WEIGHTED INEQUALITIES FOR GENERALIZED FRACTIONAL OPERATORS

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Abstract. In this note we present weighted Coifman type estimates, and two-weight estimates of strong and weak type for general fractional operators. We give applications to fractional operators given by an homogeneous function, and by a Fourier multiplier. The complete proofs of these results appear in the work [5] done jointly with Ana L. Bernardis and María Lorente.

1. Introduction and preliminaries

I would like to dedicate this note in memory of `Dr.Carlos Segovia`. First we will give some basic definitions and preliminaries needed to state the results. Let us recall some of the background on Orlicz spaces. (See [24] and [21] to complete this topic.)

A function $\mathcal{A} : [0,\infty) \to [0,\infty)$ is a Young function if it is continuous, convex, increasing and satisfies $\mathcal{A}(0) = 0$ and $\mathcal{A}(t) \to \infty$ as $t \to \infty$.

Given a Young function $\mathcal{A}$, we define the $\mathcal{A}$-mean Luxemburg norm of a function f on a cube (or a ball) Q in $\mathbb{R}^n$ by

$$||f||_{\mathcal{A},Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \mathcal{A}\left(\frac{|f|}{\lambda}\right) \leq 1\right\}. \tag{1.1}$$

It is well known that if $\mathcal{A}(t) \leq C\mathcal{B}(t)$ for all $t \geq t_0$ then $||f||_{\mathcal{A},Q} \leq C||f||_{\mathcal{B},Q}$, for all cubes Q and functions f. Thus, the behavior of $\mathcal{A}(t)$ for $t \leq t_0$ is not important. If $\mathcal{A} \approx \mathcal{B}$, that is there are constants $t_0, c_1, c_2 > 0$ such that $c_1\mathcal{A}(t) \leq \mathcal{B}(t) \leq c_2\mathcal{A}(t)$ for $t \geq t_0$, the latter estimate implies that $||f||_{\mathcal{A},Q} \approx ||f||_{\mathcal{B},Q}$.
Each Young function $\mathcal{A}$ has an associated complementary Young function $\overline{\mathcal{A}}$ satisfying

$$t \leq \mathcal{A}^{-1}(t)\overline{\mathcal{A}}^{-1}(t) \leq 2t,$$

for all $t > 0$. There is a generalization of Hölder's inequality

$$\frac{1}{|Q|}\int_Q |fg| \leq ||f||_{\mathcal{A},Q}||g||_{\overline{\mathcal{A}},Q}. \tag{1.2}$$

A further generalization of Hölder's inequality (see [21]) that will be useful later is the following: If $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are Young functions and

$$\mathcal{A}^{-1}(t)\mathcal{B}^{-1}(t) \leq \mathcal{C}^{-1}(t)$$

then

$$\|fg\|_{\mathcal{C},Q} \leq 2\|f\|_{\mathcal{A},Q}\|g\|_{\mathcal{B},Q}. \tag{1.3}$$

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When $\mathcal{A}(t) = t$ we understand that $\overline{\mathcal{A}}(t) = 0$ if $0 \le t \le 1$ and $\overline{\mathcal{A}}(t) = \infty$ otherwise. Then $\overline{\mathcal{A}}$ is not a Young function, but $L^{\overline{\mathcal{A}}} = L^\infty$ and the latter inequalities make sense if one of the functions is $\mathcal{A}$ or $\overline{\mathcal{A}}$.

For each locally integrable function f and $0 \le \alpha < n$, the fractional maximal operator associated to the Young function $\mathcal{A}$ is defined by

$$M_{\alpha,\mathcal{A}} f(x) = \sup_{Q \ni x} |Q|^{\alpha/n} ||f||_{\mathcal{A},Q}.$$

For $\alpha = 0$ we write $M_{\mathcal{A}}$ instead of $M_{0,\mathcal{A}}$. When $\mathcal{A}(t) = t$ then $M_{\alpha,\mathcal{A}} = M_\alpha$ is the classical fractional maximal operator. For $\alpha = 0$ and $\mathcal{A}(t) = t$ we obtain $M_{0,\mathcal{A}} = M$, the Hardy-Littlewood maximal operator. Consider the case $\alpha = 0$, for

$$\mathcal{A}(t) = t^r, \qquad \mathcal{A}(t) = t^r(1+\log^+(t))^\beta, \qquad \mathcal{A}(t) = t(1+\log^+(t))^k,$$

the maximal operators associated to these Young functions are:

$$M_r(f) = M(|f|^r)^{1/r}, \qquad M_{L^r(\log^+ L)^\beta}(f) \quad \text{and} \quad M_{L(\log^+ L)^k}(f) \quad .$$

If $k \ge 0$, $k \in \mathbb{Z}$, then $M_{L(\log^+ L)^k}$ is pointwise equivalent to M^{k+1}, where $M^k = \overbrace{M \circ \ldots \circ M}^{k \text{ times}}$, $k \in \mathbb{N}$. It is also easy to check that if $k > 0$ and $r > 1$, then

$$Mf(x) \le C M_{L(\log^+ L)^k} f(x) \le C M_r f(x) \quad .$$

The good weights for M are those in the A_p classes of Muckenhoupt (see [19] and also [26] and [18] for the one-sided case).

The good weights for the M_α maximal operator are the $A(p,q)$ classes. It is proved in [20] (see [1] for the one sided version) that $||(M_\alpha f)w||_q \le C||fw||_p$ if and only if $w \in A(p,q)$, for $1 < p < q$, $1/p - 1/q = \alpha/n$, where

$$\left(\frac{1}{|Q|}\int_{|Q|} w^q\right)^{1/q} \left(\frac{1}{|Q|}\int_{|Q|} w^{-p'}\right)^{1/p'} \le C, \qquad\qquad A(p,q)$$

for all cube Q.

Also observe that for the case $p = q$, $w \in A(p,p)$ is equivalent to say that $w^p \in A_p$.

Let us define a generalization of the Hörmander condition, for a given kernel K. We used the notation: $|x| \sim s$ for $s < |x| \le 2\,s$ and $\|f\|_{\mathcal{A},|x|\sim s} = \|f\,\chi_{\{|x|\sim s\}}\,\|_{\mathcal{A},B(0,2\,s)}$.

Definition 1.1. *Let $\mathcal{A}$ be a Young function and let $0 \le \alpha < n$. The kernel K_α is said to satisfy the $L^{\alpha,\mathcal{A}}$-Hörmander type condition, we write $K_\alpha \in H_{\alpha,\mathcal{A}}$, if there exist $c \ge 1$, $C > 0$ such that for any $y \in \mathbb{R}^n$ and $R > c\,|y|$,*

$$\sum_{m=1}^{\infty} (2^m\,R)^{n-\alpha}\,\|K_\alpha(\cdot - y) - K_\alpha(\cdot)\|_{\mathcal{A},|x|\sim 2^m\,R} \le C.$$

We say that $K_\alpha \in H_{\alpha,\infty}$ if K_α satisfies the previous condition with $\|\cdot\|_{L^\infty,|x|\sim 2^m\,R}$ in place of $\|\cdot\|_{\mathcal{A},|x|\sim 2^m\,R}$.

Definition 1.2. *The kernel K_α is said to satisfy the $H^*_{\alpha,\infty}$ condition, if there exist $c \ge 1$, $C > 0$ such that*

$$|K_\alpha(x-y) - K_\alpha(x)| \le C\frac{|y|}{|x|^{n+1-\alpha}}, \qquad |x| > c|y|\,.$$

Observe that when $\alpha = 0$ we obtain that $H_{0,\mathcal{A}} = H_{\mathcal{A}}$ defined in [16].

If $\mathcal{A}(t) = t^r$, for $r \geq 1$, then we write $H_{\alpha,\mathcal{A}} = H_{\alpha,r}$. This $H_{\alpha,r}$ condition appears implicitly in [12]. On the other hand, since $t \leq C\,\mathcal{A}(t)$ for $t \geq 1$ we have that $H_{\alpha,\mathcal{A}} \subset H_{\alpha,1}$. Also, it is easy to see that $H^*_{\alpha,\infty} \subset H_{\alpha,\infty} \subset H_{\alpha,\mathcal{A}}$.

Suppose that T is an operator given by convolution with a kernel K which satisfies some regularity condition and suppose that we know some behavior of T with respect to the Lebesgue's measure (weak or strong type inequalities for T). Sometimes, in order to know how is the behavior of T when we change the measure, (i.e., when we consider the measure $w(x)dx$ where w is a weight, ($0 \leq w \in L^1_{loc}(\mathbb{R}^n)$)) the following inequality is useful (we call it a Coifman type inequality)

$$\int |Tf|^p w \leq C \int (\mathcal{M}_T f)^p w\,. \tag{1.4}$$

Here $\mathcal{M}_T$ is a maximal operator related to the operator T which is normally easier to deal with. In general, $\mathcal{M}_T$ is strongly related with the kernel K and its size is inverse to the smoothness of K: the rougher the kernel, the bigger the maximal.

For T a Calderón-Zygmund singular integral operator (i.e., $K \in H^*_\infty$, see Definition 1.2, for $\alpha = 0$) inequality (1.4) holds with $\mathcal{M}_T = M$, the Hardy-Littlewood maximal function, $0 < p < \infty$, and $w \in A_\infty$ (see [8]).

If T is a singular integral operator with less regular kernel, (see [13]) for example if the kernel K satisfies an L^r-Hörmander condition (Definition 1.1, for $\mathcal{A}(t) = t^r$ and $\alpha = 0$), then inequality (1.4) holds with $\mathcal{M}_T = M_{r'}$, with $1/r + 1/r' = 1$, for all $0 < p < \infty$, and $w \in A_\infty$ (see [25]).

For a Young function $\mathcal{A}$, the $L^{\mathcal{A}}$-Hörmander condition is introduced in [16], which generalized in the scale of the Orlicz spaces the L^r-Hörmander condition. In [16] the authors showed that, if the kernel $K \in H_{\mathcal{A}}$ (Definition 1.1, for $\alpha = 0$), then inequality (1.4) holds with $\mathcal{M}_T = M_{\overline{\mathcal{A}}}$, where $\overline{\mathcal{A}}$ is the complementary function of $\mathcal{A}$, for all $0 < p < \infty$, and $w \in A_\infty$.

The differential transform operator was studied in [11] and [3]. In [14] it is proved an inequality of the type (1.4), by showing that the kernel satisfies the L^A-Hörmander condition for $\mathcal{A}(t) = e^{(t^{\frac{1}{1+\epsilon}})} - 1$, ($\epsilon > 0$). Therefore, this operator satisfies inequality (1.4) with the maximal operator $M_{\overline{\mathcal{A}}}$, where $\overline{\mathcal{A}}(t) = t(\log(e+t))^{1+\epsilon}$ (actually they obtain a smaller operator since T is a one-sided operator, because supp$K \subset (-\infty, 0)$).

The Coifman type inequality allows us to obtain, for general linear operators, two-weight inequalities of the type

$$\int |Tf|^p w \leq C \int |f|^p \mathcal{M}_T w\,, \tag{1.5}$$

for $1 < p < \infty$ and in the endpoint case $p = 1$,

$$w\left(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}\right) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| \mathcal{M}_T w(x)\, dx, \tag{1.6}$$

for every weight w, with no assumptions on w. The operators $\mathcal{M}_T$ are again suitable maximal operators related with T and not necessarily the same for inequalities (1.4), (1.5) and (1.6).

There is a great amount of works that deal with inequalities of the type (1.5) and (1.6). When T is a Calderón-Zygmund operator (with kernel $K \in H^*_\infty$), inequality (1.5) holds with $\mathcal{M}_T = M^{[p]+1}$, where $[p]$ is the integer part of p (see [22]). In the endpoint case $p = 1$, inequality (1.6) for Calderón-Zygmund operators hold with $\mathcal{M}_T = M_{L\,(\log L)^\varepsilon}$, for any $\varepsilon > 0$, where $M_{L\,(\log L)^\varepsilon}$ is the maximal function associated to the Young function $\mathcal{A}(t) = t(1 + \log^+ t)^\varepsilon$. This result was proved by Carlos Pérez in [22]. For T a singular integral associated to a kernel K satisfying a general Hörmander's condition given by a Young function A, the corresponding results, that include as particular cases those of C. Pérez, has been proved in [14] and [15].

In 1974, Muckenhoupt and Wheeden [20] proved inequality (1.4) for T the classical Riesz potential I_α and $\mathcal{M}_T$ the fractional maximal function M_α, defined for $0 < \alpha < n$ and locally integrable function f by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

Observe that using the mean value theorem we can prove that, the kernel of the fractional integral, I_α, $K_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$, belongs to $H^*_{\alpha,\infty}$. For I_α, inequality (1.5) holds with $\mathcal{M}_T = M_{\alpha p}(M^{[p]})$ (this result is also due to Carlos Pérez, see [23]). On the other hand, inequality (1.6) for I_α holds with $\mathcal{M}_T = M_\alpha(M_{L\,(\log L)^\varepsilon})$ in [6] (see also [2]).

There are fractional integrals with less regular kernel than the Riesz transform (see for example [7], [12], [27], [20], [9], [10]). Suppose that Ω is homogeneous of degree zero and $\Omega \in L^s(S^{n-1})$, where S^{n-1} denotes the sphere of $\mathbb{R}^n$ and $s > 1$. Define the fractional integral associated to Ω by

$$T_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(y/|y|)}{|y|^{n-\alpha}} f(x-y)\, dy.$$

In [10] inequalities with weights were established for this operator, which generalized the corresponding inequalities for I_α given by Muckenhoupt and Wheeden in [20]. In a more general context and with an aditional condition in Ω, that is, Ω satisfying the $L^s(S^{n-1})$-Dini smoothness condition, Segovia and Torrea [27], studied the good weights for this operator and its commutators, using extrapolation theorems.

In this note we state and briefly sketch the proofs of the corresponding results for general fractional integrals T_α, $0 < \alpha < n$, given by convolution with a kernel K_α which satisfy a $H_{\alpha,\mathcal{A}}$ condition, for appropriate Young functions $\mathcal{A}$ (see Theorems 2.1, 2.3 and 2.5).

From now on, for $0 < \alpha < n$, T_α will be a fractional operator bounded from $L^p(dx)$ to $L^q(dx)$, for all $1 < p < q < \infty$ satisfying $1/p - 1/q = \alpha/n$.

2. Statements of the Results

Theorem 2.1. *Let $T_\alpha f = K_\alpha * f$ be a fractional operator given by a kernel K_α. Suppose T_α is of weak-type $(1, \frac{n}{n-\alpha})$.*

(*a*) *If $\mathcal{A}$ be a Young function and $K_\alpha \in H_{\alpha,\mathcal{A}}$, then for any $0 < p < \infty$ and $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} |T_\alpha f(x)|^p \, w(x) \, dx \le C \int_{\mathbb{R}^n} M_{\alpha,\overline{\mathcal{A}}} f(x)^p w(x) \, dx, \qquad f \in L_c^\infty . \tag{2.1}$$

(*b*) *Moreover, if the kernel K_α is supported in $(-\infty, 0)$, then for any $0 < p < \infty$, $w \in A_\infty^+$, it follows that (2.1) holds with $M^+_{\alpha,\overline{\mathcal{A}}} f$ in place of $M_{\alpha,\overline{\mathcal{A}}} f$ where $M^+_{\alpha,\overline{\mathcal{A}}} f(x) = \sup_{x<b} (b-x)^\alpha \|f\|_{\overline{\mathcal{A}},(x,b)}$.*

Remark 2.2. Observe that we can apply the theorem to I_α and I_α^+(respectively) obtaining the result in [20] and [17], for $\overline{\mathcal{A}}(t) = t$.

Proof. To prove this Theorem we use the sharp operator of $T_\alpha f$. Given $x \in \mathbb{R}^n$ and a cube $Q \ni x$, decompose $f = f_1 + f_2$, where $f_1 = f\chi_{2Q}$ and $f_2 = f - f_1$. For $T_\alpha f_1$ we use Kolmogorov and that T_α is of weak-type $(1, \frac{n}{n-\alpha})$. For the global part we use that T_α is the convolution with the kernel $K_\alpha \in H_{\mathcal{A}}$ and the generalized Hölders inequality. □

Theorem 2.3. *Let $\mathcal{A}$ be a Young function and $1 < p < \infty$. Suppose that there exist Young functions $\mathcal{E}$, $\mathcal{D}$ such that $\mathcal{E} \in B_{p'}$ and $\mathcal{E}^{-1}(t)\, \mathcal{F}^{-1}(t) \le \overline{\mathcal{A}}^{-1}(t)$ with $\mathcal{F}(t) = \mathcal{D}(t^p)$. Let T_α be a linear operator such that its adjoint T_α^* satisfies*

$$\int_{\mathbb{R}^n} |T_\alpha^* f(x)|^q \, w(x) \, dx \le C \int_{\mathbb{R}^n} M_{\alpha,\overline{\mathcal{A}}} f(x)^q \, w(x) \, dx, \qquad f \in L_c^\infty \tag{2.2}$$

for all $0 < q < \infty$ and $w \in A_\infty$. Then, for $p > 1$ and for any weight u,

$$\int_{\mathbb{R}^n} |T_\alpha f(x)|^p \, u(x) \, dx \le C \int_{\mathbb{R}^n} |f(x)|^p \, M_{\alpha p,\mathcal{D}} u(x) \, dx, \qquad f \in L_c^\infty . \tag{2.3}$$

Remark 2.4. For the applications below, and since all our operators are of convolution type, proving (2.2) for T_α^* or T_α turns out to be equivalent.

Proof. To prove this Theorem we use duality and apply Theorem 2.1. To do this we need the fact that the weight $(\mathcal{M}_T w)^\delta$ belongs to A_1, for all $0 < \delta < 1$ and any w. For the maximal operators $\mathcal{M}_T$ that appears in this proof, has been proved in [4]. □

Theorem 2.5. *Let $T_\alpha f = K_\alpha * f$ be a fractional operator. Suppose that there exists $\delta > 0$ such that for any $p \in (1, 1+\delta)$, there exists a Young function $\mathcal{D}_p$ satisfying*

$$\int_{\mathbb{R}^n} |T_\alpha f|^p u \le C \int_{\mathbb{R}^n} |f|^p M_{\alpha p,\mathcal{D}_p} u, \tag{2.4}$$

for all weight u. If $K_\alpha \in H_{\alpha,\mathcal{A}}$, then for any weight u,

$$u\left(\{x \in \mathbb{R}^n : |T_\alpha f(x)| > \lambda\}\right) \le \frac{C}{\lambda} \int_{\mathbb{R}^n} |f| (Mu + M_{\alpha,\overline{\mathcal{A}}}\, u + M_{\alpha p,\mathcal{D}_p} u) , \tag{2.5}$$

for all $\lambda > 0$.

3. Applications

3.1. **Fractional integrals associated to a homogeneous function.** Denote by S^{n-1} the unit sphere on $\mathbb{R}^n$. For $x \neq 0$, we write $x' = x/|x|$. Let us consider $\Omega \in L^1(S^{n-1})$. This function can be extended to $\mathbb{R}^n \setminus \{0\}$ as $\Omega(x) = \Omega(x')$ (abusing on the notation we call both functions Ω). Thus Ω is a function homogeneous of degree 0. Let $0 < \alpha < n$, and let $\mathcal{A}$ be a Young function such that $\mathcal{B}(t) = \mathcal{A}(t^{\frac{n-\alpha}{n}})$ is also a Young function. Let $\Omega \in L^{\mathcal{A}}(S^{n-1})$ and satisfying the $L^{\mathcal{A}}(S^{n-1})$-Dini smoothness condition, i.e.,

$$\int_0^1 \varpi_{\mathcal{A}}(t)\,\frac{dt}{t} < \infty, \tag{3.1}$$

where

$$\varpi_{\mathcal{A}}(t) = \sup_{|y|\leq t} \|\Omega(\cdot + y) - \Omega(\cdot)\|_{\mathcal{A},S^{n-1}}.$$

Consider the fractional operator

$$T_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^{n-\alpha}} f(x-y)dy.$$

Using Hölder's inequality with $\mathcal{B}$ and $\overline{\mathcal{B}}$ it is easy to see that $\Omega \in L^{\mathcal{A}}(S^{n-1})$ implies $\Omega \in L^{\frac{n}{n-\alpha}}(S^{n-1})$. Then, by the result in [7], $T_{\Omega,\alpha}$ is of weak type $(1, \frac{n}{n-\alpha})$, with respect to the Lebesgue's measure and is bounded from $L^p(dx)$ to $L^q(dx)$, whenever $1/p - 1/q = \alpha/n$, $1 < p < q < \infty$. We can prove (as in [14]) that the kernel $K_\alpha(x) = \frac{\Omega(x)}{|x|^{n-\alpha}}$ satisfies the $H_{\alpha,\mathcal{A}}$ condition. Therefore Theorems 2.1, 2.3 and 2.5 can be applied to the operator $T_{\Omega,\alpha}$.

In the particular case that $\mathcal{A}(t) = t^r$ with $r \geq \frac{n}{n-\alpha}$ we get the following:

Theorem 3.1. *Let $\Omega \in L^r(S^{n-1})$ be as above and satisfying the L^r-Dini condition.*

(*a*) *If $0 < p < \infty$ and $w \in A_\infty$, then*

$$\int_{\mathbb{R}^n} |T_{\Omega,\alpha} f(x)|^p w(x)dx \leq C \int_{\mathbb{R}^n} (M_{\alpha,r'} f)^p w(x)dx, \qquad f \in L_c^\infty. \tag{3.2}$$

(*b*) *If $1 < p < r$ and u a weight, then*

$$\int_{\mathbb{R}^n} |T_{\Omega,\alpha} f(x)|^p u(x)dx \leq C \int_{\mathbb{R}^n} |f|^p M_{\alpha p,\mathcal{D}_p} u(x)dx, \qquad f \in L_c^\infty. \tag{3.3}$$

(*c*) *If $1 < p < r$ and u is a weight, then*

$$u\{x \in \mathbb{R}^n : |T_{\Omega,\alpha} f(x)| > \lambda\} \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)|(M_{r'} u(x) + M_{\alpha p,\mathcal{D}_p} u(x))\,dx. \tag{3.4}$$

In both cases $\mathcal{D}_p(t) = t^{(r/p)'}(1+\log^+ t)^{(r/p)'(p-1)+\epsilon}$ and $\epsilon > 0$ is small enough.

Proof. We only have to apply the theorems with the following Young functions: $\mathcal{A}(t) = t^r$, $\mathcal{E}(t) = t^{p'}(1+\log^+ t)^{-1-\varepsilon}$, and $\mathcal{F}(t) = t^{\frac{rp}{r-p}}(1+\log^+ t)^{(r/p)'(p-1)+\epsilon}$, where $\varepsilon > 0$ is some small enough number that is related with $\epsilon > 0$. Observe that in part (c) we obtain $Mu + M_{\alpha,r'} u + M_{\alpha p,\mathcal{D}_p} u$ on the right hand side, but it is easy to see that $M_{\alpha,r'} u \leq M_{r'} u + M_{\alpha p,\mathcal{D}_p} u$ and $Mu \leq M_{r'} u$. □

For $T_{\Omega,\alpha}$ as above, we obtain the following weighted inequality as in [10] (see also [27]).

Corollary 3.2. *Suppose that we are under the same hypothesis as in Theorem 3.1. Let* $r' < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ *and* $w^{r'} \in A(p/r', q/r')$. *Then*

$$\left(\int_{\mathbb{R}^n} |T_{\Omega,\alpha} f(x)|^q w^q(x)dx\right)^{1/q} \le C \left(\int_{\mathbb{R}^n} |f|^p w^p(x)dx\right)^{1/p}.$$

Proof. First of all observe that $w^{r'} \in A(p/r', q/r')$ implies $w^q \in A_\infty$. Then by part (a) of Theorem 3.1

$$\int_{\mathbb{R}^n} |T_{\Omega,\alpha} f(x)|^q w^q(x)dx \le C \int_{\mathbb{R}^n} (M_{\alpha,r'} f)^q w^q(x)dx.$$

Since $M_{\alpha,r'} f(x) = \left(M_{\alpha r'}|f|^{r'}(x)\right)^{1/r'}$ and $w^{r'} \in A(p/r', q/r')$ (see [20]) we have that

$$\int_{\mathbb{R}^n} (M_{\alpha,r'} f)^q w^q = \int_{\mathbb{R}^n} \left(M_{\alpha r'}|f|^{r'}\right)^{q/r'} (w^{r'})^{q/r'} \le C \left(\int_{\mathbb{R}^n} \left(|f|^{r'}\right)^{p/r'} (w^{r'})^{p/r'}\right)^{q/p}.$$

□

3.2. **Fractional integrals associated to a multiplier.** Let $0 < \alpha < n$. Given a function m defined in $\mathbb{R}^n$ we consider the multiplier operator T_α defined *a priori* for functions f in the Schwartz class by $\widehat{T_\alpha f}(\xi) = m(\xi)\,\widehat{f}(\xi)$. Given $1 < s \le 2$ and $0 \le l \in \mathbb{N}$ we say that $m \in M(s, l, \alpha)$ if there exists a constant B such that $|m(x)| \le B|x|^{-\alpha}$ and

$$\sup_{R>0} R^{|\beta|+\alpha} \|D^\beta m\|_{L^s,|\xi|\sim R} < +\infty, \qquad \text{for all } |\beta| \le l.$$

In [12], Kurtz proved that if $n/s < l \le n$ and $m \in M(s, l, \alpha)$ then T_α is bounded from $L^p(dx)$ to $L^q(dx)$, for $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. If K_α is the kernel of T_α, he proved that $K_\alpha \in H_{\alpha,r}$ for all $1 < r < (n/l)'$ and, as a consequence, he obtained the following Coifman type inequality: for all $\varepsilon > 0$, $0 < p < \infty$ and $w \in A_\infty$,

$$\int_{\mathbb{R}^n} |T_\alpha f(x)|^p \, w(x)\, dx \le C \int_{\mathbb{R}^n} M_{\alpha,n/l+\varepsilon} f(x)^p \, w(x)\, dx. \tag{3.5}$$

Now we can apply Theorems 2.3 and 2.5 to this operator.

Theorem 3.3. *If* $1 < p < r < (n/l)'$ *and* u *a weight, then*

$$\int_{\mathbb{R}^n} |T_\alpha f(x)|^p u(x)dx \le C \int_{\mathbb{R}^n} |f|^p M_{\alpha p,\mathcal{D}_p} u(x)dx, \qquad f \in L^\infty_c. \tag{3.6}$$

and

$$u\{x \in \mathbb{R}^n : |T_\alpha f(x)| > \lambda\} \le \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)|(M_{r'} u(x) + M_{\alpha p,\mathcal{D}_p} u(x))\, dx, \tag{3.7}$$

where $\mathcal{D}_p(t) = t^{(r/p)'}(1 + \log^+ t)^{(r/p)'(p-1)+\epsilon}$ *and* $\epsilon > 0$ *is small enough.*

Observe that as ϵ is at our choice, we can write $\mathcal{D}_p(t) = t^{(r/p)'}(1+\log^+ t)^{(r/p)'(p-1)+\epsilon} \lesssim t^{(\tilde{r}/p)'}$, for all $1 < p < \tilde{r} < (n/l)'$. Therefore, we can write $\mathcal{D}_p(t) = t^{(r/p)'}$ in (3.6) and (3.7).

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