1}{|B|}\int_B \mathcal{A}\left(\frac{|f(x)|}{\lambda}\right)\,dx \le C + \frac{C}{|B|}\int_B \mathcal{B}\left(\frac{|f(x)|}{\lambda}\right)\,dx \tag{2.1}$$

and so $\|f\|_{\mathcal{A},B} \le C\,\|f\|_{\mathcal{B},B}$. Thus, we observe that the behavior of $\mathcal{A}(t)$ for $t \le t_0$ does not matter: if $\mathcal{A}(t) \approx \mathcal{B}(t)$ for $t \ge t_0$ the last estimate implies that $\|f\|_{\mathcal{A},B} \approx \|f\|_{\mathcal{B},B}$. This means that in most cases we will not be concerned about the value of the Young functions for t small.

Denoting by $\overline{\mathcal{A}}$ the complementary function associated to $\mathcal{A}$ one has the generalized Hölder inequality

$$\frac{1}{|B|}\int_B |f\,g| \le 2\,\|f\|_{\mathcal{A},B}\|g\|_{\overline{\mathcal{A}},B}.$$

There is a further generalization that turns out to be useful for our purposes, see [17]: If $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are Young functions such that $\mathcal{A}^{-1}(t)\,\mathcal{B}^{-1}(t)\,\mathcal{C}^{-1}(t) \le t$, for all $t \ge 1$, then

$$\|f\,g\,h\|_{L^1,B} \le C\,\|f\|_{\mathcal{A},B}\,\|g\|_{\mathcal{B},B}\,\|h\|_{\mathcal{C},B}. \tag{2.2}$$

Note that this implies

$$\|f\,g\|_{\overline{\mathcal{C}},B} \le C\,\|f\|_{\mathcal{A},B}\,\|g\|_{\mathcal{B},B} \qquad \text{and} \qquad \|f\|_{\overline{\mathcal{C}},B} \le C\,\|f\|_{\mathcal{A},B}. \tag{2.3}$$

The first estimate is obtained by duality and for the second one takes $g \equiv 1$.

Remark 2.1. Let us observe that when $\mathcal{D}(t) = t$, which gives L^1, then $\overline{\mathcal{D}}(t) = 0$ if $s \le 1$ and $\overline{\mathcal{D}}(t) = \infty$ otherwise. Note that $\overline{\mathcal{D}}$ is not a Young function but one has $L^{\overline{\mathcal{D}}} = L^\infty$. Besides, the (generalized) inverse is $\overline{\mathcal{D}}^{-1}(t) \equiv 1$ and the previous Hölder inequalities make sense if one of the three functions is $\overline{\mathcal{D}}$.

Remark 2.2. The convexity of $\mathcal{A}$ implies that $\mathcal{A}(t)/t$ is increasing and so $t \le \mathcal{A}(t)$ for all $t \ge 1$. This yields that $\|f\|_{L^1,B} \le C\,\|f\|_{\mathcal{A},B}$ for all Young functions $\mathcal{A}$.

We can now define the Hardy-Littlewood maximal function associated with $L^{\mathcal{A}}$ as

$$M_{\mathcal{A}} f(x) = \sup_{B \ni x} \|f\|_{\mathcal{A},B}.$$

Abusing on the notation if $\mathcal{A}(t) = t^r$, $\mathcal{A}(t) = e^{t^\alpha} - 1$ or $\mathcal{A}(t) = t^r (1 + \log^+ t)^\alpha$, the Orlicz norms are respectively written as $\|\cdot\|_r = \|\cdot\|_{L^r}$, $\|\cdot\|_{\exp L^\alpha}$, $\|\cdot\|_{L^r\,(\log L)^\alpha}$ and the corresponding maximal operators as $M_r = M_{L^r}$, $M_{\exp L^\alpha}$ and $M_{L^r\,(\log L)^\alpha}$.

For $k \ge 0$, it is known that $M_{L(\log L)^k} f(x) \approx M^{k+1} f(x)$ where M^k is the k-times iterated of M (see [20], [24] and [5]).

For $1 < p < \infty$, a Young function $\mathcal{A}$ is said to belong to B_p if there exists $c > 0$ such that

$$\int_c^\infty \frac{\mathcal{A}(t)}{t^p}\,\frac{dt}{t} < \infty.$$

This condition appears first in [21] and it was shown that $\mathcal{A} \in B_p$ if and only if $M_{\mathcal{A}}$ is bounded on $L^p(\mathbb{R}^n)$.

When $n = 1$, we can also define the one-sided maximal functions associated with a given Young function $\mathcal{A}$:

$$M^+_{\mathcal{A}} f(x) = \sup_{b > x} \|f\|_{\mathcal{A},(x,b)} \qquad \text{and} \qquad M^-_{\mathcal{A}} f(x) = \sup_{a < x} \|f\|_{\mathcal{A},(a,x)}.$$

3. Main results

Let T be a singular integral operator with kernel K. We assume different smoothness conditions on K. The weakest one is the so called Hörmander condition: we say that $K \in H_1$ (or that K satisfies the L^1-Hörmander condition) if there exist $c > 1$ and $C_1 > 0$ so that

$$\int_{|x| > c\,|y|} |K(x-y) - K(x)|\,dx \le C_1, \qquad y \in \mathbb{R}^n.$$

The classical Lipschitz condition is called H^*_∞ (this notation is not standard but we keep H_∞ for a weaker L^∞-condition, see the definition below). We say that $K \in H^*_\infty$ if there are $\alpha, C > 0$ and $c > 1$ such that

$$|K(x-y) - K(x)| \le C\,\frac{|y|^\alpha}{|x|^{\alpha+n}}, \qquad |x| > c\,|y|.$$

Clearly $H^*_\infty \subset H_1$ and, in between them, one can consider classes associated with L^r. Let us introduce some notation: $|x| \sim s$ means $s < |x| \le 2\,s$. Given a Young function $\mathcal{A}$ we write

$$\|f\|_{\mathcal{A},|x|\sim s} = \|f\,\chi_{\{|x|\sim s\}}\|_{\mathcal{A},B(0,2\,s)}.$$

The same is applied to the space L^∞.

Given $1 \le r \le \infty$ we say that $K \in H_r = H_{t^r}$ (or K satisfies the L^r-Hörmander condition) if there exist $c \ge 1$, $C_r > 0$ such that for any $y \in \mathbb{R}^n$ and $R > c\,|y|$

$$\sum_{m=1}^{\infty} (2^m R)^n \,\|K(\cdot - y) - K(\cdot)\|_{L^r,|x|\sim 2^m R} \le C_r.$$

Notice that H_1 coincides with the definition above an that one has

$$H^*_\infty \subset H_r \subset H_s \subset H_1, \qquad\qquad 1 < s < r < \infty.$$

These classes appeared implicit in the work [10] where it is shown that classical L^r-Dini condition for K implies $K \in H_r$ (see also [25] and [29]).

In [12] extensions of these classes were introduced replacing L^r by more general Orlicz spaces (see Section 2.2 for the precise definitions and the needed background): given a Young function $\mathcal{A}$, the kernel K is said to satisfy the $L^{\mathcal{A}}$-Hörmander condition (we write $K \in H_{\mathcal{A}}$), if there exist $c \ge 1$, $C_{\mathcal{A}} > 0$ such that for any $y \in \mathbb{R}^n$ and $R > c\,|y|$,

$$\sum_{m=1}^{\infty} (2^m R)^n \,\|K(\cdot - y) - K(\cdot)\|_{\mathcal{A},|x|\sim 2^m R} \le C_{\mathcal{A}}.$$

Note that if $\mathcal{A}(t) = t^r$ then $H_{\mathcal{A}} = H_r$. On the other hand, since $t \le \mathcal{A}(t)$ for $t \ge 1$ by convexity we have that $H_{\mathcal{A}} \subset H_1$ which implies that the classical Calderón-Zygmund theory applies to T. Thus, T is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ and T is also of weak-type $(1,1)$.

In [12] it was shown that a given singular integral operator, with kernel $K \in H_{\mathcal{A}}$, is controlled by $M_{\overline{\mathcal{A}}}$ improving the previous results in [10], [25] and [29]:

Theorem 3.1 ([12]). *Let $\mathcal{A}$ be a Young function and let T be a singular integral operator with kernel $K \in H_{\mathcal{A}}$. Then for any $0 < p < \infty$ and $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} |Tf(x)|^p\, w(x)\, dx \le C \int_{\mathbb{R}^n} M_{\overline{\mathcal{A}}} f(x)^p w(x)\, dx, \qquad f \in L^\infty_c$$

whenever the left-hand side is finite.

Similar results can be proved for vector valued operators or one-sided operators. (See [12].)

Next, we define new classes of kernels depending on a Young function $\mathcal{A}$ and some exponent $k \ge 0$, which will be related with the order of the commutator, —when $k = 0$, $H_{\mathcal{A},0}$ coincides with the class $H_{\mathcal{A}}$ introduced in [12]—:

Definition 3.2. *Let $\mathcal{A}$ be a Young function and $k \in \mathbb{N}$. We say that the kernel K satisfies the $L^{\mathcal{A},k}$-Hörmander condition (we write $K \in H_{\mathcal{A},k}$), if there exist $c \ge 1$ and $C > 0$ (depending on $\mathcal{A}$ and k) such that for all $y \in \mathbb{R}^n$ and $R > c\,|y|$*

$$\sum_{m=1}^{\infty} (2^m R)^n\, m^k\, \|K(\cdot - y) - K(\cdot)\|_{\mathcal{A},|x|\sim 2^m R} \le C.$$

We say that $K \in H_{\infty,k}$ if K satisfies the previous condition with $\|\cdot\|_{L^\infty,|x|\sim 2^m R}$ in place of $\|\cdot\|_{\mathcal{A},|x|\sim 2^m R}$

Let us mention that we have written our definition in terms of dyadic dilations but one can equivalently use a-adic annuli with $a > 1$.

The classes $H_{\mathcal{A},k}$ satisfy the following: for any Young function $\mathcal{A}$ and $k \geq 0$ we have

$$H_\infty^* \subset H_{\infty,k} \subset H_{\mathcal{A},k} \subset H_{\mathcal{A},k-1} \subset \cdots \subset H_{\mathcal{A},1} \subset H_1.$$

Also, if $\mathcal{A}(t) \leq C\,\mathcal{B}(t)$ for $t > t_0$ then

$$H_\infty^* \subset H_{\infty,k} \subset H_{\mathcal{B},k} \subset H_{\mathcal{A},k} \subset H_{1,k} \subset H_1.$$

In the particular case on which we consider the L^r-Hörmander conditions it follows that for $1 < r < s < \infty$

$$H_\infty^* \subset H_{\infty,k} \subset H_{s,k} \subset H_{r,k} \subset H_{1,k} \subset H_1.$$

All these properties follow easily and the proofs are left to the reader (see Remark 2.2 to obtain $H_{\mathcal{A},k} \subset H_{1,k}$).

Now we are ready to state our main results.

3.1. Coifman type estimates.

Theorem 3.3. *Let $b \in \mathrm{BMO}$ and $k \geq 0$.*

(a) *Let $\mathcal{A}$, $\mathcal{B}$ be Young functions, such that $\overline{\mathcal{A}}^{-1}(t)\,\mathcal{B}^{-1}(t)\,\overline{\mathcal{C}}_k^{-1}(t) \leq t$ with $\overline{\mathcal{C}}_k(t) = e^{t^{1/k}}$ for $t \geq 1$. If T is a singular integral operator with kernel $K \in H_{\mathcal{B}} \cap H_{\mathcal{A},k}$ (or, in particular, $K \in H_{\mathcal{B},k}$), then for any $0 < p < \infty$, $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} |T_b^k f(x)|^p\, w(x)\, dx \leq C\, \|b\|_{\mathrm{BMO}}^{p\,k} \int_{\mathbb{R}^n} M_{\overline{\mathcal{A}}} f(x)^p\, w(x)\, dx, \qquad f \in L_c^\infty, \tag{3.1}$$

whenever the left-hand side is finite. If one further assumes that $\overline{\mathcal{A}}$ is submultiplicative, then for all $w \in A_\infty$ and $\lambda > 0$,

$$w\big\{x \in \mathbb{R}^n : |T_b^k f(x)| > \lambda\big\} \leq C \int_{\mathbb{R}^n} \overline{\mathcal{A}}\left(\frac{\|b\|_{\mathrm{BMO}}^k\, |f(x)|}{\lambda}\right) Mw(x)\, dx. \tag{3.2}$$

(b) *If T is a singular integral operator with kernel $K \in H_\infty \cap H_{e^{t^{1/k}},k}$ (or, in particular, $K \in H_{\infty,k}$), then for any $0 < p < \infty$, $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} |T_b^k f(x)|^p\, w(x)\, dx \leq C\, \|b\|_{\mathrm{BMO}}^{p\,k} \int_{\mathbb{R}^n} M^{k+1} f(x)^p\, w(x)\, dx, \qquad f \in L_c^\infty \tag{3.3}$$

whenever the left-hand side is finite. As a consequence, for all $w \in A_\infty$ and $\lambda > 0$

$$w\big\{x \in \mathbb{R}^n : |T_b^k f(x)| > \lambda\big\} \leq C \int_{\mathbb{R}^n} \varphi_k\left(\frac{\|b\|_{\mathrm{BMO}}^k\, |f(x)|}{\lambda}\right) Mw(x)\, dx, \tag{3.4}$$

where $\varphi_k(t) = t\,(1 + \log^+ t)^k$

(c) *Let $\mathcal{A}$ and $\mathcal{B}$ be as in (a) and assume that T^+ is a one-sided singular integral operator with kernel K supported in $(-\infty, 0)$. If $K \in H_{\mathcal{B}} \cap H_{\mathcal{A},k}$ (or, in particular, $K \in H_{\mathcal{B},k}$), then for any $0 < p < \infty$, $w \in A_\infty^+$, it follows that (3.1) holds with $M_{\overline{\mathcal{A}}}^+ f$ in place of $M_{\overline{\mathcal{A}}} f$. If one further assumes that $\overline{\mathcal{A}}$ is submultiplicative, then for all $w \in A_\infty^+$, $T_b^{+,k}$ satisfies (3.2) with $M^- w$ in place of Mw.*

(*d*) *Let T^+ be a one-sided singular integral operator with kernel K whose support is contained in $(-\infty,0)$. If $K\in H_\infty\cap H_{e^{t^{1/k}},k}$ (or, in particular, $K\in H_{\infty,k}$), then for any $0<p<\infty$, $w\in A_\infty^+$, it follows that (3.3) holds with $(M^+)^{k+1}f$ in place of $M^{k+1}f$. As a consequence, for all $w\in A_\infty^+$, $T_b^{+,k}$ satisfies (3.4) with M^-w in place of Mw.*

Remark 3.4. We would like to emphasize that parts (*c*) and (*d*) improve respectively (*a*) and (*b*). Observe that one-sided operators are singular integral operators with the additional hypothesis that the kernels are supported in $(-\infty,0)$ so, in particular, we can apply (*a*) and (*b*) to them. In parts (*c*) and (*d*) we extend the class of weights ($A_\infty \subsetneq A_\infty^+$) and write pointwise smaller maximal operators in the right-hand side since $M_{\mathcal{B}}^+f(x)\le M_{\mathcal{B}}f(x)$ and $(M^+)^{k+1}f(x)\le M^{k+1}f(x)$.

Remark 3.5. Notice that in (*a*) and (*c*), if $K\in H_{\mathcal{B},k}$ then $K\in H_{\mathcal{B}}\cap H_{\mathcal{A},k}$. Indeed, $H_{\mathcal{B},k}\subset H_{\mathcal{B}}$ and also $H_{\mathcal{B},k}\subset H_{\mathcal{A},k}$ since (2.3) gives $\|h\|_{\mathcal{A},B}\le C\,\|h\|_{\mathcal{B},B}$. On the other hand, in (*b*) and (*d*) if $K\in H_{\infty,k}$ one obtains that $K\in H_\infty$ and also $K\in H_{e^{t^{1/k}},k}$ since $\|h\|_{e^{t^{1/k}},B}\le\|h\|_{L^\infty,B}$.

To understand the difference between these two conditions, we concentrate on (*b*) and take $K\in H_\infty$. If one is able to show that $K\in H_{\infty,k}$ then we get (3.3). Alternatively, the same estimate holds from the weaker but more-difficult-to-check condition $K\in H_{e^{t^{1/k}},k}$. It may happen that we just know that $K\in H_{\mathcal{A},k}$ for some Young function which can be worse than $e^{t^{1/k}}$. In this case, a careful examination of the proof would lead us to obtain (3.3) with $M_{\overline{\mathcal{A}}}f+M^{k+1}f$ in the righthand side . Notice that when $\mathcal{A}(t)\approx e^{t^{1/k}}$ both maximal operators are comparable. In general, $\mathcal{A}$ might be worse and then the maximal operator $M_{\overline{\mathcal{A}}}$ would be bigger than M^{k+1} (this means that $M_{\overline{\mathcal{A}}}$ is the maximal operator that controls the commutator). The same occurs in (*a*), (*c*) and (*d*): details and proofs of these alternative formulations are left to the reader. For examples of this, see Section 4.5 and in particular Remarks 4.7, 4.10.

3.2. **Vector-valued and one-weight estimates.** Once we have the Coifman type inequality just stated, vector-valued estimates follow by extrapolation. Indeed, as it is shown in [6], estimate (3.1) (analogously (3.3)) yields that for every $0<p,q<\infty$ and $w\in A_\infty$

$$\Big\|\Big(\sum_j|T_b^kf_j|^q\Big)^{\frac1q}\Big\|_{L^p(w)}\le C\,\Big\|\Big(\sum_j(M_{\overline{\mathcal{A}}}f_j)^q\Big)^{\frac1q}\Big\|_{L^p(w)}.$$

Let us emphasize that this is nothing specific of commutators or singular integral operators. Whenever an estimate like (3.1) holds with an operator in each side, for one (equivalently for all) $0<p<\infty$ and for all $w\in A_\infty$, the extrapolation techniques in [6] give vector-valued inequalities as before. Furthermore, as it is shown in [5], all these estimates (vector-valued or not) also hold for any "reasonable" quasi-Banach rearrangement invariant function space $\mathbb{X}(w)$. Examples of these $\mathbb{X}$'s are $L^{p,q}$, $L^{p,q}(\log L)^\alpha$, Orlicz spaces, Marcinkiewicz spaces, ... Also, weak and strong modular estimates hold and we will use them to get (3.2).

As explained in the introduction, Coifman type estimates are generally used to control an operator with some degree of singularity by a maximal operator which, in principle, is easier to handle. For instance, in the case of classical Calderón-Zygmund operators with regular kernels (in our notation kernels in H^*_∞) one has (1.1) and, as a consequence, it follows that T is bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A_p$ as M is. Indeed the extrapolation results mentioned before (see [6], [5]) state a much deeper fact, T and M behave almost the same on weighted function spaces and in the sense of weighted modular estimates (here "almost" is because somehow one needs to be apart from L^∞ as M is bounded on L^∞ and T is not, see [6] and [5] for more details). In this way, starting from (3.1) (analogously (3.3)), we have that T_b^k behave as $M_{\overline{\mathcal{A}}}$. Thus, most of the inequalities that one can show for the maximal operator (which in general are easier) will hold for the commutator. Let us notice that this is indeed what happens in (3.2) or (3.4). These estimates are satisfied by $M_{\overline{\mathcal{A}}}$ or M^{k+1} and, by extrapolation the commutators verify them.

We state some known weighted norm-estimates that maximal operators associated with Orlicz functions satisfy:

Theorem 3.6. *Let $\mathcal{A}$ be a Young function, $r \geq 1$ and $p > 1$. If $\overline{\mathcal{A}}(t)^r \in B_p$ then, $M_{\overline{\mathcal{A}}}$ is bounded on $L^p(w)$ for all $w \in A_r$. Analogously, $M^+_{\overline{\mathcal{A}}}$ is bounded on $L^p(w)$ for all $w \in A^+_r$.*

In the one-sided case, this result is proved in [24]. The general case follows the same way and we sketch the proof in Section 5.

Remark 3.7. The reader should notice that, as a consequence of (3.1) and under the same hypothesis, T_b^k and $T_b^{+,k}$ satisfy the same estimates. Notice also, that in Theorem (3.3) part (*b*), one can trivially prove that T_b^k is bounded on $L^p(w)$ for any $w \in A_p$ (as M^{k+1} is). The same happens in (*d*) with $T_b^{+,k}$ with weights $w \in A^+_p$. Precise statements and details are left to the reader

Let us notice that as explained before, weighted vector-valued estimates can be proved for the commutators, once we have them for the maximal operators (and in many cases the latter ones are also obtained by extrapolation). Here we do not want to get into this matter.

Remark 3.8. We would like to point out that in the applications below, for conciseness, we will just write the scalar Coifman type estimates on weighted Lebesgue spaces. As we have explained in this section, this estimates can be proved for other function spaces and in the sense of modular inequalities. Also, all of them admit vector-valued extensions (see [6] and [5] for more details of this technique and for potential applications). On the other hand, we can get boundedness of commutators on weighted Lebesgue spaces from Theorem 3.6. The precise statements and the details are left to the interested reader.

3.3. **Two-weight norm inequalities.** Next, we obtain two-weight norm inequalities for operators such that their adjoints satisfy a Coifman type inequality. Here, the weights are no longer in A_∞. In order to simplify, we use the following notation: we use w for weights in A_∞ or $w \in A^\pm_\infty$ and u for arbitrary weights, that is, for $0 \leq u \in L^1_{\text{loc}}(\mathbb{R}^n)$ and we do not assume that u is a Muckenhoupt weight.

Theorem 3.9. *Let $\mathcal{A}$ be a Young function and $1 < p < \infty$. Suppose that there exist Young functions $\mathcal{E}$, $\mathcal{F}$ such that $\mathcal{E} \in B_{p'}$ and $\mathcal{E}^{-1}(t)\,\mathcal{F}^{-1}(t) \leq \overline{\mathcal{A}}^{-1}(t)$. Set $\mathcal{D}(t) = \mathcal{F}(t^{1/p})$.*

(*a*) *Let T be a linear operator such that its adjoint T^* satisfies*

$$\int_{\mathbb{R}^n} |T^* f(x)|^q\, w(x)\, dx \leq C \int_{\mathbb{R}^n} M_{\overline{\mathcal{A}}} f(x)^q\, w(x)\, dx, \tag{3.5}$$

for all $0 < q < \infty$ and $w \in A_\infty$. Then for any weight u, that is, $0 \leq u \in L^1_{\text{loc}}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |T f(x)|^p\, u(x)\, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p\, M_{\mathcal{D}} u(x)\, dx. \tag{3.6}$$

(*b*) *Let T^+ be a one-sided linear operator such that its adjoint, T^-, satisfies (3.5) for all $0 < q < \infty$, $w \in A^-_\infty$ and with $M^-_{\overline{\mathcal{A}}} f$ on the righthand side. Then, for any weight u, that is, $0 \leq u \in L^1_{\text{loc}}(\mathbb{R})$, it follows that T^+ verifies (3.6) with $M^-_{\mathcal{D}} u$ in place of $M_{\mathcal{D}} u$.*

Let us point out that estimates assumed for T^* or T^- are assumed to hold for all $f \in L^\infty_c(\mathbb{R}^n)$ such that the lefthand side is finite.

Remark 3.10. For the applications below, and since all our operators are of convolution type, proving (3.5) for T^* or T turns out be equivalent: T^* is a convolution operator given by the kernel $\tilde{K}(x) = \overline{K(-x)}$ and so $\tilde{K} \in H_{\mathcal{A},k}$ if and only if $K \in H_{\mathcal{A},k}$. We do not mention this below although we use it repeatedly. The same applies to the commutators and also to the one-sided operators with the appropriate changes.

Next we present some examples of different $M_{\mathcal{D}}$ that can be obtained from the last result. In all of them, we have taken $\mathcal{E}(t) = t^{p'}\,(1 + \log^+ t)^{-1-\tilde{\varepsilon}} \in B_{p'}$ where $\tilde{\varepsilon} > 0$ is some small enough number that is related to ε appearing in each example. One can be a little bit sharper by taking $\mathcal{E}(t) = t^{p'}\,(1+\log^+ t)^{-1}(1+\log^+ \log^+ t)^{-1-\tilde{\varepsilon}} \in B_{p'}$, we leave this to the reader.

$M_{\overline{\mathcal{A}}}$	Range of p's	$M_{\mathcal{D}}$	Iterations
$M = M_{L^1}$	$1 < p < \infty$	$M_{L\,(\log L)^{p-1+\varepsilon}}$	$M^{[p]+1}$
$M^{k+1} \approx M_{L(\log L)^k}$	$1 < p < \infty$	$M_{L\,(\log L)^{(k+1)\,p-1+\varepsilon}}$	$M^{[(k+1)\,p]+1}$
$M_{L(\log L)^{k+\varepsilon'}}$	$1 < p < \infty$	$M_{L\,(\log L)^{(k+1)\,p-1+\varepsilon}}$	$M^{[(k+1)\,p]+1}$
$M_{L^{r'}}$	$1 < p < r$	$M_{L^{(\frac{r}{p})'}\,(\log L)^{(\frac{r}{p})'\,(p-1)+\varepsilon}}$	–
$M_{L^{r'}(\log L)^{k\,r'}}$	$1 < p < r$	$M_{L^{(\frac{r}{p})'}\,(\log L)^{(\frac{r}{p})'\,((k+1)\,p-1)+\varepsilon}}$	–
$(M_{L^{r'}})^{k+1} \approx M_{L^{r'}(\log L)^k}$	$1 < p < r$	$M_{L^{(\frac{r}{p})'}\,(\log L)^{(\frac{r}{p})'\,((\frac{k}{r'}+1)\,p-1)+\varepsilon}}$	–

TABLE 1. Examples of two-weight estimates

Remark 3.11. In the third example we assume that $\varepsilon' > 0$ is small enough. Notice that passing to iterations we have $M_{\overline{\mathcal{A}}} \lesssim M^{k+2}$. Having done so, by the second example, we would have a weight $M^{[(k+2)\,p]+1} u$ in place of what we get. This will be applied to the differential transform operator considered in Section 4.5.2. On the other hand, the last three cases are motivated by the examples considered in Section 4.4, see Table 2.

Remark 3.12. The previous examples can be adapted easily to the one-sided case with $M^-_{(\cdot)}$ in place of $M_{(\cdot)}$.

In the following applications these examples can be used to derive two-weight estimates, we leave the precise statements to the reader.

4. Applications

Next we present the applications of our main results. In what follows for $k \geq 0$, let $\mathcal{C}_k(t) = t\,(1+\log^+ t)^k$ and then $\overline{\mathcal{C}}_k(t) \approx e^{t^{1/k}}$ for $t \geq 1$. In the case $k=0$, $\overline{\mathcal{C}}_0(t)$ is understood to be ∞ for all $t \geq 1$ and so $\overline{\mathcal{C}}_0^{-1}(t) = 1$ for all $t \geq 1$.

4.1. **Homogeneous Singular Integrals.** Denote by $\Sigma = \Sigma_{n-1}$ the unit sphere on $\mathbb{R}^n$. For $x \neq 0$, we write $x' = x/|x|$. Let us consider $\Omega \in L^1(\Sigma)$. This function can be extended to $\mathbb{R}^n \setminus \{0\}$ as $\Omega(x) = \Omega(x')$ (abusing on the notation we call both functions Ω). Thus Ω is a function homogeneous of degree 0. We assume that $\int_\Sigma \Omega(x')d\sigma(x') = 0$. Set $K(x) = \Omega(x)/|x|^n$ and let T be the operator associated with the kernel K.

Given a Young function $\mathcal{B}$ we define the $L^{\mathcal{B}}$-modulus of continuity of Ω as

$$\varpi_{\mathcal{B}}(t) = \sup_{|y|\leq t} \|\Omega(\cdot+y) - \Omega(\cdot)\|_{\mathcal{B},\Sigma}.$$

Theorem 4.1. *Let $\Omega \in L^{\mathcal{B}}(\Sigma)$ and T be as above. Let $k \geq 0$ and $\mathcal{A}$, $\mathcal{B}$ be Young functions such that $\overline{\mathcal{A}}^{-1}(t)\,\mathcal{B}^{-1}(t)\,\overline{\mathcal{C}}_k^{-1}(t) \leq t$ for all $t \geq 1$. If*

$$\int_0^1 \Big(1+\log\frac{1}{t}\Big)^k \varpi_{\mathcal{B}}(t)\,\frac{dt}{t} < \infty, \tag{4.1}$$

then T_b^k satisfies (3.1). Furthermore if (4.1) holds for ϖ_{L^∞} in place of $\varpi_{\mathcal{B}}$ then T_b^k satisfies (3.3).

The proof of this result follows at once from Theorem 3.3 combined with the following:

Proposition 4.2. *Let T and Ω be as above and $\mathcal{B}$ be a Young function. If (4.1) holds, then $K \in H_{\mathcal{B},k}$. Furthermore, if (4.1) holds with ϖ_{L^∞} in place of $\varpi_{\mathcal{B}}$ then $K \in H_{\infty,k}$.*

Remark 4.3. Notice that the above result states that $K \in H_{\mathcal{B},k}$. However, by Theorem 3.3, the same estimate can be obtained with the weaker hypothesis $K \in H_{\mathcal{B}} \cap H_{\mathcal{A},k}$ which follows from

$$\int_0^1 \varpi_{\mathcal{B}}(t)\,\frac{dt}{t} + \int_0^1 \Big(1+\log\frac{1}{t}\Big)^k \varpi_{\mathcal{A}}(t)\,\frac{dt}{t} < \infty, \tag{4.2}$$

and this relaxes (4.1). The same can be done with ϖ_{L^∞} replacing $\varpi_{\mathcal{B}}$ and with $\mathcal{A}(t) = e^{t^{1/k}}$. Details are left to the reader.

Remark 4.4. Theorem 4.1 with $\mathcal{B}(t) = t^r$ with $1 < r < \infty$ is implicit in [10] (see also [29]) for $k=0$ and in [26] for $k \geq 1$. In these cases, $\mathcal{A}(t) = t^{r'}\,(1+\log^+ t)^{k\,r'}$ for $k \geq 0$. Here we improve such results since, as we have mentioned in the previous remark, for (3.1) it suffices to assume (4.2) which is weaker than (4.1). On the other hand, (4.1) with the L^∞-modulus of continuity of Ω (or the corresponding (4.2)) gives (3.3). This relaxes the classical and much stronger condition H^*_∞.

Notice that as explained before in Section 3.2 from Theorem 4.1 some scalar and vector-valued weighted estimates can be proved. Also, using the examples in Table 1 we can get two-weight norm inequalities for these homogeneous singular integrals. The precise conditions assumed on the kernel in terms of (4.1) or (4.2) are left to the reader (Table 2 below eases this task).

4.2. **Fourier Multipliers.** Let $m \in L^\infty(\mathbb{R}^n)$ and consider the multiplier operator T defined *a priori* for functions f in the Schwartz class by $\widehat{Tf}(\xi) = m(\xi)\,\widehat{f}(\xi)$. Given $1 < s \le 2$ and $0 \le l \in \mathbb{N}$ we say that $m \in M(s,l)$ if

$$\sup_{R>0} R^{|\alpha|}\, \|D^\alpha m\|_{L^s,|\xi|\sim R} < +\infty, \qquad \text{for all } |\alpha| \le l.$$

Theorem 4.5. *Let $m \in M(s,l)$, with $1 < s \le 2$, $0 \le l \le n$ and with $l > n/s$. Then for all $k \ge 0$ and any $\varepsilon > 0$ we have that for all $0 < p < \infty$ and $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} |T_b^k f(x)|^p\, w(x)\, dx \le C \int_{\mathbb{R}^n} M_{n/l+\varepsilon} f(x)^p\, w(x)\, dx. \tag{4.3}$$

The proof of this result relies on showing that an appropriate truncation of K belongs to $H_{L^r(\log L)^{kr},k}$ with $r' = n/l + \varepsilon$, see Proposition 6.2 below.

The fourth example in Table 1 gives us two-weight estimates from (4.3). However, as ε is at our choice, we can write $M_{\mathcal{D}} = M_{(r/p)'}$ for any $1 < r < (n/l)'$, in other words, $M_{\mathcal{D}} = M_{((n/l)'/p)'+\varepsilon}$ for any $\varepsilon > 0$.

4.3. **Operators with "smooth" kernels.** When T is a Calderón-Zygmund operator with regular kernel, that is, with $K \in H_\infty^*$, it is well known that T satisfies (1.1) and also that the first order commutator verifies (1.3). Analogously, there is a Coifman type estimate establishing that T_b^k is controlled by M^{k+1} as (3.3). As a consequence of all this, some sharp two-weight estimates as (1.2) and (1.4) are known.

For every $k \ge 1$, we assume that $K \in H_\infty \cap H_{e^{t^{1/k}},k}$, and observe that this happens, in particular, if $K \in H_{\infty,k}$ or, even more, if $K \in H_\infty^*$. When $k = 0$, we just assume $K \in H_\infty$. Applying Theorem 3.3 Part (b) it follows (3.3) and also the weak modular weighted estimate (3.4).

From (3.3) and using Table 1, we can obtain the following two-weight estimate

$$\int_{\mathbb{R}^n} |T_b^k f(x)|^p\, u(x)\, dx \le C \int_{\mathbb{R}^n} |f(x)|^p\, M^{[(k+1)\,p]+1} u(x)\, dx.$$

Notice that in the particular case $K \in H_\infty^*$ we recover the results proved by C. Pérez in [18] for $k = 0$ and in [19] for $k \ge 1$.

The same can be done with one-sided operators and the last estimate holds with $(M^-)^{[(k+1)\,p]+1} u(x)$ in place of $M^{[(k+1)\,p]+1} u(x)$. Thus we improve the results in [1], [24] for $k = 0$, and [11] for $k \ge 1$.

4.4. **Kernels related to H_{t^r} and M_{L^r}.** Implicit in [25] (see also [10], [29]) and as it was observed in [13] when $K \in H_{t^r}$, that is, when the kernel satisfies the L^r-Hörmander condition, then one obtains that T is controlled by $M_{L^{r'}}$. Here we want to consider different extensions of this inequality for the higher order commutators. First, we see what happens when $K \in H_{t^r,k}$ or when $K \in H_{t^r}$. Second, we seek for conditions that guarantee that all the commutators are controlled by $M_{L^{r'}}$ as happened with the multipliers studied before. Finally, we give conditions on the kernel that lead us to iterations of $M_{L^{r'}}$ (as happens with classical Calderón-Zygmund operators with $r = \infty$). In what follows, $1 < r < \infty$. Following the notation of Theorem 3.3 these are the different conditions and maximal operators obtained:

$H_{\mathcal{B},k}$	$H_{\mathcal{B}} \cap H_{\mathcal{A},k}$	$M_{\overline{\mathcal{A}}} f$
$H_{t^r,k}$	$H_{t^r} \cap H_{t^r (\log t)^{-k\,r},k}$	$M_{L^{r'} (\log L)^{k\,r'}} f$
$H_{t^r (\log t)^{k\,r},k}$	$H_{t^r (\log L)^{k\,r}} \cap H_{t^r,k}$	$M_{L^{r'}} f$
$H_{t^r (\log t)^{k},k}$	$H_{t^r (\log t)^{k}} \cap H_{t^r (\log t)^{-k\,(r-1)},k}$	$M_{L^{r'} (\log L)^{k}} f \approx (M_{L^{r'}})^{k+1}$

TABLE 2. Examples of different H_{t^r}-conditions

These examples and Table 1 allow us to establish two-weight estimates, details are left to the reader.

4.5. **One-sided operators.**

4.5.1. *One-sided kernels: H_∞ and $H_{e^{t^{1/k}},k}$.* We consider the one-sided operator

$$T^+ f(x) = \sum_{j\in\mathbb{Z}} (-1)^j \big(D_j f(x) - D_{j-1} f(x)\big),$$

where

$$D_j f(x) = \frac{1}{(1+j^2)\,2^j} \int_x^{x+2^j} f(t)\,dt = \frac{a_j}{2^j} \int_x^{x+2^j} f(t)\,dt.$$

with $a_j = (1+j^2)^{-1}$. Observe that $T^+ f = K * f$ with K supported in $(-\infty, 0)$ and

$$K(x) = \sum_{j\in\mathbb{Z}} (-1)^j \left(\frac{a_j}{2^j}\,\chi_{(-2^j,0)}(x) - \frac{a_{j-1}}{2^{j-1}}\,\chi_{(-2^{j-1},0)}(x) \right).$$

We show in Corollary 6.6 part (b) that $K \in H_\infty \cap H_{e^{t^{1/k}},k}$ and then we have the following result:

Theorem 4.6. *Under the previous conditions, $K \in H_\infty \cap H_{e^{t^{1/k}},k}$. Therefore, for all $w \in A_\infty^+$ and $0 < p < \infty$ we have*

$$\int_{\mathbb{R}} |T_b^{+,k} f(x)|^p\, w(x)\,dx \le C \int_{\mathbb{R}} (M^+)^{k+1} f(x)^p\, w(x)\,dx,$$

$$w\big\{x \in \mathbb{R}^n : |T_b^{+,k} f(x)| > \lambda\big\} \le C \int_{\mathbb{R}^n} \varphi_k\left(\frac{\|b\|_{\mathrm{BMO}}^k\,|f(x)|}{\lambda}\right) M^- w(x)\,dx,$$

where $\varphi_k(t) = t\,(1+\log^+ t)^k$. Moreover, and for any weight u and $1 < p < \infty$

$$\int_{\mathbb{R}} |T_b^{+,k} f(x)|^p\, u(x)\,dx \le C \int_{\mathbb{R}} |f(x)|^p\,(M^-)^{[(k+1)\,p]+1} u(x)\,dx.$$

Remark 4.7. Let us emphasize that Proposition 6.4 implies that $K \notin H_{\infty,k}$, so here it is crucial to have a formulation of Theorem 3.3 with the weaker hypothesis $K \in H_\infty \cap H_{e^{t^{1/k}},k}$. On the other hand, it is also very important to take into account that $K \in H_\infty$: if one only uses that $K \in H_{e^{t^{1/k}},k}$, then by Theorem 3.3 Part (c) with

$\mathcal{B}(t) = e^{t^{1/k}}$ it follows that $M_{\overline{\mathcal{A}}}^{+} = M_{L(\log L)^{2k}}^{+} \approx (M^{+})^{2k+1}$, that is, we get k extra iterations. Let us mention that in this case the two-weight estimates would be for the pair of weights $\big(u, (M^{-})^{[p(2k+1)]+1}u\big)$.

Remark 4.8. The same result can be obtained considering a slightly worse operator T^{+} associated with the sequence $a_j = (1+|j|^{1+\alpha})^{-1}$ for some $0 < \alpha < 1$ (the case just studied corresponds to $\alpha = 1$). We can repeat the computations in Corollary 6.6 part (b) and therefore $T_b^{+,k}$ satisfies the estimates in Theorem 4.6.

4.5.2. *The differential transform operator.* We consider the following differential transform operator studied in [9] and [3]: given $\{\nu_j\}_j \in \ell^{\infty}$,

$$T^{+}f(x) = \sum_{j\in\mathbb{Z}} \nu_j \big(D_j f(x) - D_{j-1}f(x)\big), \qquad D_j f(x) = \frac{1}{2^j}\int_x^{x+2^j} f(t)\,dt.$$

This operator appears when studying the rate of convergence of the averages $D_j f$. Let us observe that $D_j f \longrightarrow f$ a.e. when $j \to -\infty$ and $D_j f \longrightarrow 0$ when $j \to \infty$.

Note that T^{+} is a one-sided singular integral as $T^{+}f = K * f$ where K is supported in $(-\infty, 0)$ and

$$K(x) = \sum_{j\in\mathbb{Z}} \nu_j \left(\frac{1}{2^j}\,\chi_{(-2^j,0)}(x) - \frac{1}{2^{j-1}}\,\chi_{(-2^{j-1},0)}(x)\right).$$

In [3] it is proved that for f in an appropriate class (L_c^{∞}, L^p, $L^p(w)$ with $w \in A_p, \dots$)

$$T^{+}f(x) = \lim_{(N_1,N_2)\to(-\infty,\infty)} \sum_{N_1}^{N_2} \nu_j \big(D_j f(x) - D_{j-1}f(x)\big) \qquad \text{for a.e. } x \in \mathbb{R}.$$

It is also obtained that T^{+} is bounded on $L^p(w)$ for any $w \in A_p^{+}$, $1 < p < \infty$, and T^{+} maps $L^1(w)$ into $L^{1,\infty}(w)$ for all $w \in A_1^{+}$.

We choose $\nu_j = (-1)^j$. By Remark 6.5, since T^{+} is the operator associated with the sequence $a_j = 1 \in \ell^{\infty}(\mathbb{Z})$, we have that $K \in \cap_{r\ge 1} H_r$. However, $K \notin H_{\infty}$, by (a) in Corollary 6.6. Note that this result also gives $K \in H_{e^{t^{1/(1+\varepsilon+k)}},k}$.

Theorem 4.9. *Under the previous hypothesis, for each $\varepsilon > 0$ and all $k \ge 0$ we have $K \in H_{e^{t^{1/(1+\varepsilon+k)}},k}$. Therefore, for all $w \in A_{\infty}^{+}$ and $0 < p < \infty$ we have*

$$\int_{\mathbb{R}} |T_b^{+,k} f(x)|^p\, w(x)\,dx \le C \int_{\mathbb{R}} (M^{+})^{k+3} f(x)^p\, w(x)\,dx;$$

and for any weight u and $1 < p < \infty$

$$\int_{\mathbb{R}} |T_b^{+,k} f(x)|^p\, u(x)\,dx \le C \int_{\mathbb{R}} |f(x)|^p\, (M^{-})^{[(k+2)\,p]+1} u(x)\,dx.$$

Remark 4.10. In this result, fixed k, to get the first estimate (and thus the second) we use that $K \in H_{e^{t^{1/1+\varepsilon}}} \cap H_{e^{t^{1/(1+\varepsilon+k)}},k}$, that is, one condition at level $k = 0$ and another at level k. In this case, following the notation in Theorem 3.3, we take $\mathcal{B}(t) = e^{t^{1/1+\varepsilon}}$ and $\mathcal{A}(t) = e^{t^{1/(1+\varepsilon+k)}}$, which gives $M_{\overline{\mathcal{A}}}^{+} f(x) = M_{L(\log L)^{1+k+\varepsilon}}^{+} f(x) \le C\,(M^{+})^{k+3} f(x)$ for $0 < \varepsilon < 1$. Notice, that as observed in Remark 3.11, to get the two-weight estimate one has to use $M_{\overline{\mathcal{A}}}$, since using M^{k+3} we get a worse weight.

As in Remark 4.7, if we only use that $K \in H_{e^{t^{1/(1+\varepsilon+k)}},k}$ (that is we do not take into account what is known when $k=0$), then Theorem 3.3 applies with $\mathcal{B}(t)=e^{t^{1/(1+\varepsilon+k)}}$ yielding $M^+_{\overline{\mathcal{A}}}f(x)=M^+_{L\,(\log L)^{2\,k+1+\varepsilon}}\le C\,(M^+)^{2\,k+3}f(x)$ provided $0<\varepsilon<1$. So this way adds k extra iterations.

Remark 4.11. In Theorem 4.9 one can be more precise and prove that $K\in H_{\mathcal{A},k}$ with $\mathcal{A}(t)=\exp\big(t^{1/1+k}/(\log t)^{(1+\varepsilon)/(1+k)}\big)$. In this case, the maximal operator obtained is $M^+_{\overline{\mathcal{A}}}=M^+_{L\,(\log L)^{1+k}\,(\log\log L)^{1+\varepsilon}}$ which is pointwise smaller than $M^+_{L\,(\log L)^{1+k+\varepsilon'}}$. In terms of iterations, both maximal operators are controlled by $(M^+)^{k+3}$ and these estimates are sharp. Details are left to the reader.

5. Proofs of the Main results

Let us first recall some properties of BMO for later use. Given $b\in\mathrm{BMO}$, a ball B, $k\ge 0$ and $q>0$, by John-Nirenberg's theorem we have

$$\|(b-b_B)^k\|_{L^q,B}\le\|(b-b_B)^k\|_{\overline{\mathcal{C}}_k,B}=\|b-b_B\|^k_{\exp L,B}\le C\,\|b\|^k_{\mathrm{BMO}}. \tag{5.1}$$

On the other hand, for every $j\ge 1$ and $b\in\mathrm{BMO}$, we have

$$|b_B-b_{2^jB}|\le\sum_{m=1}^{j}|b_{2^{m-1}B}-b_{2^mB}|\le 2^n\sum_{m=1}^{j}\|b-b_{2^mB}\|_{L^1,2^mB}\le 2^n\,j\,\|b\|_{\mathrm{BMO}}. \tag{5.2}$$

5.1. **Theorem 3.3, Part** (a). We need the following auxiliary result:

Lemma 5.1. *Under the hypotheses of Theorem 3.3 Part* (a)*, for any* $b\in\mathrm{BMO}$*,* $0<\delta<\varepsilon<1$ *and* $k\ge 1$*, there exists* $C=C_{\delta,\varepsilon}>0$ *such that*

$$M^{\#}_\delta\big(T^k_bf\big)(x)\le C\sum_{j=0}^{k-1}\|b\|^{k-j}_{\mathrm{BMO}}M_\varepsilon(T^j_bf)(x)+C\,\|b\|^k_{\mathrm{BMO}}M_{\overline{\mathcal{A}}}f(x).$$

Remark 5.2. The case $k=0$ was already considered in [12] to obtain Theorem 3.1 above. In particular the following estimate was proved: if $K\in H_{\mathcal{A}}$ then

$$M^{\#}_\delta(Tf)(x)\le C_\delta\,M_{\overline{\mathcal{A}}}f(x).$$

Proof. By homogeneity we may assume that $\|b\|_{\mathrm{BMO}}=1$: for any $b\in\mathrm{BMO}$ we can write $\widetilde{b}=b/\|b\|_{\mathrm{BMO}}$ whose BMO norm is 1 and in this case $T^k_bf(x)=\|b\|^k_{\mathrm{BMO}}\,T^k_{\widetilde{b}}f(x)$. As mentioned in Remark 3.5 we only need to consider the case $K\in H_{\mathcal{B}}\cap H_{\mathcal{A},k}$. In both conditions and for simplicity we assume that $c=1$. Then, as in [19] or [11], for any constant λ we can write

$$T^k_bf(x)=T\big((\lambda-b)^kf\big)(x)+\sum_{m=0}^{k-1}C_{k,m}\big(b(x)-\lambda\big)^{k-m}T^m_bf(x), \tag{5.3}$$

where we remind the reader that $T^0_b=T$. Let us fix $x\in\mathbb{R}^n$, and a ball $B\ni x$ with radius R and center x_B. We write $\widetilde{B}=2\,B$, $\lambda=b_{\widetilde{B}}$ and $f=f_1+f_2=f\,\chi_{\widetilde{B}}+f\,\chi_{\widetilde{B}^c}$. Let a_B be a constant to be chosen later and observe that

$$\left(\frac{1}{|B|}\int_B\big||T^k_bf(y)|^\delta-|a_B|^\delta\big|\,dy\right)^{\frac{1}{\delta}}\le\left(\frac{1}{|B|}\int_B\big|T^k_bf(y)-a_B\big|^\delta\,dy\right)^{\frac{1}{\delta}}$$

$$\leq C \Bigg[\sum_{m=0}^{k-1} \left(\frac{1}{|B|} \int_B |b(y) - b_{\widetilde{B}}|^{(k-m)\delta} |T_b^m f(y)|^\delta \, dy \right)^{\frac{1}{\delta}} + \left(\frac{1}{|B|} \int_B |T((b_{\widetilde{B}} - b)^k f_1)(y)|^\delta \, dy \right)^{\frac{1}{\delta}} + \left(\frac{1}{|B|} \int_B |T((b_{\widetilde{B}} - b)^k f_2)(y) - a_B|^\delta \, dy \right)^{\frac{1}{\delta}} \Bigg]$$
$$= C\,(I + II + III). \tag{5.4}$$

We estimate I: as $0 < \delta < \varepsilon < 1$, by Hölder's inequality with $q = \varepsilon/\delta > 1$ and (5.1)

$$I \leq \sum_{m=0}^{k-1} \left(\frac{1}{|B|} \int_B |b(y) - b_{\widetilde{B}}|^{(k-m)\delta q'} \, dy \right)^{\frac{1}{\delta q'}} \left(\frac{1}{|B|} \int_B |T_b^m f(y)|^{\delta\, q} dy \right)^{\frac{1}{\delta\, q}}$$
$$\leq C \sum_{m=0}^{k-1} \|b\|_{\mathrm{BMO}}^{k-m} \, M_\varepsilon(T_b^m f)(x) \leq C \sum_{m=0}^{k-1} M_\varepsilon(T_b^m f)(x). \tag{5.5}$$

For II, as mentioned before $H_{\mathcal{B}} \subset H_1$ and, therefore, T is of weak type $(1,1)$. Then Kolmogorov's inequality, the generalized Hölder inequality for $\overline{\mathcal{A}}$, $\mathcal{B}$, $\overline{\mathcal{C}}_k$ and (5.1) yield

$$II \leq C \, \frac{1}{|B|} \int_{\widetilde{B}} |b(y) - b_{\widetilde{B}}|^k \, |f(y)| \, dy \leq C \, \|(b - b_{\widetilde{B}})^k\|_{\overline{\mathcal{C}}_k, \widetilde{B}} \|f\|_{\overline{\mathcal{A}}, \widetilde{B}} \leq C \, M_{\overline{\mathcal{A}}} f(x). \tag{5.6}$$

Next, we estimate III: Let us take $a_B = T((b_{\widetilde{B}} - b)^k f_2)(x_B)$. Then, by Jensen's inequality,

$$III \leq \frac{1}{|B|} \int_B |T((b_{\widetilde{B}} - b)^k f_2)(y) - T((b_{\widetilde{B}} - b)^k f_2)(x_B)| \, dy.$$

For $j \geq 1$, let $S_j = 2^{j+1} B \setminus 2^j B$ and $B_j = 2^{j+1} B$. For every $y \in B$, we have by (5.2)

$$|T((b_{\widetilde{B}} - b)^k f_2)(y) - T((b_{\widetilde{B}} - b)^k f_2)(x_B)|$$
$$\leq \sum_{j=1}^{\infty} \int_{S_j} |b(z) - b_{\widetilde{B}}|^k \, |K(y - z) - K(x_B - z)| \, |f(z)| \, dz$$
$$\leq C \sum_{j=1}^{\infty} (2^j R)^n \frac{1}{|B_j|} \int_{B_j} |b(z) - b_{B_j}|^k \, |K(y - z) - K(x_B - z)| \; \chi_{S_j}(z) \, |f(z)| \, dz$$
$$+ C \sum_{j=2}^{\infty} (2^j R)^n \, j^k \, \frac{1}{|B_j|} \int_{B_j} |K(y - z) - K(x_B - z)| \; \chi_{S_j}(z) \, |f(z)| \, dz$$
$$= C\,(IV + V). \tag{5.7}$$

By the generalized Hölder inequality for $\overline{\mathcal{A}}$, $\mathcal{B}$, $\overline{\mathcal{C}}_k$, (5.1) and as $K \in H_{\mathcal{B}}$ we obtain

$$IV \leq C \sum_{j=1}^{\infty} (2^j R)^n \, \|(b - b_{B_j})^k\|_{\overline{\mathcal{C}}_k, B_j} \, \|f\|_{\overline{\mathcal{A}}, B_j} \, \|(K(y - \cdot) - K(x_B - \cdot)) \, \chi_{S_j}\|_{\mathcal{B}, B_j}$$
$$\leq C \, M_{\overline{\mathcal{A}}} f(x) \sum_{j=1}^{\infty} (2^j R)^n \, \|K(\cdot - (x_B - y)) - K(\cdot)\|_{\mathcal{B}, |z| \sim 2^j R}$$
$$\leq C \, M_{\overline{\mathcal{A}}} f(x),$$

where we have used that $x \in B \subset B_j$ and that $|x_B - y| < R$ since $y \in B$. Besides, since $K \in H_{\mathcal{A},k}$ we use again the generalized Hölder inequality for $\mathcal{A}$ and thus

$$\begin{aligned} V &\leq C \sum_{j=1}^{\infty} (2^j R)^n \, j^k \, \|(K(y-\cdot) - K(x-\cdot))\,\chi_{S_j}\|_{\mathcal{A},B_j} \, \|f\|_{\overline{\mathcal{A}},B_j} \\ &\leq C\, M_{\overline{\mathcal{A}}} f(x) \sum_{j=1}^{\infty} (2^j\, R)^n \, j^k \, \|K(\cdot - (x_B - y)) - K(\cdot)\|_{\mathcal{A},|z|\sim 2^j\, R} \\ &\leq C\, M_{\overline{\mathcal{A}}} f(x). \end{aligned}$$

Plugging the obtained estimates into (5.4) we conclude

$$M_\delta^{\#}\big(T_b^k f\big)(x) \leq C\, M_{\overline{\mathcal{A}}} f(x) + C \sum_{m=0}^{k-1} M_\varepsilon(T_b^m f)(x).$$

□

Proof of Theorem 3.3, Part (a). By the extrapolation results obtained in [6], estimate (3.1) holds for all $0 < p < \infty$ and all $w \in A_\infty$ if and only if it holds for some fixed exponent $0 < p_0 < \infty$ and all $w \in A_\infty$. Therefore, we show (3.1) for p_0 that is taken so that $1 < p_0 < \infty$ (this will make some computations cleaner and avoid some technicalities). We first consider the case on which w and $b \in L^\infty$. By homogeneity, we assume that $\|b\|_{\mathrm{BMO}} = 1$. We proceed by induction.

When $k = 0$ then $T_b^k = T$. As $K \in H_{\mathcal{A},0} = H_{\mathcal{A}}$, Theorem 3.1 (proved in [12]) implies that T is controlled by $M_{\overline{\mathcal{A}}}$ as desired

Next, we assume that the result holds for all $0 \leq j \leq k-1$ and let us see how to derive the case k. We fix $\mathcal{A}$ and $\mathcal{B}$ so that $\overline{\mathcal{A}}^{-1}(t)\, \mathcal{B}^{-1}(t)\, \overline{\mathcal{C}}_k^{-1}(t) \leq t$ and $K \in H_{\mathcal{B}} \cap H_{\mathcal{A},k}$. Let $f \in L_c^\infty$ and, without loss of generality, we assume that both $\|M_{\overline{\mathcal{A}}} f\|_{L^{p_0}(w)}$, $\|T_b^k f\|_{L^{p_0}(w)}$ are finite. Let $w \in A_\infty$, then there exists $r > 1$ (that can be taken greater than p_0) such that $w \in A_r$. Observe that for all $0 < \delta < p_0/r < 1$, we have that $r < p_0/\delta$ and thus, $w \in A_{p_0/\delta}$. Fefferman-Stein's inequality, see [8], states that for all $0 < p < \infty$ and $w \in A_\infty$

$$\int_{\mathbb{R}^n} Mf(x)^p w(x)\, dx \leq C \int_{\mathbb{R}^n} M^{\#} f(x)^p w(x)\, dx \tag{5.8}$$

for all functions such that the left-hand side is finite. We want to use this inequality and to do so we need to check that $\|M_\delta(T_b^k f)\|_{L^{p_0}(w)}$ is finite. Notice that since $w \in A_{p_0/\delta}$ with $p_0/\delta > 1$ we have

$$\|M_\delta(T_b^k f)\|_{L^{p_0}(w)} = \big\|M\big(|T_b^k f|^\delta\big)\big\|_{L^{\frac{p_0}{\delta}}(w)}^{\frac{1}{\delta}} \leq C\, \|T_b^k f\|_{L^{p_0}(w)} < \infty,$$

by assumption. Then by (5.8) and Lemma 5.1, for all ε with $\delta < \varepsilon < 1$, we have

$$\begin{aligned} \|T_b^k f\|_{L^{p_0}(w)} &\leq \|M_\delta(T_b^k f)\|_{L^{p_0}(w)} \leq C\, \|M_\delta^{\#}(T_b^k f)\|_{L^{p_0}(w)} \\ &\leq C \sum_{j=0}^{k-1} \|M_\varepsilon(T_b^j f)\|_{L^{p_0}(w)} + C\, \|M_{\overline{\mathcal{A}}} f\|_{L^{p_0}(w)}. \end{aligned} \tag{5.9}$$

Since $\delta < p_0/r < 1$ we can take $\varepsilon > 0$ such that $\delta < \varepsilon < p_0/r < 1$ and so $w \in A_{p_0/\varepsilon}$. Hence

$$\|M_\varepsilon(T_b^j f)\|_{L^{p_0}(w)} = \big\|M\big(|T_b^j f|^\varepsilon\big)\big\|_{L^{\frac{p_0}{\varepsilon}}(w)}^{\frac{1}{\varepsilon}} \le C\,\|T_b^j f\|_{L^{p_0}(w)}.$$

Notice that for $0 \le j \le k-1$ and for all $t \ge e$ we have

$$\overline{\mathcal{A}}^{-1}(t)\,\mathcal{B}^{-1}(t)\,\overline{\mathcal{C}}_j^{-1}(t) = \overline{\mathcal{A}}^{-1}(t)\,\mathcal{B}^{-1}(t)\,\overline{\mathcal{C}}_k^{-1}(t)\,(\log t)^{j-k} \le t.$$

Besides, $K \in H_{\mathcal{B}} \cap H_{\mathcal{A},k} \subset H_{\mathcal{B}} \cap H_{\mathcal{A},j}$. Thus, the induction hypothesis implies that, for any $0 \le j \le k-1$,

$$\|M_\varepsilon(T_b^j f)\|_{L^{p_0}(w)} \le C\,\|T_b^j f\|_{L^{p_0}(w)} \le C\,\|M_{\overline{\mathcal{A}}} f\|_{L^{p_0}(w)}$$

provided the middle term is finite. Assume for the moment that this is the case. Plugging the last estimate into (5.9) it follows that

$$\|T_b^k f\|_{L^{p_0}(w)} \le C\,\|M_{\overline{\mathcal{A}}} f\|_{L^{p_0}(w)}. \tag{5.10}$$

Observe that so far we have not used that w and $b \in L^\infty$, this will be needed in the following argument to show that some quantities are finite.

We still have to see that $\|T_b^j f\|_{L^{p_0}(w)} < \infty$ for all $0 \le j \le k-1$. As $w \in L^\infty$, it suffices to see that $\|T_b^j f\|_{L^{p_0}} < \infty$. Observe that since $p_0 > 1$ and $K \in H_{\mathcal{A},k} \subset H_1$ it follows that T is a Calderón-Zygmund operator and so bounded on L^{p_0}. Thus, since $f \in L^\infty_c$

$$\|T_b^j f\|_{L^{p_0}} = \Big\|\sum_{m=0}^{j} C_{m,j}\, b^{j-m}\, T(b^m f)\Big\|_{L^{p_0}} \le C\,\|b\|_{L^\infty}^j\,\|f\|_{L^{p_0}} < \infty.$$

In this way, we have shown that (5.10) holds assuming that w and $b \in L^\infty$ with $\|b\|_{\mathrm{BMO}} = 1$. By homogeneity, we have that

$$\|T_b^k f\|_{L^{p_0}(w)} \le C\,\|b\|_{\mathrm{BMO}}^k\,\|M_{\overline{\mathcal{A}}} f\|_{L^{p_0}(w)}, \tag{5.11}$$

for all $b \in L^\infty$ and any $w \in A_\infty \cap L^\infty$, where C does not depend on $\|b\|_{L^\infty}$ and $\|w\|_{L^\infty}$ (C depends on the A_∞ constant of w, p_0, k, T).

We remove the restriction $b \in L^\infty$: let us take $b \in \mathrm{BMO}$ and for any $N > 0$ we define $b_N(x) = b(x)$ if $-N \le b(x) \le N$, $b_N(x) = N$ if $b(x) > N$ and $b_N(x) = -N$ if $b(x) < -N$. It is not hard to prove that $|b_N(x) - b_N(y)| \le |b(x) - b(y)|$ and hence $\|b_N\|_{\mathrm{BMO}} \le 2\,\|b\|_{\mathrm{BMO}}$. Therefore, as $b_N \in L^\infty$ we can use (5.11) with b_N in place of b and so for any $w \in A_\infty$ with $w \in L^\infty$,

$$\|T_{b_N}^k f\|_{L^{p_0}(w)} \le C\,\|b_N\|_{\mathrm{BMO}}^k\,\|M_{\overline{\mathcal{A}}} f\|_{L^{p_0}(w)} \le C\,\|b\|_{\mathrm{BMO}}^k\,\|M_{\overline{\mathcal{A}}} f\|_{L^{p_0}(w)}, \tag{5.12}$$

where C does not depend on N. Since $f \in L^\infty_c$ it follows that for $0 \le m \le k$, $(b_N)^m f \longrightarrow b^m f$ as $N \to \infty$ in L^q for $q > 1$. The fact that T is bounded on L^q implies $T((b_N)^m f) \longrightarrow T(b^m f)$ as $N \to \infty$ in L^q. Passing to a subsequence the convergence is almost everywhere and so using that

$$T_{b_N}^k f(x) = \sum_{m=0}^{k} C_{m,k}\, b_N(x)^{k-m}\, T(b_N^m f)(x)$$

it follows that $T^k_{b_{N_j}} f(x) \longrightarrow T^k_b(x)$ for a.e. $x \in \mathbb{R}^n$ as $j \to \infty$. Thus, using Fatou's lemma and (5.12)

$$\begin{aligned}\int_{\mathbb{R}^n} |T^k_b f(x)|^{p_0}\, w(x)\, dx &= \int_{\mathbb{R}^n} \lim_{j\to\infty} |T^k_{b_{N_j}} f(x)|^{p_0}\, w(x)\, dx \\ &\le \liminf_{j\to\infty} \int_{\mathbb{R}^n} |T^k_{b_{N_j}} f(x)|^{p_0}\, w(x)\, dx \le C\, \|b\|^{k\,p_0}_{\mathrm{BMO}} \int_{\mathbb{R}^n} M_{\overline{\mathcal{A}}} f(x)^{p_0}\, w(x)\, dx,\end{aligned}$$

and this shows (5.11) with the only restriction that $w \in L^\infty$.

Next, we remove the assumption $w \in L^\infty$: take any $w \in A_\infty$ and for any $N > 0$ we define $w_N = \min\{w, N\}$. Then $w_N \in A_\infty$ and also $[w_N]_{A_\infty} \le C\,[w]_{A_\infty}$ with C independent of N. Since $w_N \in L^\infty$ then (5.11) holds with w_N and C does not depend on N. Letting $N \to \infty$ and using the monotone convergence theorem we conclude that (5.11) holds for any $w \in A_\infty$.

In this way we have concluded that (3.1) holds for $p = p_0$ and for all $w \in A_\infty$. Thus, as mentioned, using the extrapolation results obtained in [6], (3.1) holds for all $0 < p < \infty$ and all $w \in A_\infty$.

Next we show (3.2). Note that it suffices to consider $\lambda = 1$ (the general case follows by applying the result to the function f/λ). We may also assume that $\|b\|_{\mathrm{BMO}} = 1$. Set $\Phi(\lambda) = 1/\overline{\mathcal{A}}(1/\lambda)$, and note that since $\overline{\mathcal{A}}$ is submultiplicative then $\Phi \in \Delta_2$, that is, $\Phi(2\,t) \le C\,\Phi(t)$. Using standard arguments, namely a Vitali covering lemma, one can show the following endpoint modular estimate for $M_{\overline{\mathcal{A}}}$:

$$w\{x \in \mathbb{R}^n : M_{\overline{\mathcal{A}}} f(x) > \lambda\} \le C \int_{\mathbb{R}^n} \overline{\mathcal{A}}\left(\frac{|f(x)|}{\lambda}\right) Mw(x)\, dx. \tag{5.13}$$

Therefore using [5, Theorem 3.1], from (3.1) and the fact that $\overline{\mathcal{A}}$ is submultiplicative, it follows that

$$\begin{aligned}& w\big\{x \in \mathbb{R}^n : |T^k_b f(x)| > 1\big\} \le \sup_{\lambda>0} \Phi(\lambda)\, w\big\{x : |T^k_b f(x)| > \lambda\big\} \\ &\le \sup_{\lambda>0} \Phi(\lambda)\, w\big\{x : M_{\overline{\mathcal{A}}} f(x) > \lambda\big\} \le C \sup_{\lambda>0} \Phi(\lambda) \int_{\mathbb{R}^n} \overline{\mathcal{A}}\left(\frac{|f(x)|}{\lambda}\right) Mw(x)\, dx \\ &\le C \sup_{\lambda>0} \Phi(\lambda)\, \overline{\mathcal{A}}(1/\lambda) \int_{\mathbb{R}^n} \overline{\mathcal{A}}\big(|f(x)|\big)\, Mw(x)\, dx \le C \int_{\mathbb{R}^n} \overline{\mathcal{A}}\big(|f(x)|\big)\, Mw(x)\, dx.\end{aligned}$$

□

5.2. **Theorem 3.3, Part** (b)**.** We proceed as in (a), with some little changes in the proof of Lemma 5.1. Namely, I is handled in the same way. For II we apply the generalized Hölder inequality for $\mathcal{C}_k$ and $\overline{\mathcal{C}}_k$ (in place of the one for $\overline{\mathcal{A}}$, $\mathcal{B}$, $\overline{\mathcal{C}}_k$). Thus we get that $II \le C\, M_{\mathcal{C}_k} f(x) = M_{L\,(\log L)^k} f(x) \approx M^{k+1} f(x)$. To estimate IV we use the generalized Hölder inequality for $\mathcal{C}_k$ and $\overline{\mathcal{C}}_k$, and (5.1):

$$\begin{aligned}IV &\le C \sum_{j=1}^{\infty} (2^j\,R)^n\, \|(b - b_{\widetilde{B}})^k\|_{\overline{\mathcal{C}}_k, B_j}\, \|f\|_{\mathcal{C}_k, B_j} \sup_{z\in S_j} |K(y-z) - K(x_B - z)| \\ &\le C\, \|b\|^k_{\mathrm{BMO}}\, M_{\mathcal{C}_k} f(x) \sum_{j=2}^{\infty} (2^j\,R)^n\, \|K(\cdot - (x_B - y)) - K(\cdot)\|_{L^\infty, |z|\sim 2^j R}\end{aligned}$$

$$\le C\, M^{k+1} f(x),$$

where we have used that $K \in H_\infty$. To estimate V, since $K \in H_{\overline{\mathcal{C}}_k,k}$, we have for all $x \in B$

$$\begin{aligned} V &\le C \sum_{j=1}^{\infty} (2^j R)^n j^k \|f\|_{\mathcal{C}_k,B_j} \|K(\cdot - (x_B - y)) - K(\cdot)\|_{\overline{\mathcal{C}}_k,|z|\sim 2^j R} \\ &\le C\, M_{\mathcal{C}_k} f(x) \sum_{j=1}^{\infty} (2^j R)^n j^k \|K(\cdot - (x_B - y)) - K(\cdot)\|_{\overline{\mathcal{C}}_k,|z|\sim 2^j R} \\ &\le C\, M^{k+1} f(x). \end{aligned}$$

In this way we have obtained Lemma 5.1 with $M^{k+1} \approx M_{L(\log L)^k}$ in place of $M_{\overline{\mathcal{A}}}$. From here we just need to repeat the steps in (a) to get the desired estimate. The modular estimate is obtained as before using the Young function $\mathcal{C}_k(t) = t\,(1+\log^+ t)^k$ (in place of $\overline{\mathcal{A}}$) which is submultiplicative and has the property that $M_{\mathcal{C}_k} f(x) \approx M^{k+1} f(x)$.

5.3. **Theorem 3.3, Part** (c)**.** This part is proved essentially as Part (a): the main change consists of taking the corresponding one-sided sharp operator (see [12] for more details). For the analog of (5.13) with $M^+_{\overline{\mathcal{A}}}$ in place of $M_{\overline{\mathcal{A}}}$ and with $M^- w$ in place of Mw see [24]. The extrapolation results needed here follow as in [5], see [7].

5.4. **Theorem 3.3, Part** (d)**.** This part is proved essentially as Part (b): the main change consists of taking the corresponding one-sided sharp operator (see [12] for more details).

5.5. **Proof of Theorem 3.6.** In the one-sided case, this result appears in [24]. The argument can be adapted *mutatis mutandis* to the general case as follows. First, by Vitali's covering Lemma and by using that $w \in A_r$ one gets, for any $\lambda > 0$,

$$w\{x \in \mathbb{R}^n : M_{\overline{\mathcal{A}}} f(x) > \lambda\} \le C \int_{\{x:|f(x)|>\lambda/2\}} \overline{\mathcal{A}}\left(\frac{|f(x)|}{\lambda}\right)^r w(x)\,dx.$$

Integrating this against $p\,\lambda^{p-1}\,d\lambda$ on $(0,\infty)$, we use Fubini and the fact $\overline{\mathcal{A}}(t)^r \in B_p$ to derive the desired estimate.

5.6. **Proof of Theorem 3.9.** We start with (a). By duality, (3.6) turns out to be equivalent to

$$\int_{\mathbb{R}^n} |T^* f(x)|^{p'} M_{\mathcal{D}} w(x)^{1-p'}\,dx \le C \int_{\mathbb{R}^n} |f(x)|^{p'} w(x)^{1-p'}\,dx.$$

We use that $M_{\mathcal{D}} w(x)^{1-p'} \in A_\infty$ (see [18]), and so we apply (3.5). This and the generalized Hölder inequality for $\overline{\mathcal{A}}$, $\mathcal{E}$ and $\mathcal{F}$ with $\mathcal{F}(t^{1/p}) = \mathcal{D}(t)$ yield

$$\begin{aligned} \int_{\mathbb{R}^n} |T^* f(x)|^{p'} M_{\mathcal{D}} w(x)^{1-p'}\,dx &\le C \int_{\mathbb{R}^n} M_{\overline{\mathcal{A}}} f(x)^{p'} M_{\mathcal{D}} w(x)^{1-p'}\,dx \\ &\le C \int_{\mathbb{R}^n} M_{\mathcal{E}}(f\,w^{-\frac{1}{p}})(x)^{p'} M_{\mathcal{F}}(w^{\frac{1}{p}})(x)^{p'} M_{\mathcal{D}} w(x)^{1-p'}\,dx \\ &= C \int_{\mathbb{R}^n} M_{\mathcal{E}}(f\,w^{-\frac{1}{p}})(x)^{p'} M_{\mathcal{D}} w(x)^{\frac{p'}{p}} M_{\mathcal{D}} w(x)^{1-p'}\,dx \end{aligned}$$

$$= C \int_{\mathbb{R}^n} M_{\mathcal{E}}(f\, w^{-\frac{1}{p}})(x)^{p'}\, dx \le C \int_{\mathbb{R}^n} |f(x)\, w(x)^{-\frac{1}{p}}|^{p'}\, dx$$
$$= C \int_{\mathbb{R}^n} |f(x)|^{p'}\; w(x)^{1-p'}\, dx,$$

where we have used that $\mathcal{E} \in B_{p'}$ and so $M_{\mathcal{E}}$ is bounded on $L^{p'}$ (see [21]).

Part (b) follows almost identically, the only thing to observe is that $M_{\mathcal{D}}^{-} w(x)^{1-p'} \in A_\infty^{-}$ (see [24] and [15]).

6. Proofs related to the applications

6.1. **Homogeneous Singular Integrals.** Notice that Theorem 4.1 follows from Theorem 3.3 combined with Proposition 4.2 whose proof is given next.

Proof of Proposition 4.2. We show that $K \in H_{\mathcal{B},k}$ —the case $K \in H_{\infty,k}$ follows in the same way and is left to the reader—. We proceed as in [10]. Without loss of generality we assume that $\|\Omega\|_{\mathcal{B},\Sigma} = 1$. We first show that for each $s > 0$

$$\|K(\cdot - y) - K(\cdot)\|_{\mathcal{B},|x|\sim s} \le C\, s^{-n} \left(\frac{|y|}{s} + \varpi_{\mathcal{B}}\left(\frac{|y|}{s}\right)\right), \qquad |y| < \frac{s}{2}. \tag{6.1}$$

Note that if $|x| \sim s$ and $|y| < s/2$ then $s/2 < |x-y| < 5\,s/2$ and therefore

$$|K(x-y) - K(x)| \le C\, s^{-n} \Big(|\Omega(x-y) - \Omega(x)| + \frac{|y|}{s}\, |\Omega(x)|\Big).$$

On the other hand,

$$\frac{1}{|B(0,2\,s)|} \int_{|x|\sim s} \mathcal{B}\left(\frac{|\Omega(x)|}{\lambda}\right) dx \le \frac{1}{\sigma(\Sigma)} \int_{\Sigma} \mathcal{B}\left(\frac{|\Omega(x')|}{\lambda}\right) d\sigma(x')$$

which implies that $\|\Omega\|_{\mathcal{B},|x|\sim s} \le \|\Omega\|_{\mathcal{B},\Sigma}$. Besides, let $\lambda > \varpi_{\mathcal{B}}(|y|/s)$. Then, writing $z = -y/r$ with $s \le r \le 2\,s$ we have $|z| \le |y|/s$ and so

$$\frac{1}{\sigma(\Sigma)} \int_{\Sigma} \mathcal{B}\left(\frac{|\Omega(x'+z) - \Omega(x')|}{\lambda}\right) d\sigma(x') \le 1.$$

Consequently,

$$\frac{1}{|B(0,2\,s)|} \int_{|x|\sim s} \mathcal{B}\left(\frac{|\Omega(x-y) - \Omega(x)|}{\lambda}\right) dx$$
$$= \frac{n}{(2\,s)^n \sigma(\Sigma)} \int_s^{2\,s} \int_{\Sigma} \mathcal{B}\left(\frac{|\Omega(x' + (-y/r)) - \Omega(x'))|}{\lambda}\right) d\sigma(x')\, r^{n-1}\, dr \le 1,$$

for all $\lambda > \varpi_{\mathcal{B}}(|y|/s)$. This yields that $\|\Omega(\cdot - y) - \Omega(\cdot)\|_{\mathcal{B},|x|\sim s} \le \varpi_{\mathcal{B}}(|y|/s)$. Collecting the obtained estimates we conclude (6.1).

This estimate leads us to prove that $K \in H_{\mathcal{B},k}$. Indeed, let $R > 0$ and $|y| < R$. Using (6.1) with $s = 2^m\, R$ and since $|y| < R \le s/2$ we have

$$\sum_{m=1}^{\infty} (2^m\, R)^n\, m^k\, \|K(\cdot - y) - K(\cdot)\|_{\mathcal{B},|x|\sim 2^m R} \le C \sum_{m=1}^{\infty} m^k \left(\frac{|y|}{2^m\, R} + \varpi_{\mathcal{B}}\left(\frac{|y|}{2^m\, R}\right)\right)$$
$$\le\; C \sum_{m=1}^{\infty} m^k\, (2^{-m} + \varpi_{\mathcal{B}}(2^{-m})) \le C + C \int_0^1 \left(1 + \log\frac{1}{t}\right)^k \varpi_{\mathcal{B}}(t) \frac{dt}{t} < \infty,$$

where the last inequality uses (4.1). □

6.2. **Multipliers.** In order to prove Theorem 4.5 we first decompose the operator T as in [10]. Let $\phi \in C^\infty$ be supported in $\{\xi : 1/2 < |\xi| < 2\}$ so that

$$\sum_j \phi_j(\xi) = \sum_j \phi(2^{-j}\,\xi) = 1, \qquad \xi \neq 0.$$

We write $m_j(\xi) = \phi_j(\xi)\, m(\xi)$ and so $m(\xi) = \sum_j m_j(\xi)$ for $\xi \neq 0$. Notice that m_j is supported in $\{\xi : 2^{j-1} < |\xi| < 2^{j+1}\}$. Let us set $K_j = \check{m}_j$ and

$$m^N(\xi) = \sum_{|j|\le N} m_j(\xi), \qquad K^N(x) = (m^N)\check{}\,(x) = \sum_{|j|\le N} K_j(x).$$

As it is done in [10] one can show that if $m \in M(s_0, l_0)$ and $\frac{n}{s_0} < l_0 < \frac{n}{s_0} + 1$ then

$$\|K^N(\cdot - y) - K^N(\cdot)\|_{L^{s_0'},|x|\sim R} \le C\, R^{-n} \left(\frac{|y|}{R}\right)^{l_0 - \frac{n}{s_0}}, \qquad |y| < \frac{R}{2},$$

where C does not depend on N. This implies that $K^N \in H_{s_0',k}$ for all $k \ge 0$ and this happens uniformly on N: for all $R > 0$ and $|y| < R$,

$$\sum_{j=1}^{\infty} (2^j\, R)^n\, j^k\, \|K^N(\cdot - y) - K^N(\cdot)\|_{L^{s_0'},|x|\sim 2^j\, R} \le C \sum_{j=1}^{\infty} j^k \left(\frac{|y|}{2^j\, R}\right)^{l_0 - \frac{n}{s_0}}$$

$$\le C \sum_{j=1}^{\infty} j^k\, 2^{-j\,(l_0 - \frac{n}{s_0})} \le C,$$

where C does not depend on N and where we have used that $l_0 > \frac{n}{s_0}$. In short, from [10], one gets the following:

Lemma 6.1. *If $m \in M(s_0, l_0)$ and $\frac{n}{s_0} < l_0 < \frac{n}{s_0} + 1$ then $K^N \in H_{s_0',k}$ for all $k \ge 0$.*

To prove Theorem 4.5 we need the following result.

Proposition 6.2. *If $m \in M(s, l)$ with $1 < s \le 2$, $1 \le l \le n$ and with $l > n/s$, then for all $k \ge 0$ and all $1 < r < (n/l)'$ we have that $K^N \in H_{L^r\,(\log L)^{k\,r},k}$ uniformly in N.*

Proof. Fixed s, l and $1 < r < s_0' = (n/l)'$, we take $r_0 = s_0 + \varepsilon$ where $\varepsilon > 0$ is small enough so that

$$s_0 < r_0 < \min\{r', s\}, \qquad \frac{n}{s_0} < \frac{n}{r_0} + 1.$$

This can be done since $s_0 < r'$ and $s_0 < s$ by assumption. Note that as $r_0 < s$ and $m \in M(s, l)$ then $m \in M(r_0, l)$. Note also that our choice of r_0 guarantees that $\frac{n}{r_0} < l < \frac{n}{r_0} + 1$. Thus, we can apply Lemma 6.1 and obtain that $K^N \in H_{r_0',k}$ for all $k \ge 0$. Notice that $r_0 < r'$ and so $r < r_0'$. Setting $\mathcal{A}(t) = t^r(1 + \log^+ t)^{k\,r}$ we have that $\mathcal{A}(t) \le C\, t^{r_0'}$ for all $t \ge 1$. This implies, using (2.1), that $\|\cdot\|_{\mathcal{A},B} \le C\,\|\cdot\|_{L^{r_0'},B}$ for every ball B. Therefore $K^N \in H_{r_0',k}$ implies that $K^N \in H_{L^r\,(\log L)^{k\,r},k}$. □

Proof of Theorem 4.5. We take $N > 1$ and consider the operator T^N whose kernel is K^N. We write $r' = n/l + \varepsilon$ and observe that $1 < r < (n/l)'$. Set $\mathcal{B}(t) = t^r(1 + \log^+ t)^{k\,r}$ and $\mathcal{A}(t) = t^r$, then $\overline{\mathcal{A}}^{-1}(t)\,\mathcal{B}^{-1}(t)\,\overline{\mathcal{C}}_k^{-1}(t) \le C\,t$. Since Proposition 6.2 implies that $K^N \in H_{\mathcal{B},k}$ we can use Theorem 3.3 and therefore (3.1) holds with $M_{\overline{\mathcal{A}}} = M_{L^{r'}}$ with a constant independent of N. A standard approximation argument as in [10] leads to the desired estimate for T. □

6.3. **One-sided operators.** Theorems 4.6 follows from (d) in Theorem 3.3 after showing that $K \in H_\infty \cap H_{e^{t^{1/k}},k}$. On the other hand, to prove Theorem 4.9, we obtain that $K \in H_{e^{t^{1/(1+\varepsilon+k)}},k}$ and by Theorem 3.3 part (c) with $\mathcal{B}(t) = e^{t^{1/(1+\varepsilon)}}$ and $\mathcal{A}(t) = e^{t^{1/(1+\varepsilon+k)}}$ we get the desired estimate.

So in both cases, everything reduces to get appropriate estimates for the kernels. To do so, we are going to write both operators in a more general way: we take a sequence of positive numbers $\{a_j\}_{j\in\mathbb{Z}}$ so that $\{a_j\}_j \in \ell^\infty$ and

$$T^+ f(x) = \sum_{j\in\mathbb{Z}} (-1)^j \big(D_j f(x) - D_{j-1} f(x)\big),$$

where

$$D_j f(x) = \frac{a_j}{2^j} \int_x^{x+2^j} f(t)\, dt.$$

Notice that the operator in Section 4.5.1 corresponds to $a_j = (1+j^2)^{-1}$ and the one in Section 4.5.2 to $a_j \equiv 1$.

It can bee seen that $T^+ f(x)$ exists almost everywhere and also that $T^+ f = K * f$ where

$$K(x) = \sum_{j\in\mathbb{Z}} (-1)^j \left(\frac{a_j}{2^j}\, \chi_{(-2^j,0)}(x) - \frac{a_{j-1}}{2^{j-1}}\, \chi_{(-2^{j-1},0)}(x) \right).$$

Observe that for each $x \in \mathbb{R}$, the series defining $K(x)$ is absolutely convergent. Let us notice that K is supported on $(-\infty, 0)$ so T^+ is a one-sided operator. The following result is essentially contained in [28] and can be obtained following the same ideas, so we omit the proof.

Lemma 6.3. *Let $2^j < |x| \le 2^{j+1}$ and $|y| \le 2^i$ with $j > i$. Then*

$$|K(x-y) - K(x)| = \begin{cases} 0 & \text{if} \quad x > 0 \\ 2\, a_j\, 2^{-j}\, \chi_{(-2^j+y,-2^j)}(x) & \text{if} \quad x < 0,\ y \le 0 \\ 2\, a_{j+1}\, 2^{-(j+1)}\, \chi_{(-2^{j+1},-2^{j+1}+y]}(x) & \text{if} \quad x < 0,\ y \ge 0. \end{cases}$$

Proposition 6.4. *Let $\mathcal{A}$ be a Young function, then*

$$K \in H_{\mathcal{A},k} \qquad \Longleftrightarrow \qquad \sup_{j\in\mathbb{Z}} \sum_{m=1}^\infty \frac{m^k}{\mathcal{A}^{-1}(8\,c\,2^m)}\, a_{m+j} < \infty, \quad \text{for some} \quad c > 1.$$

Analogously, $K \in H_{\infty,k}$ if and only if $\sup_{j\in\mathbb{Z}} \sum_{m=1}^\infty m^k\, a_{m+j} < \infty$.

Remark 6.5. We assumed before that $\{a_j\}_j \in \ell^\infty(\mathbb{Z})$. We have done so since this is equivalent to $K \in H_1$: It is clear that $\{a_j\}_j \in \ell^\infty(\mathbb{Z})$ implies that $K \in H_1$. On the other hand, if we assume that $K \in H_1$ then

$$\infty > \sup_{j\in\mathbb{Z}} \sum_{m=1}^\infty \frac{1}{2^m}\, a_{m+j} \ge \frac{1}{2} \sup_{j\in\mathbb{Z}} a_{1+j} = \frac{1}{2}\, \|\{a_j\}\|_{\ell^\infty(\mathbb{Z})}.$$

Let us point out that once we have assumed that $\{a_j\}_j \in \ell^\infty(\mathbb{Z})$ it follows that $K \in H_{r,k}$ for all $1 \le r < \infty$ and all $k \ge 0$.

Proof. We leave the case $H_{\infty,k}$ to the reader. Assume that $K \in H_{\mathcal{A},k}$. We use this condition with $R = 2^{j_0}$ for any fixed $j_0 \in \mathbb{Z}$, and Lemma 6.3 with $i = j_0$, $j = m+j_0 > i$ and $y \leq 0$ to obtain

$$\begin{aligned} C_{\mathcal{A}} &\geq \sup_{-R/(2\,c)<y\leq 0} \sum_{m=1}^{\infty} 2^{m+j_0}\, m^k\, \|2\, a_{m+j_0}\, 2^{-(m+j_0)}\, \chi_{(-2^{m+j_0}+y,-2^{m+j_0})}(\cdot)\|_{\mathcal{A},|x|\sim 2^{m+j_0}} \\ &\geq 2 \sup_{-R/(2\,c)<y\leq -R/(4\,c)} \sum_{m=1}^{\infty} \frac{m^k\, a_{m+j_0}}{\mathcal{A}^{-1}(2^{m+j_0+1}/|y|)} \geq 2 \sum_{m=1}^{\infty} \frac{m^k\, a_{m+j_0}}{\mathcal{A}^{-1}(8\,c\,2^m)}. \end{aligned}$$

Note that the last estimate holds for every $j_0 \in \mathbb{Z}$ and so we conclude with this part.

Let us show the converse. We first see that the $H_{\mathcal{A},k}$ condition holds for $R = 2^{j_0}$ for any $j_0 \in \mathbb{Z}$ and with C independent of j_0. If $y < 0$ and $|y| < R/(4\,c)$ we have $|y| < R/2 = 2^{j_0-1}$ and so for $m \geq 1$ we can use Lemma 6.3 as before

$$\begin{aligned} &\sum_{m=1}^{\infty} 2^m\, R\, m^k\, \|K(\cdot - y) - K(\cdot)\|_{\mathcal{A},|x|\sim 2^m\, R} \\ &\quad = \sum_{m=1}^{\infty} 2^{m+j_0}\, m^k\, \|2\, a_{m+j_0}\, 2^{-(m+j_0)}\, \chi_{(-2^{m+j_0}+y,-2^{m+j_0})}(\cdot)\|_{\mathcal{A},|x|\sim 2^{m+j_0}} \\ &\quad \leq 2 \sum_{m=1}^{\infty} \frac{m^k\, a_{m+j_0}}{\mathcal{A}^{-1}(8\,c\,2^m)} \leq 2 \sup_{j\in\mathbb{Z}} \sum_{m=1}^{\infty} \frac{m^k}{\mathcal{A}^{-1}(8\,c\,2^m)}\, a_{m+j} < \infty. \end{aligned} \tag{6.2}$$

A similar argument can be carried out when $y > 0$, the details are left to the reader.

Let us see now what to do for a general R. Let $j_0 \in \mathbb{Z}$ be such that $2^{j_0-1} < R \leq 2^{j_0}$. We write $\widetilde{R} = 2^{j_0-1}$. If $|y| < R/(8\,c)$ then $|y| < \widetilde{R}/(4\,c)$. On the other hand for every function h we have

$$\|h\|_{\mathcal{A},|x|\sim 2^m\, R} \leq 2\,(\|h\|_{\mathcal{A},|x|\sim 2^m\, \widetilde{R}} + \|h\|_{\mathcal{A},|x|\sim 2^{m+1}\, \widetilde{R}}).$$

Therefore, for $y < 0$, by (6.2) we conclude that

$$\begin{aligned} \sum_{m=1}^{\infty} 2^m\, R\, m^k\, \|K(\cdot - y) - K(\cdot)\|_{\mathcal{A},|x|\sim 2^m\, R} &\leq 4 \sum_{m=1}^{\infty} 2^m\, \widetilde{R}\, m^k\, \|K(\cdot - y) - K(\cdot)\|_{\mathcal{A},|x|\sim 2^m\, \widetilde{R}} \\ &\leq 8 \sup_{j\in\mathbb{Z}} \sum_{m=1}^{\infty} \frac{m^k}{\mathcal{A}^{-1}(8\,c\,2^m)}\, a_{m+j} < \infty. \end{aligned}$$

We can do the same when $y > 0$ and so $K \in H_{\mathcal{A},k}$. □

Next, we state the promised estimates for the kernels of the two one-sided operators in question.

Corollary 6.6.

(*a*) *If* $a_j = 1$ *for all* $j \in \mathbb{Z}$ *then for each* $k \geq 0$, $K \notin H_\infty$ *and* $K \in H_{e^{t^{1/(1+\varepsilon+k)}},k}$ *for all* $\varepsilon > 0$.

(*b*) *If* $a_j = (1+|j|)^{-2}$ *then* $K \in H_\infty \cap H_{e^{t^{1/k}},k}$.

Proof. We use the characterization given in Proposition 6.4. In (a), we obviously have that $K \notin H_\infty$. Besides, $K \in H_{e^{t^{1/(1+\varepsilon+k)}},k}$ since

$$\sup_{j\in\mathbb{Z}} \sum_{m=1}^{\infty} \frac{m^k}{\mathcal{A}^{-1}(2^m)}\, a_{m+j} = \sum_{m=1}^{\infty} \frac{m^k}{\mathcal{A}^{-1}(2^m)} = \sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}} < \infty.$$

For (b) we observe that $\{a_j\}_j \in \ell^1(\mathbb{Z})$ and so $K \in H_\infty$. On the other hand, if $\mathcal{A}(t) = e^{t^{1/k}}$ we have that $K \in H_{\mathcal{A},k}$, since

$$\sup_{j\in\mathbb{Z}} \sum_{m=1}^{\infty} \frac{m^k}{\mathcal{A}^{-1}(2^m)}\, a_{m+j} = \sup_{j\in\mathbb{Z}} \sum_{m=1}^{\infty} a_{m+j} = \sum_{j\in\mathbb{Z}} a_j < \infty.$$

□

7. Further extensions: multilinear commutators

In this section we extend the obtained results to the multilinear commutators considered in [22]. Given $k \ge 1$, a singular integral operator T with kernel K and a vector $\vec{b} = (b_1, \dots, b_k)$ of locally integrable functions, the multilinear commutator is defined as

$$T_{\vec{b}} f(x) = \int_{\mathbb{R}^n} \Big(\prod_{j=1}^{k} \big(b_j(x) - b_j(y)\big) \Big) K(x,y)\, f(y)\, dy.$$

When $k = 0$ we understand that $T_{\vec{b}} = T$. Notice that if $k = 1$ and $\vec{b} = b$ then $T_{\vec{b}} = T_b^1$. For $k \ge 1$ if $b_1 = \dots = b_k = b$ then $T_{\vec{b}} = T_b^k$.

For standard commutators, one assumes that $b \in \mathrm{BMO}$, and by John-Nirenberg's inequality we have that $\|b\|_{\mathrm{BMO}} \approx \sup_Q \|b - b_Q\|_{\exp L, Q}$. This can be seen as a supremum of the oscillations of b on the space $\exp L$.

As it was done in [22], when dealing with multilinear commutators, the symbols b_j are assumed to be in one of these oscillation spaces. Given $s \ge 1$ we set

$$\|f\|_{\mathrm{Osc}(\exp L^s)} = \sup_Q \|f - f_Q\|_{\exp L^s, Q}$$

and the space $\mathrm{Osc}(\exp L^s)$ is the set of measurable functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $\|f\|_{\mathrm{Osc}(\exp L^s)} < \infty$. Let us notice that $\mathrm{Osc}(\exp L^s) \subset \mathrm{Osc}(\exp L^1) = \mathrm{BMO}$.

We assume that, for each $1 \le j \le k$, $b_j \in \mathrm{Osc}(\exp L^{r_j})$ with $r_j \ge 1$. We set

$$\frac{1}{r} = \frac{1}{r_1} + \dots + \frac{1}{r_k}, \qquad \|\vec{b}\| = \prod_{j=1}^{k} \|b_j\|_{\mathrm{Osc}(\exp L^{r_j})}. \tag{7.1}$$

As done before with the standard commutators, we obtain estimates for $T_{\vec{b}}$ assuming different conditions on the kernel K. In contrast with Theorem 3.3, we only state the conclusions obtained by assuming that $K \in H_{\mathcal{B},k}$. Nevertheless, one can relax this hypothesis imposing conditions in the spirit of $K \in H_{\mathcal{B}} \cap H_{\mathcal{A},k}$. Details are left to the reader.

Theorem 7.1. *Let $k \ge 1$ and $\vec{b} = (b_1, \dots, b_k)$ such that $b_j \in \mathrm{Osc}(\exp L^{r_j})$ with $r_j \ge 1$ for $j = 1, \dots, k$. Let r be given by (7.1).*

(*a*) *Let $\mathcal{A}$, $\mathcal{B}$ be Young functions, such that $\overline{\mathcal{A}}^{-1}(t)\,\mathcal{B}^{-1}(t)\,\overline{\mathcal{C}}_{1/r}^{-1}(t) \le t$ with $\overline{\mathcal{C}}_{1/r}(t) = e^{t^r}$. If T is a singular integral operator with kernel $K \in H_{\mathcal{B},k}$, then for any $0 < p < \infty$, $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} |T_{\vec{b}} f(x)|^p\, w(x)\, dx \le C\, \|\vec{b}\|^p \int_{\mathbb{R}^n} M_{\overline{\mathcal{A}}} f(x)^p\, w(x)\, dx, \qquad f \in L^\infty_c, \tag{7.2}$$

whenever the left-hand side is finite. If one further assumes that $\overline{\mathcal{A}}$ is submultiplicative, then $T_{\vec{b}}$ satisfies the weak modular estimate (3.2) with $\|\vec{b}\|$ replacing $\|b\|^k_{\mathrm{BMO}}$.

(*b*) *If T is a singular integral operator with kernel $K \in H_{\infty,k}$, then for any $0 < p < \infty$, $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} |T_{\vec{b}} f(x)|^p\, w(x)\, dx \le C\, \|\vec{b}\|^p \int_{\mathbb{R}^n} M_{L\,(\log L)^{1/r}} f(x)^p\, w(x)\, dx, \qquad f \in L^\infty_c \tag{7.3}$$

whenever the left-hand side is finite. As a consequence, for all $w \in A_\infty$ and $\lambda > 0$

$$w\big\{x \in \mathbb{R}^n : |T^k_b f(x)| > \lambda\big\} \le C \int_{\mathbb{R}^n} \varphi_{1/r}\left(\frac{\|\vec{b}\|\,|f(x)|}{\lambda}\right) Mw(x)\, dx, \tag{7.4}$$

where $\varphi_{1/r}(t) = t\,(1 + \log^+ t)^{1/r}$

(*c*) *If T^+ is a one-sided singular integral operator as before, then (a) and (b) hold with the appropriate changes.*

Remark 7.2. In the previous result, taking $b_1 = \cdots = b_k = b \in \mathrm{BMO} = \mathrm{Osc}(\exp L^1)$ we have that $T_{\vec{b}} = T^k_b$. Note also that $r = 1/k$ as $r_j = 1$ for $1 \le j \le k$. Thus, we recover Theorem 3.3.

Part (*b*) was proved in [22] under the stronger assumption that K is a regular kernel (in our notation $K \in H^*_\infty$). Here we extend these estimates to less smooth kernels. The results for one-sided operators are new (even when $K \in H^*_\infty$).

Let us observe that from (7.2), two-weight estimates can be proved by means of Theorem 3.9. In (*b*), from (7.3), we can get the following two-weight estimate: for every weight $0 \le u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |T_{\vec{b}} f(x)|^p\, u(x)\, dx \le C \int_{\mathbb{R}^n} |f(x)|^p\, M^{[(1/r+1)\,p]+1} u(x)\, dx.$$

Note that, working as in Table 1, one can get a sharper result by taking on the righthand side $M_{\mathcal{D}} u$ with $\mathcal{D}(t) = t\,(1 + \log^+ t)^{(1/r+1)\,p-1+\varepsilon}$ for any $\varepsilon > 0$.

Proof. We need to introduce some notation. Given $\vec{b} = (b_1, \dots, b_k)$ we write $\bar{b} = b_1 \cdots b_k$. Let C^k_j, $1 \le j \le k$, be the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(j)\} \subset \{1, \dots, k\}$ of j different elements. In this case, we write $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$ and $\bar{b}_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$. We also set $C^k_0 = \emptyset$ in which case we understand that $T_{\vec{b}_\sigma} = T$ and $\bar{b}_\sigma = 1$. If $\sigma \in C^k_j$ we set $\sigma' = \{1, \dots, k\} \setminus \sigma$ (note that for $j = 0$ we have $\sigma' = \{1, \dots, k\}$).

By homogeneity we assume that for each $j = 1, \dots, k$, we have $\|b_j\|_{\mathrm{Osc}(\exp L^{r_j})} = 1$ and so $\|\vec{b}\| = 1$. Note that by the generalized Hölder inequality, for every $q > 0$,

$$\|f_1 \cdots f_k\|_{L^q,B} \le \|f_1 \cdots f_k\|_{\exp L^r,B} \le C\, \|f_1\|_{\exp L^{r_1},B} \cdots \|f_k\|_{\exp L^{r_k},B},$$

Using this inequality (with some of the functions identically one) one gets that for $\sigma \in C_m^k$, $1 \le m \le k$,

$$\Big\| \prod_{j=1}^m \big(b_{\sigma(j)} - (b_{\sigma(j)})_B\big)\Big\|_{L^q,B} \le \Big\| \prod_{j=1}^m \big(b_{\sigma(j)} - (b_{\sigma(j)})_B\big)\Big\|_{\exp L^r,B} \tag{7.5}$$

$$\le C \prod_{j=1}^m \|b_{\sigma(j)}\|_{\mathrm{Osc}(\exp L^{r_j})} = C. \tag{7.6}$$

We start with (a). The proof follows the ideas of Theorem 3.3 part (a) and we only give the main changes leaving the details to the reader. We obtain an analog of Lemma 5.1: if $0 < \delta < \varepsilon < 1$ then

$$M_\delta^\#\big(T_{\vec b} f\big)(x) \le C \sum_{m=0}^{k-1} \sum_{\sigma \in C_m^k} M_\varepsilon(T_{\vec b_\sigma} f)(x) + C\, M_{\overline{\mathcal{A}}} f(x). \tag{7.7}$$

Let us observe that we have normalized each b_j, otherwise as happened in Lemma 5.1, we need to introduce $\|b_j\|_{\mathrm{Osc}(\exp L^{r_j})}$. Once this estimate is shown, the induction argument used in the proof of Theorem 3.3 part (a) can be carried out with the appropriate changes and the desired estimates follows. Details are left to the reader.

To show (7.7), as in [22], for every $\vec\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$ we write

$$T_{\vec b} f(x) = T\big(\overline{(\lambda - b)}\, f\big)(x) + \sum_{m=0}^{k-1} c_{m,k} \sum_{\sigma \in C_m^k} \overline{(\lambda - b(x))}_{\sigma'}\, T_{\vec b_\sigma} f(x),$$

where $c_{m,k}$ are constants depending just on m and k. Given, $x \in \mathbb{R}^n$ and a ball $B = B(x_B, R) \ni x$ we write $\widetilde B = 2\,B$, $\lambda_j = (b_j)_{\widetilde B}$ and $f = f_1 + f_2 = f\, \chi_{\widetilde B} + f\, \chi_{\widetilde B^c}$. Let a_B be a constant to be chosen. Then as in (5.4), using the previous decomposition we have

$$\begin{aligned}
&\Big(\frac{1}{|B|}\int_B \big||T_{\vec b} f(y)|^\delta - |a_B|^\delta\big|\, dy\Big)^{\frac1\delta} \le \Big(\frac{1}{|B|}\int_B \big|T_{\vec b} f(y) - a_B\big|^\delta\, dy\Big)^{\frac1\delta} \\
&\le C \Bigg[\sum_{m=0}^{k-1} \sum_{\sigma \in C_m^k} \Big(\frac{1}{|B|}\int_B \big|\overline{(b(y) - \lambda)}_{\sigma'}\big|^\delta \big|T_{\vec b_\sigma} f(x)\big|^\delta\, dy\Big)^{\frac1\delta} \\
&\quad + \Big(\frac{1}{|B|}\int_B \big|T\big(\overline{(\lambda - b)}\, f_1\big)(y)\big|^\delta\, dy\Big)^{\frac1\delta} + \Big(\frac{1}{|B|}\int_B \big|T\big(\overline{(\lambda - b)}\, f_2\big)(y) - a_B\big|^\delta\, dy\Big)^{\frac1\delta}\Bigg] \\
&= C\,(I + II + III).
\end{aligned}$$

We estimate I as in (5.5), where we use (7.6) in place of (5.1). In this way, we obtain that I is controlled by the first term in the righthand side of (7.7).

For II we proceed as in (5.6): T is of weak-type $(1,1)$ (as $K \in H_1$), we have a generalized Hölder inequality for $\overline{\mathcal{A}}$, $\mathcal{B}$, $\overline{\mathcal{C}}_{1/r}$, and (7.6). Thus, $II \le C\, M_{\overline{\mathcal{A}}} f(x)$.

Finally, for III we take $a_B = T\big(\overline{(\lambda - b)}\, f_2\big)(x_B)$. As we are assuming that $K \in H_{\mathcal{B},k}$, we can simplify what was done in (5.7): observe that as in (5.2) we have

$$\|\overline{b - \lambda}\|_{\exp L^r, B_j} = \Big\| \prod_{m=1}^k |b_m - (b_m)_{\widetilde B}|\Big\|_{\exp L^r, B_j} \le C \prod_{m=1}^k \|b_m - (b_m)_{\widetilde B}\|_{\exp L^{r_m}, B_j}$$

$$\leq C \prod_{m=1}^{k} \big(\|b_m - (b_m)_{B_j}\|_{\exp L^{r_m}, B_j} + |(b_m)_{B_j} - (b_m)_{\widetilde{B}}| \big)$$

$$\leq C \prod_{m=1}^{k} (1 + 2^n j)\, \|b_m\|_{\mathrm{Osc}(\exp L^{r_m})} \leq C\, j^k.$$

This allows us to obtain for every $y \in B$,

$$\big|T\big(\overline{(\lambda - b)}\, f_2\big)(y) - a_B\big| = \big|T\big(\overline{(\lambda - b)}\, f_2\big)(y) - T\big(\overline{(\lambda - b)}\, f_2\big)(x_B)\big|$$

$$\leq C \sum_{j=1}^{\infty} (2^j R)^n \frac{1}{|B_j|} \int_{B_j} \big|\overline{b(z) - \lambda}\big|\, |K(y - z) - K(x_B - z)|\, \chi_{S_j}(z)\, |f(z)|\, dz$$

$$\leq C \sum_{j=1}^{\infty} (2^j R)^n \big\|\overline{b - \lambda}\big\|_{\exp L^r, B_j} \|f\|_{\overline{\mathcal{A}}, B_j} \|(K(y - \cdot) - K(x_B - \cdot))\, \chi_{S_j}\|_{\mathcal{B}, B_j}$$

$$\leq C\, M_{\overline{\mathcal{A}}} f(x) \sum_{j=1}^{\infty} (2^j R)^n j^k \|K(\cdot - (x_B - y)) - K(\cdot)\|_{\mathcal{B}, |z| \sim 2^j R}$$

$$\leq C\, M_{\overline{\mathcal{A}}} f(x),$$

where we have used that $K \in H_{\mathcal{B},k}$. This pointwise estimate implies that $III \leq C\, M_{\overline{\mathcal{A}}} f(x)$ and this concludes the proof. □

References

[1] H. Aimar, L. Forzani and F. J. Martín-Reyes, *On weighted inequalities for one-sided singular integrals*, Proc. Amer. Math. Soc. **125** (1997), 2057–2064.

[2] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, New York , 1998.

[3] A.L. Bernardis, M. Lorente, F.J. Martín-Reyes, M.T. Martínez, A. de la Torre and J.L. Torrea, *Differential transforms in weighted spaces*, J. Fourier Anal. Appl. **12** (2006), no. 1, 83–103.

[4] R. Coifman, *Distribution function inequalities for singular integrals*, Proc. Nat. Acad. Sci. U.S.A. **69** (1972), 2838–2839.

[5] G. Curbera, J. García-Cuerva, J. M. Martell and C. Pérez, *Extrapolation with Weights, Rearrangement Invariant Function Spaces, Modular inequalities and applications to Singular Integrals*, Adv. Math. **203** (2006), no. 1, 256–318.

[6] D. Cruz-Uribe, J.M. Martell and C. Pérez, *Extrapolation from A_∞ weights with applications*, J. Funct. Anal. **213** (2004), 412–439.

[7] D. Cruz-Uribe, J.M. Martell and C. Pérez, *Weights, Extrapolation and the Theory of Rubio de Francia*, Preprint 2007.

[8] C. Fefferman and E.M. Stein, *H^p spaces in several variables*, Acta Math. **129** (1972), 137–193.

[9] R.L. Jones and J. Rosenblatt, *Differential and ergodic transforms*, Math. Ann. **323** (2002), 525–546.

[10] D.S. Kurtz and R.L. Wheeden, *Results on weighted norm inequalities for multipliers*, Trans. Amer. Math. Soc. **255** (1979), 343–362.

[11] M. Lorente and M.S. Riveros, *Weighted inequalities for commutators of one-sided singular integrals*, Comment. Math. Univ. Carolinae **43** (2002), 83–101.

[12] M. Lorente, M.S. Riveros and A. de la Torre, *Weighted estimates for singular integral operators satisfying Hörmander's conditions of Young type*, J. Fourier Anal. Apl. **11** (2005), no. 5, 497–509.

[13] J.M. Martell, C. Pérez and R. Trujillo-González, *Lack of natural weighted estimates for some singular integral operators*, Trans. Amer. Math. Soc. **357** (2005), no. 1, 385–396.

[14] F.J. Martín-Reyes, P. Ortega and A. de la Torre, *Weighted inequalities for one-sided maximal functions*, Trans. Amer. Math. Soc. **319** (1990), 517–534.

[15] F.J. Martín-Reyes, L. Pick and A. de la Torre, A_∞^+ *condition*, Canad. J. Math. **45** (1993), 1231–1244.

[16] F.J. Martín-Reyes and A. de la Torre, *One Sided BMO Spaces*, J. London Math. Soc. **49** (1994), 529–542.

[17] R. O'Neil, *Fractional integration in Orlicz spaces*, Trans. Amer. Math. Soc. **115** (1963), 300–328.

[18] C. Pérez, *Weighted norm inequalities for singular integral operators*, J. London Math. Soc. **49** (1994), 296–308.

[19] C. Pérez, *Sharp estimates for commutators of singular integrals via iterations of the Hardy-Littlewood maximal function*, J. Fourier Anal. Appl. **3** (1997), 743–756.

[20] C. Pérez, *Endpoint estimates for commutators of singular integral operators*, J. Funct. Anal. **128** (1995), 163–185.

[21] C. Pérez, *On sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted* L^p*-spaces with different weights*, Proc. London Math. Soc. **71** (1995), 135–157.

[22] C. Pérez and R. Trujillo-Gonzalez, *Sharp weighted estimates for multilinear commutators*, J. London Math. Soc. (2) **65** (2002), 672–692.

[23] M.M. Rao and Z.D. Ren, *Theory of Orlicz spaces*, Monographs and Textbooks in Pure and Applied Mathematics **146**, Marcel Dekker, Inc., New York, 1991.

[24] M. S. Riveros, L. de Rosa, and A. de la Torre, *Sufficient Conditions for one-sided Operators*, J. Fourier Anal. Appl. **6** (2000), 607–621.

[25] J.L. Rubio de Francia, F.J. Ruiz and J. L. Torrea, *Calderón-Zygmund theory for vector-valued functions*, Adv. in Math. **62** (1986), 7–48.

[26] C. Segovia and J.L. Torrea, *Higher order commutators for vector-valued Calderón-Zygnund operators*, Trans. Amer. Math. Soc. **336** (1993), 537–556.

[27] E. Sawyer, *Weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Trans. Amer. Math. Soc. **297** (1986), 53–61.

[28] A. de la Torre and J.L. Torrea, *One-sided discrete square function*, Studia Math. **156** (2003), 243–260.

[29] D.K. Watson, *Weighted estimates for singular integrals via Fourier transform estimates*, Duke Math. J. **60** (1990), no. 2, 389–399.

[30] J.M. Wilson, *Weighted norm inequalities for the continuous square functions*, Trans. Amer. Math. Soc. **314** (1989), 661–692.

M. Lorente, Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, Spain

E-mail address: lorente@anamat.cie.uma.es

J. M. Martell, Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM, Consejo Superior de Investigaciones Científicas, C/ Serrano 121, 28006 Madrid, Spain

E-mail address: chema.martell@uam.es

M. S. Riveros, FaMAF, Universidad Nacional de Córdoba, CIEM (CONICET), 5000 Córdoba, Argentina

E-mail address: sriveros@mate.uncor.edu

A. de la Torre, Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, Spain

E-mail address: torre@anamat.cie.uma.es